Part 2 Fourier Analysis and Power Spectrum Density

Fourier Analysis

- Fourier Transform Pair

Fourier Transform of $g(t)$: $$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)\,dt$$

Inverse Fourier Transform of $g(t)$: $$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

- *Fourier Transform* $G(f)$ is the (*frequency*) *spectrum* content of a signal $g(t)$.
  - $|G(f)|$ magnitude spectrum
  - $\arg\{G(f)\}$ phase spectrum

Dirichlet's Condition

- *Dirichlet's condition*
  - In every finite interval, $g(t)$ has a finite number of local maxima and minima, and a finite number of discontinuity points.
- Sufficient conditions for the **existence** of Fourier transform
  - $g(t)$ satisfies Dirichlet's condition
  - Absolute integrability: $\int_{-\infty}^{\infty} |g(t)|\,dt < \infty$

Dirichlet's Condition

- "Existence" means that the Fourier transform pair is valid only for **continuity** points.
  - $$g(t) = \begin{cases} 1, & -1 < t < 1; \\ 0, & |t| \ge 1. \end{cases} \quad \text{and} \quad \bar{g}(t) = \begin{cases} 1, & -1 \le t \le 1; \\ 0, & |t| > 1. \end{cases}$$

    has the **same** Fourier transform $G(f)$.

    Note that the above two functions are not equal at $t = 1$ and $t = -1$!

Dirac Delta Function

- It is a function that exists only in principle.
- **Define** the Dirac delta function as a function δ(*t*) satisfies:

$$\delta(t)=\begin{cases}\infty, & t=0;\\ 0, & t\neq 0.\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)dt=1.$$

  - δ(*t*) can be thought of as a limit of a unit-area pulse function.

$$\lim_{n\to\infty} s_n(t)=\delta(t), \text{ where } s_n(t)=\begin{cases} n, & -\frac{1}{2n}<t<\frac{1}{2n};\\ 0, & \text{otherwise.}\end{cases}$$

Properties of Dirac Delta Function

1. Sifting property
   - If $g(t)$ is continuous at t_0, then

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$$

$$\left(\int_{-\infty}^{\infty} g(t)s_n(t-t_0)dt = \int_{t_0-1/(2n)}^{t_0+1/(2n)} g(t)\cdot n \cdot dt \to g(t_0)\right)$$

   - The sifting property is not necessarily true at t_0 if $g(t)$ is **discontinuous** at t_0.

Properties of Dirac Delta Function

2. Replication property
   - For every continuous point of $g(t)$,

$$g(t) = \int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau$$

3. Constant spectrum

$$\int_{-\infty}^{\infty} \delta(t)\exp(-j2\pi ft)dt = \int_{-\infty}^{\infty} \delta(t-0)\exp(-j2\pi ft)dt = 1.$$

Thus, the inverse Fourier transform of 1 is (by definition) $\delta(t)$.

Properties of Dirac Delta Function

4. Scaling after integration

$f(x) = g(x) \Rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} g(x)dx$???

- Although

$$\delta(t) = 2 \cdot \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

their integrations (by replication property) are different

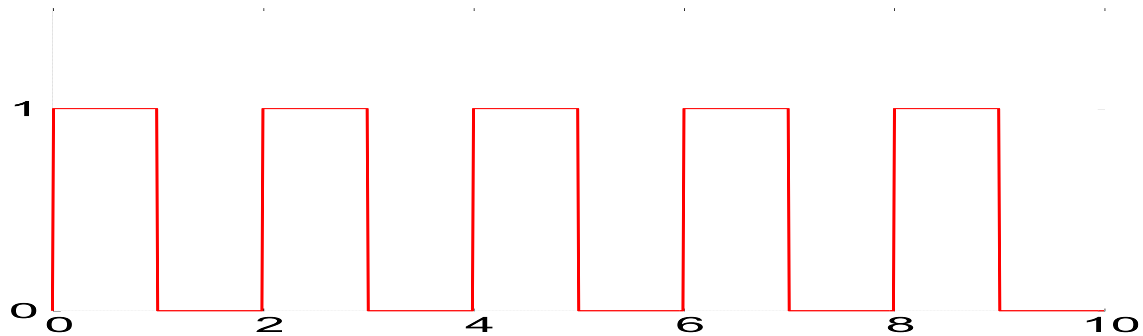
$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \quad \text{while} \quad \int_{-\infty}^{\infty} 2\delta(t)dt = 2.$$

- Hence, the "multiplicative constant" to the Dirac delta function is *significant*, and shall never be ignored!

Fourier Series

- The Fourier transform of a periodic function does not exist!
  - E.g., for integer k,

$$g(t) = \begin{cases} 1, & 2k \le t < 2k+1; \\ 0, & \text{otherwise.} \end{cases}$$



Fourier Series

- **Theorem**: If $g_T(t)$ is a bounded periodic function with period T and satisfies *Dirichlet's condition*, then

$$g_T(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \frac{2\pi n}{T} t \right)$$

at every **continuity** points of $g_T(t)$, where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} g_T(t) \exp\left(-j \frac{2\pi n}{T} t \right) dt$$

Relation between a Periodic Function and its Generating Function

- Define the *generating function* of a periodic function $g_T(t)$ with period T as:

$$g(t) = \begin{cases} g_T(t), & -T/2 \le t < T/2; \\ 0, & \text{otherwise.} \end{cases}$$

  - Then

$$g_T(t) = \sum_{m=-\infty}^{\infty} g(t - mT)$$

Relation between a Periodic Function and its Generating Function

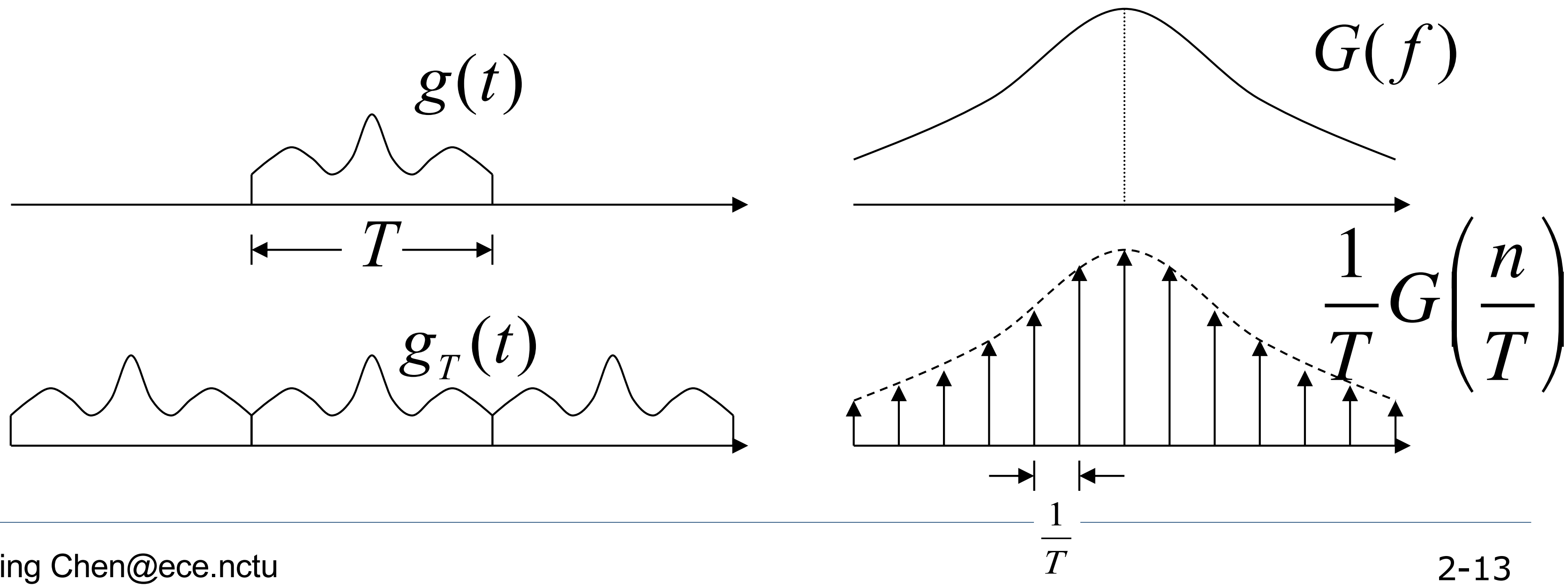
- Let $G(f)$ be the spectrum of $g(t)$ (which is assumed to exist).
- Then, from the Theorem in Slide 2-10,

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} g_T(t)\exp\left(-j2\pi\frac{n}{T}t\right)\mathrm{d}t \\ &= \frac{1}{T}\int_{-T/2}^{T/2} g(t)\exp\left(-j2\pi\frac{n}{T}t\right)\mathrm{d}t \\ &= \frac{1}{T}G\left(\frac{n}{T}\right) \end{aligned}$$

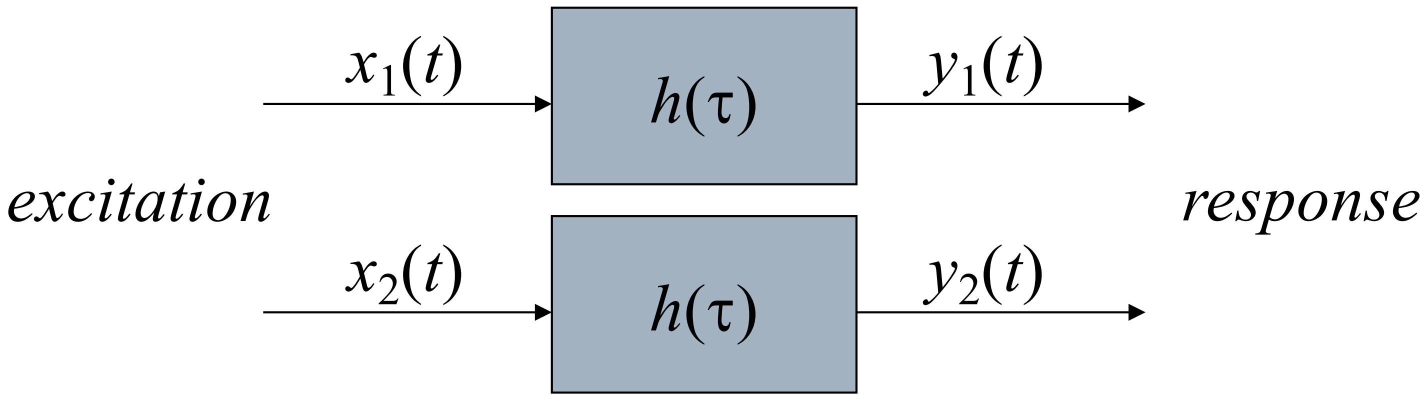
Relation between a Periodic Function and its Generating Function

- This concludes to **Poisson's sum formula**.

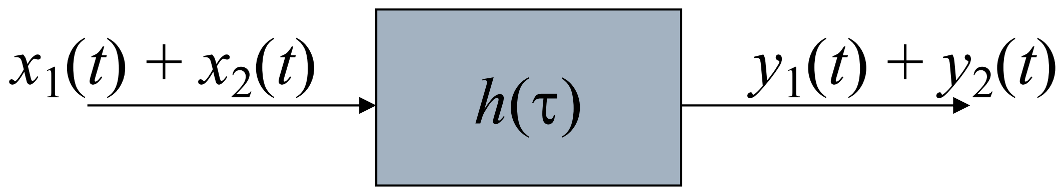
$$g_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G\left(\frac{n}{T}\right)\exp\left(j2\pi\frac{n}{T}t\right)$$



Spectrums through LTI Filter



A **linear** filter satisfies the **principle of superposition**, i.e.,

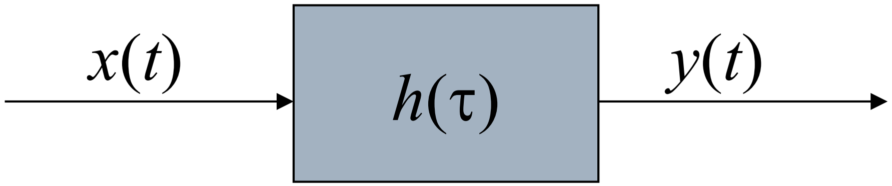


Linearity and Convolution

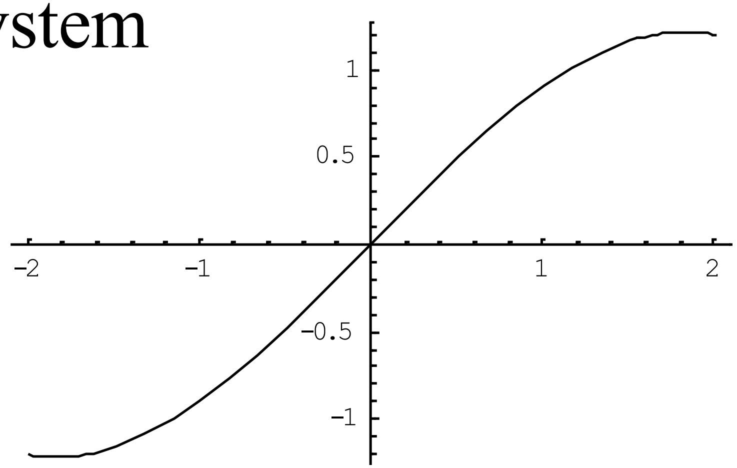
- A linear time-invariant (LTI) filter can be described by *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

  - Example of a **nonlinear** system



$$y(t) = x(t) - 0.1 \cdot x^3(t)$$



Linearity and Convolution

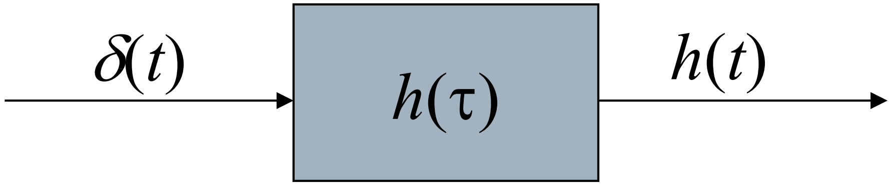
- Convolution in time = Multiplication in Spectrum

$$\boxed{y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau} \text{ and } \begin{cases} x(t) = \int_{-\infty}^{\infty} x(f)\exp(j2\pi ft)df \\ h(\tau) = \int_{-\infty}^{\infty} H(f)\exp(j2\pi f\tau)df \end{cases}$$

$$
\begin{aligned}
y(f) &= \int_{-\infty}^{\infty} y(t)\exp(-j2\pi ft)dt \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]\exp(-j2\pi ft)dt \\
&= \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t-\tau)\exp(-j2\pi ft)dt\right]d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(s)\exp(-j2\pi f(s+\tau))ds\right]d\tau, \;\; s = t-\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\exp(-j2\pi f\tau)\left[\int_{-\infty}^{\infty} x(s)\exp(-j2\pi fs)ds\right]d\tau \\
&= x(f)\int_{-\infty}^{\infty} h(\tau)\exp(-j2\pi f\tau)d\tau \\
&= \underline{x(f)H(f)}
\end{aligned}
$$

Impulse Response of LTI Filter

- Impulse response = Filter response to Dirac delta function (an application of the replication property)

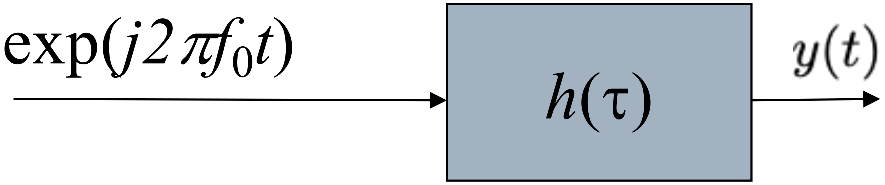


$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)\delta(t-\tau)d\tau=h(t).$$

provided $h(\tau)$ is continuous at $\tau=t$.

Frequency Response of LTI Filter

- Frequency response = Filter response to a complex exponential input of unit amplitude and of frequency f_0



$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\exp(j2\pi f_0(t-\tau))d\tau \\
&= \exp(j2\pi f_0 t)\int_{-\infty}^{\infty} h(\tau)\exp(-j2\pi f_0\tau)d\tau \\
&= \exp(j2\pi f_0 t)H(f_0)
\end{aligned}
$$

Measures for Frequency Response

Expression 1

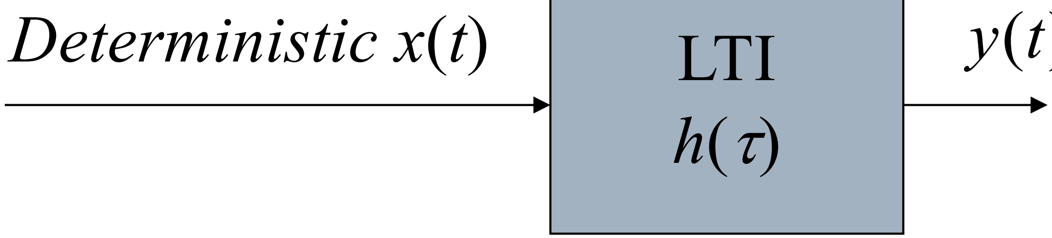
$$H(f) = |H(f)| \cdot \exp[j\beta(f)], \quad \text{where} \quad \begin{cases} |H(f)| & \text{amplitude response} \\ \beta(f) & \text{phase response} \end{cases}$$

Expression 2

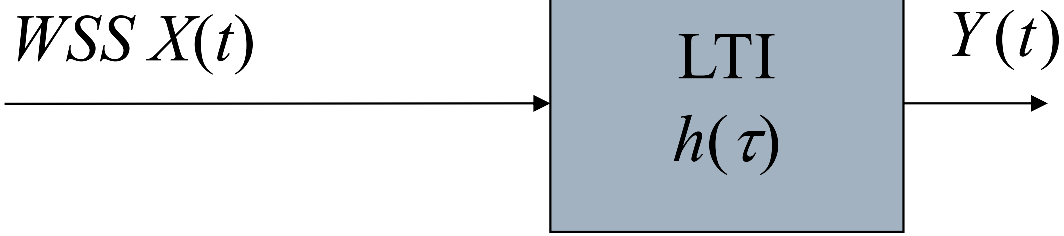
$$\begin{aligned} \log H(f) &= \log|H(f)| + j\beta(f) \\ &= \alpha(f) + j\beta(f) \end{aligned} \quad \text{where} \quad \begin{cases} \alpha(f) & \text{gain} \\ \beta(f) & \text{phase response} \end{cases}$$

$$\begin{aligned} \alpha(f) &= \ln|H(f)| \ \text{nepers} \\ &= 20\log_{10}|H(f)| \,\text{dB} \end{aligned}$$

Power Spectral Density



$$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$



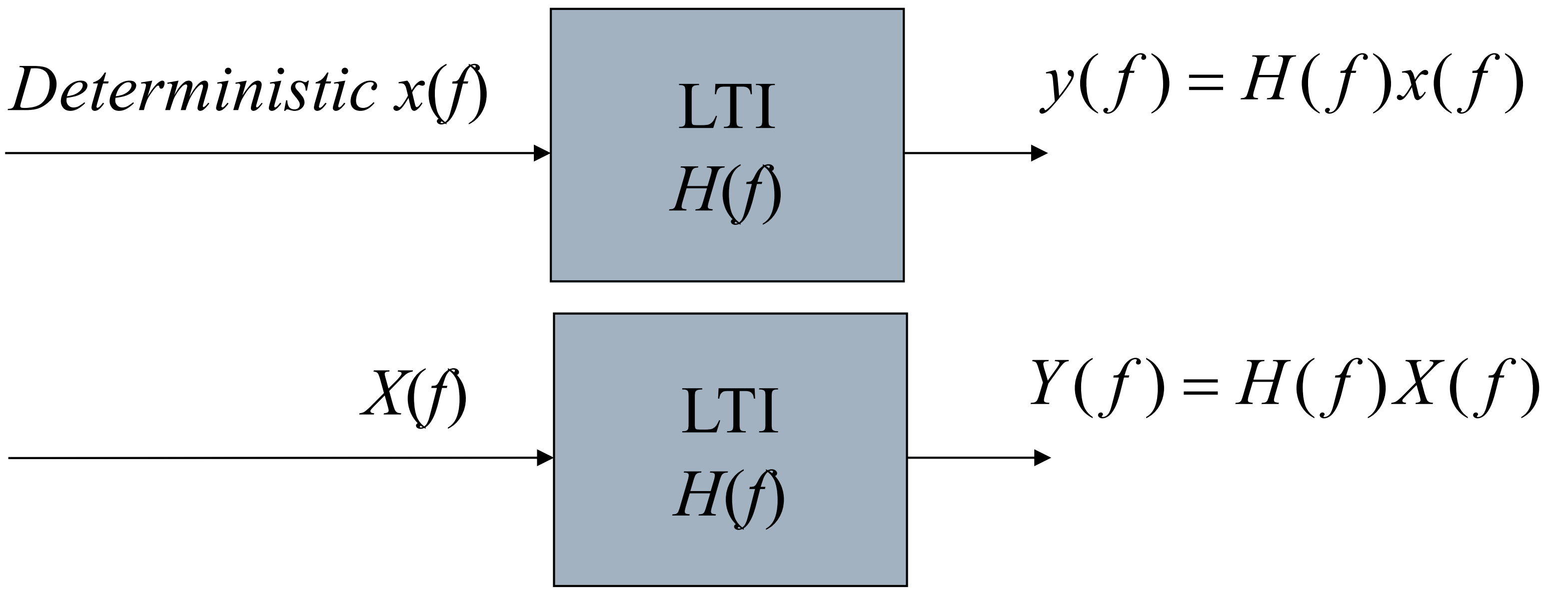
$$Y(t)=\int_{-\infty}^{\infty}h(\tau)X(t-\tau)d\tau$$

$$\mu_Y(t)=\int_{-\infty}^{\infty}h(\tau)\mu_X(t-\tau)d\tau$$

$$R_Y(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(\tau_1)h^*(\tau_2)R_X(\tau-\tau_1+\tau_2)d\tau_2 d\tau_1$$

Power Spectral Density

- How about the spectrum relation between filter input and filter output?
  - An apparent relation is:



Power Spectral Density

- This is however not adequate for a random process.
  - For a *random* process, what concerns us is the relation between the *input statistic* and *output statistic*.

Power Spectral Density

- How about the relation of the first two moments between filter input and output?
  - Spectrum relation of mean processes

$$\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau)\mu_X(t-\tau)d\tau$$

$$\Rightarrow \mu_Y(f) = \mu_X(f)H(f)$$

Time-Average Autocorrelation Function

- For a **non-stationary** process, we can use the *time-average autocorrelation function* to define the *average power correlation* for a given time difference.

$$\bar{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau)X^*(t)]dt$$

  - It is implicitly assumed that $\bar{R}_X(\tau)$ is independent of the location of the integration window. Hence,

$$\bar{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{3T/2} E[X(t+\tau)X^*(t)]dt$$

Time-Average Autocorrelation Function

- E.g., for a WSS process,

$$\bar{R}_X(\tau) = E[X(t+\tau)X^*(t)]$$

- E.g., for a deterministic function,

$$\begin{aligned}\bar{R}_X(\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[x(t+\tau)x^*(t)]dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t+\tau)x^*(t)dt\end{aligned}$$

Time-Average Autocorrelation Function

- E.g., for a cyclostationary process,

$$\bar{R}_X(\tau) = \frac{1}{2T}\int_{-T}^{T} E[X(t+\tau)X^*(t)]dt$$

where T is the cyclostationary period of $X(t)$.

Time-Average Autocorrelation Function

- The *time-average power spectral density* is the Fourier transform of the *time-average autocorrelation function.*

$$
\begin{aligned}
\bar{S}_X(f) &= \int_{-\infty}^{\infty}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E[X(t+\tau)X^*(t)]\mathrm{d}t\right)e^{-j2\pi f\tau}\mathrm{d}\tau \\
&= \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}X(t+\tau)X_{2T}^*(t)\mathrm{d}t\right)e^{-j2\pi f\tau}\mathrm{d}\tau\right] \\
&= \lim_{T\to\infty}\frac{1}{2T}E[X(f)X_{2T}^*(f)],\ \text{where } X_{2T}\triangleq X(t)\cdot\mathbf{1}\{|t|\le T\}.
\end{aligned}
$$

Time-Average Autocorrelation Function

- For a WSS process, $\bar{S}_X(f) = S_X(f)$.
- For a deterministic process,

$$\bar{S}_X(f) = \lim_{T\to\infty} \frac{1}{2T} x(f) x_{2T}^*(f).$$

Power Spectral Density

- Relation of time-average PSDs

$$
\begin{aligned}
R_Y(t+\tau,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)R_X(t+\tau-\tau_1,t-\tau_2)d\tau_2 d\tau_1\\
\bar{R}_Y(\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)R_X(t+\tau-\tau_1,t-\tau_2)d\tau_2 d\tau_1 dt\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_X(t+\tau-\tau_1,t-\tau_2)dt\right)d\tau_2 d\tau_1\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_2)\bar{R}_X(\tau-\tau_1+\tau_2)d\tau_2 d\tau_1
\end{aligned}
$$

$$
\begin{aligned}
\bar{S}_Y(f) &= \int_{-\infty}^{\infty} \bar{R}_Y(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \bar{R}_X(\tau - \tau_1 + \tau_2) d\tau_2 d\tau_1 \right) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \left(\int_{-\infty}^{\infty} \bar{R}_X(\tau - \tau_1 + \tau_2) e^{-j2\pi f\tau} d\tau \right) d\tau_2 d\tau_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h^*(\tau_2) \left(\int_{-\infty}^{\infty} \bar{R}_X(u) e^{-j2\pi f(u+\tau_1-\tau_2)} du \right) d\tau_2 d\tau_1 \\
& \quad \text{(We let } u = \tau - \tau_1 + \tau_2.) \\
&= \left(\int_{-\infty}^{\infty} h(\tau_1) e^{-j2\pi f\tau_1} d\tau_1 \right) \left(\int_{-\infty}^{\infty} h^*(\tau_2) e^{j2\pi f\tau_2} d\tau_2 \right) \left(\int_{-\infty}^{\infty} \bar{R}_X(u) e^{-j2\pi fu} du \right) \\
&= \left(\int_{-\infty}^{\infty} h(\tau_1) e^{-j2\pi f\tau_1} d\tau_1 \right) \left(\int_{-\infty}^{\infty} h(\tau_2) e^{-j2\pi f\tau_2} d\tau_2 \right)^* \left(\int_{-\infty}^{\infty} \bar{R}_X(u) e^{-j2\pi fu} du \right) \\
&= H(f) H^*(f) \bar{S}_X(f) \\
&= |H(f)|^2 \bar{S}_X(f)
\end{aligned}
$$

Power Spectral Density under WSS Input

- For a WSS filter input,

$$\mu_X(t) = \text{constant} = \mu_X$$
$$\Rightarrow \mu_X(f) = \int_{-\infty}^{\infty} \mu_X \exp(-j2\pi f t)dt = \mu_X \delta(f)$$

$$\bar{R}_X(\tau) = R_X(\tau)$$
$$\Rightarrow S_Y(f) = \bar{S}_Y(f) = |H(f)|^2 \bar{S}_X(f) = |H(f)|^2 S_X(f)$$

Power Spectral Density under WSS Input

- Observation

$$E[|Y(t)|^2] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df$$

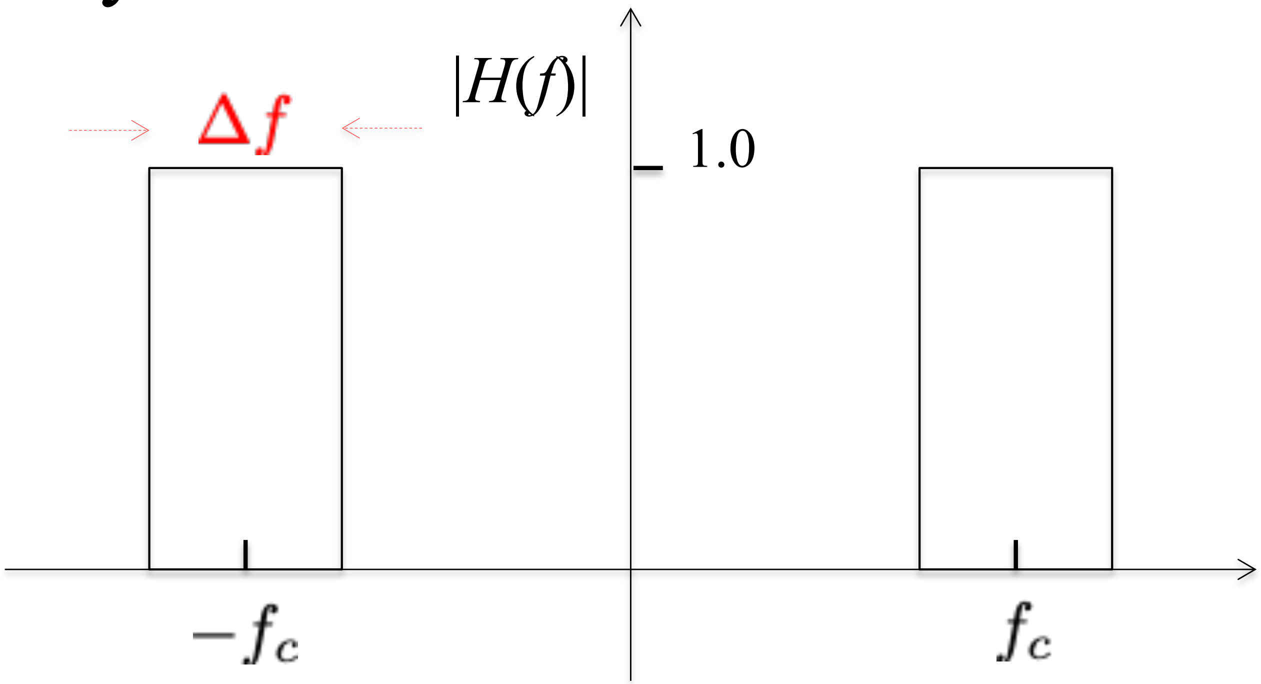
  - $E[|Y(t)|^2]$ is generally viewed as the *average power* of the WSS filter output process $Y(t)$.
  - This *average power* distributes over each spectrum frequency f through $S_Y(f)$. (Hence, the *total average power* is equal to the integration of $S_Y(f)$.)
  - Thus, $S_Y(f)$ is named the **power spectral density (PSD)** of $Y(t)$.

Power Spectral Density under WSS Input

- The unit of $E[|Y(t)|^2]$ is, e.g., **Watt**.
- So the unit of $S_Y(f)$ is therefore **Watt per Hz**.

Operational Meaning of PSD

- Example. Assume $h(\tau)$ is real, and $|H(f)|$ is given by:



Operational Meaning of PSD

Then,
$$
\begin{aligned}
E[|Y(t)|^2] &= R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df \\
&= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df \\
&= \int_{f_c-\Delta f/2}^{f_c+\Delta f/2} S_X(f)df + \int_{-f_c-\Delta f/2}^{-f_c+\Delta f/2} S_X(f)df \\
&\approx \Delta f \cdot [S_X(f_c) + S_X(-f_c)]
\end{aligned}
$$

The filter passes only those frequency components of the input random process $X(t)$, which lie inside a *narrow frequency band* of width Δf, centered about the frequency f_c and $-f_c$.

Properties of PSD

Property 0. Wiener-Khintchine-Einstein relation

- Relation between autocorrelation function and PSD of a WSS process $X(t)$

$$\begin{cases} S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau \\ R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df \end{cases}$$

Properties of PSD

Property 1. Power density at zero frequency

$$S_X(0) \text{ [Watt/Hz]} = S_X(0) \quad \text{[Watt-Second]}$$
$$= \int_{-\infty}^{\infty} R_X(\tau) \text{ [Watt]} \, d\tau \text{ [Second]}$$

Property 2: Average power

$$E[|X(t)|^2] \text{ [Watt]} = \int_{-\infty}^{\infty} S_X(f) \text{ [Watt/Hz]} \, df \text{ [Hz]}$$

Properties of PSD

<u>Property 3</u>: PSD is real.

Proof.

$$\begin{aligned}
S_X^*(f) &= \left(\int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau\right)^* \\
&= \int_{-\infty}^{\infty} R_X^*(\tau)e^{j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty} R_X(-\tau)e^{j2\pi f\tau}d\tau \quad (R_X^*(\tau) = R_X(-\tau)) \\
&= \int_{-\infty}^{\infty} R_X(s)e^{-j2\pi f s}d\tau \quad (s = -\tau) \\
&= S_X(f)
\end{aligned}$$

Q.E.D.

Properties of PSD

Property 4: If $R_X(\tau)$ is real, PSD is an even function:

$$S_X(f) = S_X(-f).$$

Proof.

$$\begin{aligned}
S_X^*(f) &= \left(\int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau}\, d\tau\right)^* \\
&= \int_{-\infty}^{\infty} R_X^*(\tau) e^{j2\pi f\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} R_X(-\tau) e^{j2\pi f\tau}\, d\tau \quad (R_X^*(\tau) = R_X(-\tau)) \\
&= \int_{\infty}^{-\infty} R_X(s) e^{-j2\pi f s}\, d(-s) \quad (s = -\tau) \\
&= \int_{-\infty}^{\infty} R_X(s) e^{-j2\pi f s}\, ds = S_X(f)
\end{aligned}$$

Q.E.D.

Properties of PSD

<u>Property 5</u>: Non-negativity for WSS processes

$$S_X(f) \ge 0$$

Proof: Pass $X(t)$ through a filter with transfer function satisfying $|H(f)|$^2=$\delta(f - f_c)$.

Properties of PSD

$$\begin{aligned} E[|Y(t)|^2] &= \int_{-\infty}^{\infty} S_Y(f)df \\ &= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df \\ &= \int_{-\infty}^{\infty} \delta(f-f_c) S_X(f)df \\ &= S_X(f_c) \end{aligned}$$

This step requires that $S_X(f)$ is continuous at $f = f_c$.

Therefore, by passing through a proper filter,

$$S_X(f_c) = E[|Y(t)|^2] \geq 0$$

for any f_c.

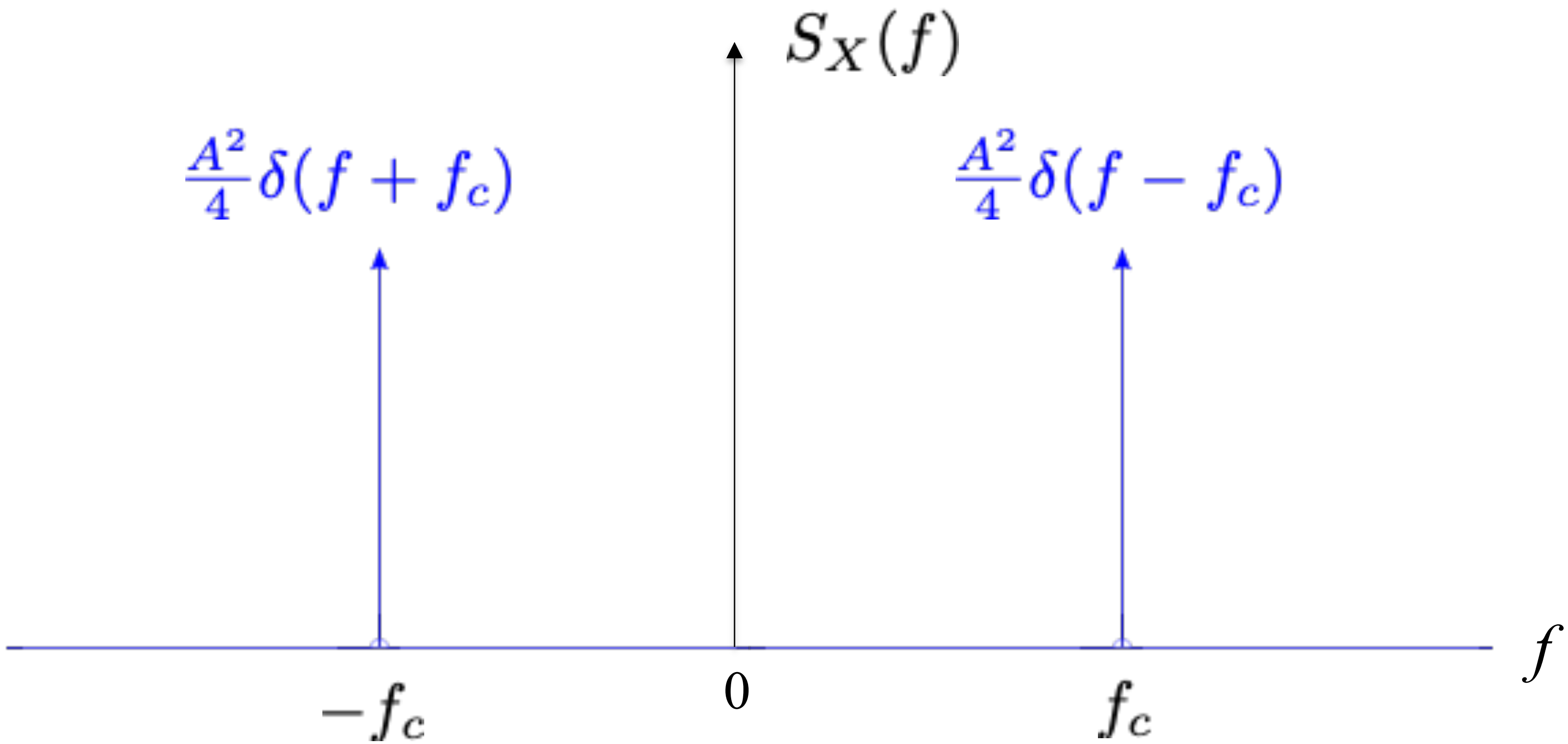
Example: Signal with Random Phase (See Slide 1-28)

- Let $X(t) = A\cos(2\pi f_c t + \Theta)$, where Θ is uniformly distributed over $[-\pi, \pi)$.

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau).$$

$$\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} \frac{A^2}{2}\cos(2\pi f_c \tau) e^{-j2\pi f\tau} d\tau \\
&= \frac{A^2}{4}\int_{-\infty}^{\infty}\left[e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}\right] e^{-j2\pi f\tau} d\tau \\
&= \frac{A^2}{4}\left[\int_{-\infty}^{\infty} e^{-j2\pi(f+f_c)\tau} d\tau + \int_{-\infty}^{\infty} e^{-j2\pi(f-f_c)\tau} d\tau\right] \\
&= \frac{A^2}{4}\left(\delta(f+f_c) + \delta(f-f_c)\right)
\end{aligned}$$

Example: Signal with Random Phase



Example: Signal with Random Delay (See Slide 1-33)

- Let

$$X(t) = \sum_{n=-\infty}^{\infty} A \cdot I_n \cdot p(t - nT - t_d)$$

where …, I_{-2}, I_{-1}, I_0, I_1, I_2, … are independent, and each I_j is either −1 or +1 with equal probability, and

$$p(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

Example: Signal with Random Delay

$$R_X(\tau) = \begin{cases} A^2\left(1 - \dfrac{|\tau|}{T}\right), & |\tau| < T \\ 0, & \text{otherwise} \end{cases}$$

$$\int u \cdot dv = uv\Big| - \int v \cdot du$$

$$S_X(f) = \int_{-T}^{T} A^2\left(1 - \frac{|\tau|}{T}\right) e^{-j2\pi f\tau}\, d\tau$$

$$= A^2\left(1 - \frac{|\tau|}{T}\right)\left(\frac{1}{-j2\pi f} e^{-j2\pi f\tau}\right)\Bigg|_{-T}^{T} - \int_{-T}^{T} A^2\left(-\frac{1}{T}\operatorname{sgn}(\tau)\right)\left(\frac{1}{-j2\pi f} e^{-j2\pi f\tau}\right) d\tau$$

$$= -\frac{A^2}{j2\pi f T}\int_{-T}^{T} \operatorname{sgn}(\tau) e^{-j2\pi f\tau}\, d\tau$$

(Continue from the previous slide.)

$$S_X(f) = -\frac{A^2}{j2\pi fT}\int_{-T}^{T}\text{sgn}(\tau)e^{-j2\pi f\tau}d\tau$$

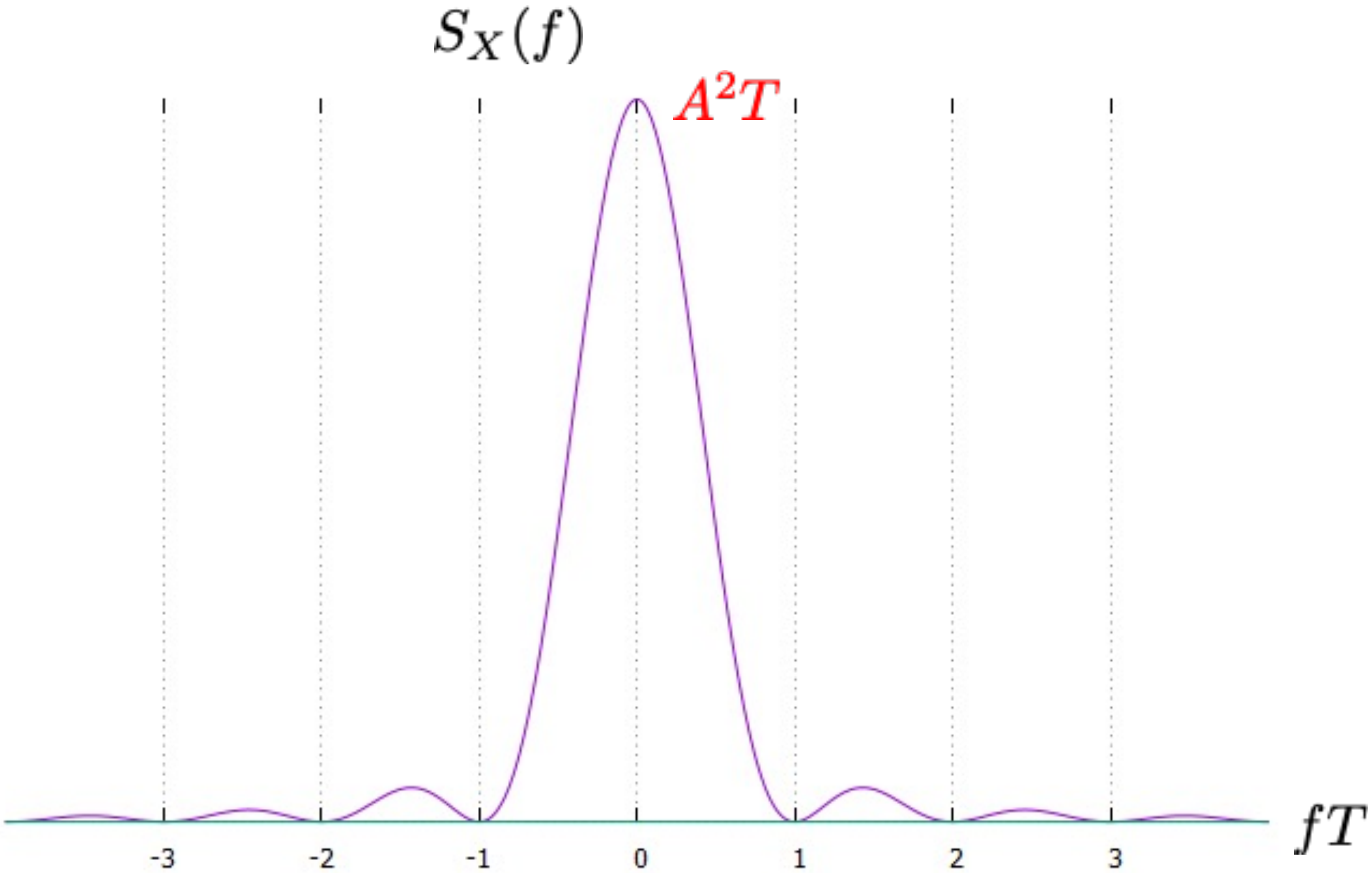
$$= -\frac{A^2}{(j2\pi fT)(-j2\pi f)}\left(\int_0^T(-j2\pi f)e^{-j2\pi f\tau}d\tau - \int_{-T}^{0}(-j2\pi f)e^{-j2\pi f\tau}d\tau\right)$$

$$= -\frac{A^2}{4\pi^2 f^2 T}\left(\left(e^{-j2\pi f\tau}\Big|_0^T\right) - \left(e^{-j2\pi f\tau}\Big|_{-T}^0\right)\right)$$

$$= -\frac{A^2}{4\pi^2 f^2 T}\left(e^{-j2\pi fT} - 1 - 1 + e^{j2\pi fT}\right)$$

$$= \frac{A^2}{4\pi^2 f^2 T}\left(2 - 2\cos(2\pi fT)\right)$$

$$= \frac{A^2}{\pi^2 f^2 T}\sin^2(\pi fT) = A^2 T\,\text{sinc}^2(fT)$$


$S_X(f)$
A^2T
fT
-3
-2
-1
0
1
2
3

Energy Spectral Density

- Energy of a (deterministic) function $p(t)$ is given by $\int_{-\infty}^{\infty} |p(t)|^2\, dt$.
  - Recall that the average power of $p(t)$ is given by

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |p(t)|^2\, dt.$$

  - Observe that

$$\int_{-\infty}^{\infty} |p(t)|^2\, dt = \int_{-\infty}^{\infty} p(t) p^*(t) dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} p(f) e^{j2\pi f t} df \right) \left(\int_{-\infty}^{\infty} p(f') e^{j2\pi f' t} df' \right)^* dt$$

(Continue from the previous slide.)

$$\begin{aligned}\int_{-\infty}^{\infty} |p(t)|^2\, dt &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} p(f)e^{j2\pi ft}df\right)\left(\int_{-\infty}^{\infty} p^*(f')e^{-j2\pi f't}df'\right)dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(f)p^*(f')\left(\int_{-\infty}^{\infty} e^{-j2\pi(f'-f)t}dt\right)dfdf' \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(f)p^*(f')\delta(f'-f)dfdf' \\ &= \int_{-\infty}^{\infty} p(f)p^*(f)df \\ &= \int_{-\infty}^{\infty} |p(f)|^2\, df\end{aligned}$$

For the same reason as PSD, $|p(f)|^2$ is named **energy spectral density** (ESD) of $p(t)$.

Example

- The ESD of a rectangular pulse of amplitude A and duration T is given by

$$E_g(f) = \left| \int_0^T A e^{-j2\pi f t} dt \right|^2 = A^2 T^2 \operatorname{sinc}^2(fT)$$

Example: Quadrature-Modulated Random Processes (See Slide 1-45)

- Let $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$, where Θ is uniformly distributed over $[-\pi, \pi)$, and $X(t)$ is WSS and independent of Θ.

$$
\begin{aligned}
R_Y(t,u) &= E[X(t)X(u)\cos(2\pi f_c t + \Theta)\cos(2\pi f_c u + \Theta)] \\
&= E[X(t)X(u)]E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c u + \Theta)] \\
&= R_X(t-u)\frac{\cos(2\pi f_c (t-u))}{2}
\end{aligned}
$$

$$\Rightarrow S_Y(f) = \frac{1}{4}\left[S_X(f - f_c) + S_X(f + f_c)\right]$$

How to Measure PSD?

- If $X(t)$ is not only (strictly) stationary but also ergodic, then any (deterministic) observation sample $x(t)$ in $[-T, T)$ satisfies:

$$\underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)dt}_{\text{Sample average}} = \underbrace{E[X(t)]}_{\text{Ensemble average}} = \mu_X$$

  - Similarly,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t+\tau)x(t)dt = R_X(\tau)$$

How to Measure PSD?

- Hence, we may use the *time-limited Fourier transform* of the *time-averaged autocorrelation function*:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t+\tau)x^*(t)dt \approx \frac{1}{2T}\int_{-T}^{T}x(t+\tau)x^*(t)dt$$

to approximate the PSD.

(Notably, we only have the values of $x(t)$ for t in $[-T, T]$.)

$$\begin{aligned}
S_X(f) &\approx \int_{-T}^{T}\left[\frac{1}{2T}\int_{-T}^{T} x(t+\tau)x^*(t)dt\right]\exp(-j2\pi f\tau)d\tau \\
&= \frac{1}{2T}\int_{-T}^{T} x^*(t)\left(\int_{-T}^{T} x(t+\tau)\exp(-j2\pi f\tau)d\tau\right)dt \\
&= \frac{1}{2T}\int_{-T}^{T} x^*(t)\left(\int_{-T+t}^{T+t} x(s)\exp(-j2\pi f(s-t))ds\right)dt,\ \ s=t+\tau \\
&= \frac{1}{2T}\int_{-T}^{T} x^*(t)\exp(j2\pi ft)\left(\int_{-T+t}^{T+t} x(s)\exp(-j2\pi fs)ds\right)dt \\
&\approx \frac{1}{2T}\left(\int_{-T}^{T} x(t)\exp(-j2\pi ft)dt\right)^*\left(\int_{-T}^{T} x(s)\exp(-j2\pi fs)ds\right) \\
&= \frac{1}{2T}|x_{2T}(f)|^2
\end{aligned}$$

Assume approximately that this integration has nothing to do with t.

How to Measure PSD?

- The estimate

$$\frac{1}{2T}|x_{2T}(f)|^2$$

is named the *periodogram*.

- To summarize:

1. Observe $x(t)$ for duration $[-T, T)$.
2. Calculate $x_{2T}(f) = \int_{-T}^{T} x(t)\exp(-j2\pi ft)dt$.
3. Then $S_X(f) \approx \frac{1}{2T}|x_{2T}(f)|^2$.

Example: PSD of Sum Process

- Determine the PSD of sum process $Z(t) = X(t) + Y(t)$ of two zero-mean WSS processes $X(t)$ and $Y(t)$.
  - Answer:

$$
\begin{aligned}
R_Z(t,u) &= E[Z(t)Z^*(u)] \\
&= E[(X(t)+Y(t))(X^*(t)+Y^*(u)] \\
&= E[X(t)X^*(u)] + E[X(t)Y^*(u)] \\
&\quad +E[Y(t)X^*(u)] + E[Y(t)Y^*(u)] \\
&= R_X(t,u) + R_{X,Y}(t,u) + R_{Y,X}(t,u) + R_Y(t,u)
\end{aligned}
$$

WSS implies that

$$R_Z(\tau) = R_X(\tau) + R_{X,Y}(\tau) + R_{Y,X}(\tau) + R_Y(\tau).$$

Hence,

$$S_Z(f) = S_X(f) + S_{X,Y}(f) + S_{Y,X}(f) + S_Y(f).$$

Q.E.D.

If $X(t)$ and $Y(t)$ are uncorrelated and zero-mean, i.e., $E[X(t+\tau)Y^*(t)] = E[X(t+\tau)]E[Y^*(t)] = 0$,

$$S_Z(f) = S_X(f) + S_Y(f).$$

The PSD of a sum process of zero-mean *uncorrelated* processes is equal to the sum of their individual PSDs.

Gaussian Random Process

- **Definition**. A random variable is Gaussian distributed, if its pdf has the form

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]$$

Gaussian Random Process

- **Definition**. An n-dimensional random vector is Gaussian distributed, if its pdf has the form

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi|\Sigma|)^{n/2}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu}) \right)$$

where $\vec{\mu} = [E[X_1], E[X_2], \ldots, E[X_n]]^T$ is the mean vector, and

$$\Sigma = \begin{bmatrix} \text{Cov}\{X_1, X_1\} & \text{Cov}\{X_1, X_2\} & \cdots \\ \text{Cov}\{X_2, X_1\} & \text{Cov}\{X_2, X_2\} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}_{n\times n}$$ is the covariance matrix.

Gaussian Random Process

- For a Gaussian random vector, "uncorrelation" implies "independence."

$$\Sigma = \begin{bmatrix} \mathrm{Cov}\{X_1, X_1\} & 0 & \cdots \\ 0 & \mathrm{Cov}\{X_2, X_2\} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}_{n\times n} \Rightarrow f_{\vec{X}}(\vec{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Gaussian Random Process

- **Definition**. A (complex) random process $X(t)$ is said to be Gaussian distributed, if for every function $g(t)$ satisfying

$$\int_0^T \int_0^T g(t) g^*(u) R_X(t,u) dt du < \infty,$$

we have $Y = \int_0^T g(t) X(t) dt$ is a Gaussian random variable.

Notably, $E[|Y|^2] = \int_0^T \int_0^T g(t) g^*(u) R_X(t,u) dt du.$

Central Limit Theorem

- **Theorem** (Central Limit Theorem). For a sequence of independent and identically distributed (i.i.d.) random variables X_1, X_2, X_3, …

$$\lim_{n\to\infty} \Pr\left[\frac{(X_1-\mu_X)+\cdots+(X_n-\mu_X)}{\sigma_X\sqrt{n}} \le y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

where $\mu_X = E[X_j]$ and $\sigma_X^2 = E[X_j^2]$.

Properties of Gaussian Random Process

Property 1. The output of a stable linear filter is a Gaussian process if the input is a Gaussian process.

(This is self-justified by the definition of Gaussian Random process.)

Property 2. A finite number of samples of a Gaussian process forms a multi-dimensional Gaussian vector.

(No proof. Some books use this as the definition of Gaussian process.)

Properties of Gaussian Random Process

Property 3. A WSS Gaussian process is also strictly stationary.

White Noise

- A (often implicitly, zero-mean) noise is white if its PSD equals constant for all frequencies.
  - It is defined as: $S_W(f) = \dfrac{N_0}{2}$
- Impracticability
  - The noise has infinite power

$$E[W^2(t)] = \int_{-\infty}^{\infty} S_W(f)df = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty.$$

White Noise

- Another impracticability
  - No matter how close in time two samples are, they are uncorrelated!
- So impractical, why white noise is so popular in the analysis of communication system?
  - There do exist noise sources that have a **flat power spectral density** over a range of frequencies that is *much* larger than the bandwidths of subsequent filters (or measurement devices).

White Noise

- Some physical measurements have shown that the PSD of a certain kind of noise has the form

$$S_W(f) = kTR\frac{2\alpha^2}{\alpha^2 + (2\pi f)^2}$$

where k is Boltzmann's constant, T is the absolute temperature, α and R are the parameters of physical medium.

- When $f << \alpha$,

$$S_W(f) = kTR\frac{2\alpha^2}{\alpha^2 + (2\pi f)^2} \approx 2kTR = \frac{N_0}{2}$$

Ideal Lowpass Filtered White Noise

- After passing through a filter, the PSD of a zero-mean white noise becomes:

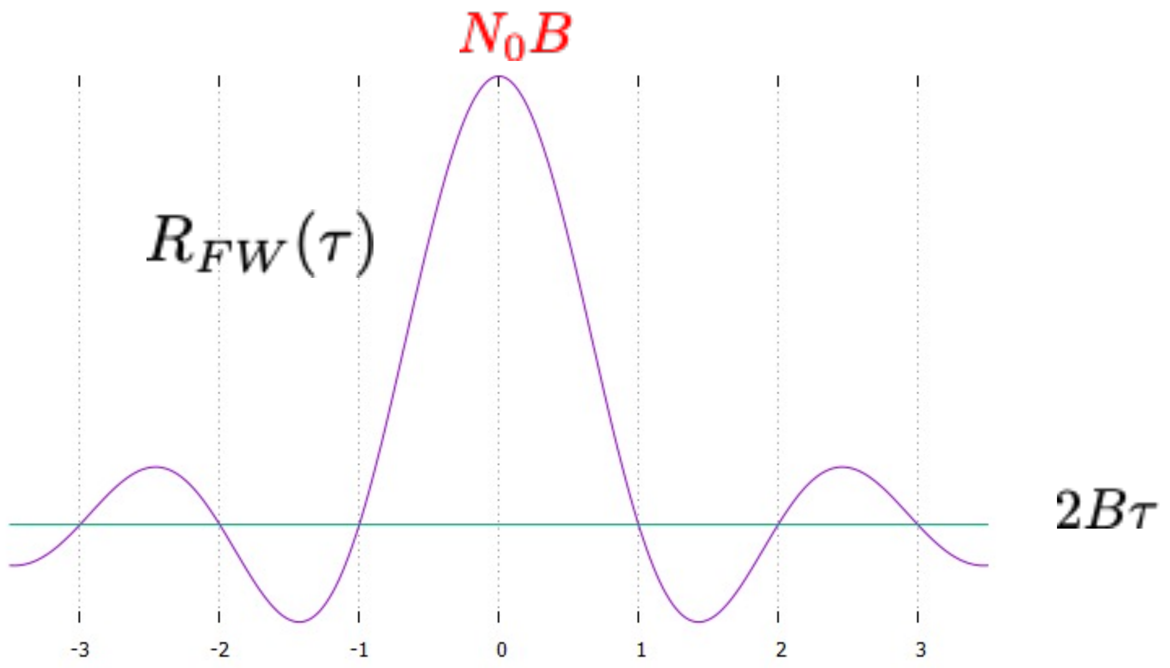
$$S_{FW}(f) = |H(f)|^2 S_W(f) = \begin{cases} \frac{N_0}{2}, & |f| < B \\ 0, & \text{otherwise} \end{cases}$$

$$R_{FW}(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f\tau)df = N_0 B \operatorname{sinc}(2B\tau)$$

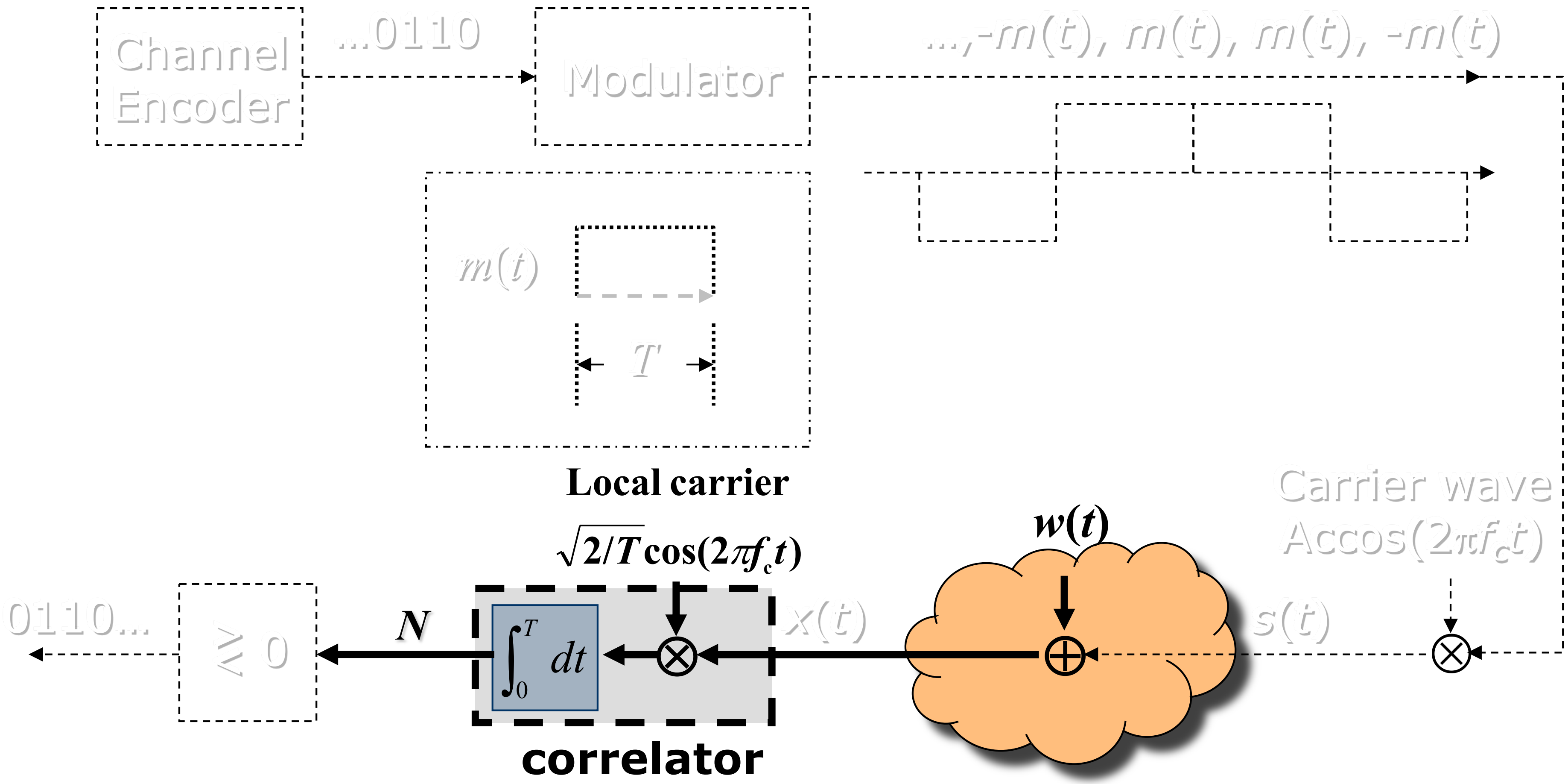
$\Rightarrow \tau = \pm k/(2B)$ for non-zero integer k implies $R_{FW}(\tau) = 0$, i.e., uncorrelated.

Ideal Lowpass Filtered White Noise

- So if we sample the noise at rate of $2B$ times per second, the resultant noise samples are uncorrelated!



Ideal "Correlated" White Noise



Ideal "Correlated" White Noise

In the previous figure, a scaling factor $\sqrt{\frac{2}{T}}$ is added to the local carrier to normalize the signal energy.

$$\begin{aligned}
\text{Signal Energy} &= \int_0^T \left(\sqrt{\frac{2}{T}}\cos(2\pi f_c t)\right)^2 dt \\
&= \int_0^T \frac{2}{T}\cos^2(2\pi f_c t)dt \\
&= \int_0^T \frac{1+\cos(4\pi f_c t)}{T}dt \\
&= 1 + \frac{\sin(4\pi f_c T)}{4\pi f_c T} \\
&= 1
\end{aligned}$$

Here, we assume f_c is a multiple of $1/T$. In practice, $f_c\ T$ is usually large; hence, the last term can be neglected.

Ideal “Correlated” White Noise

Noise $N = \int_0^T w(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt$

$$\mu_N = E\left[\int_0^T w(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt\right] = \int_0^T E[w(t)]\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt = 0.$$

$$\sigma_N^2 = E\left[\int_0^T w(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \cdot \int_0^T w(s)\sqrt{\frac{2}{T}}\cos(2\pi f_c s)ds\right]$$

$$= \frac{2}{T}\int_0^T \int_0^T E[w(t)w(s)]\cos(2\pi f_c t)\cos(2\pi f_c s)dsdt$$

(Continue from the previous slide.)

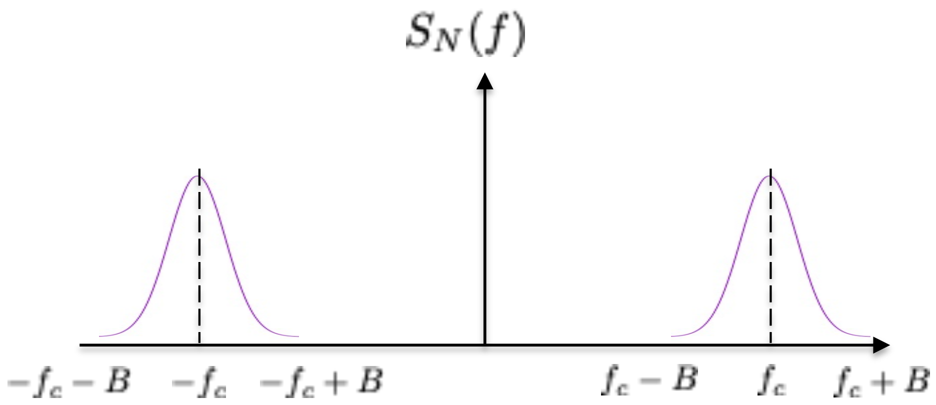
$$
\begin{aligned}
\sigma_N^2 &= \frac{2}{T}\int_0^T\int_0^T \frac{N_0}{2}\delta(t-s)\cos(2\pi f_c t)\cos(2\pi f_c s)\,ds\,dt \\
&= \frac{N_0}{T}\int_0^T \cos^2(2\pi f_c t)\,dt \\
&= \frac{N_0}{2} + \frac{N_0}{4\pi f_c T}\sin(4\pi f_c T) \\
&= \frac{N_0}{2}
\end{aligned}
$$

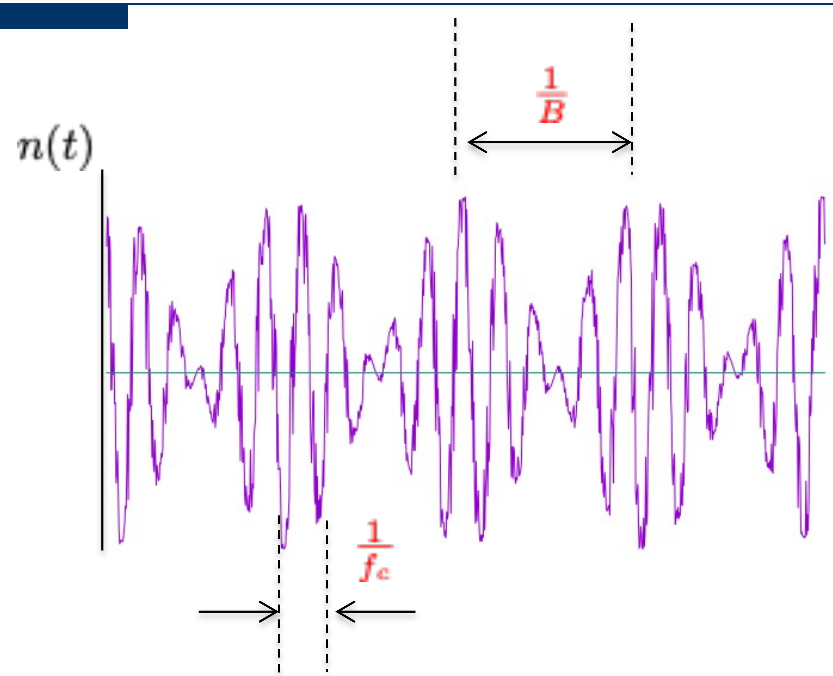
If $w(t)$ is white Gaussian, then the pdf of N is uniquely determined by the first and second moments.

Narrowband Noise

- In general, the receiver of a communication system includes a *narrowband filter* whose bandwidth is just large enough to pass the modulated component of the received signal.
- The noise is therefore also filtered by this *narrowband filter*.
- So, noise's PSD after being filtered may look like the figure in the next slide.

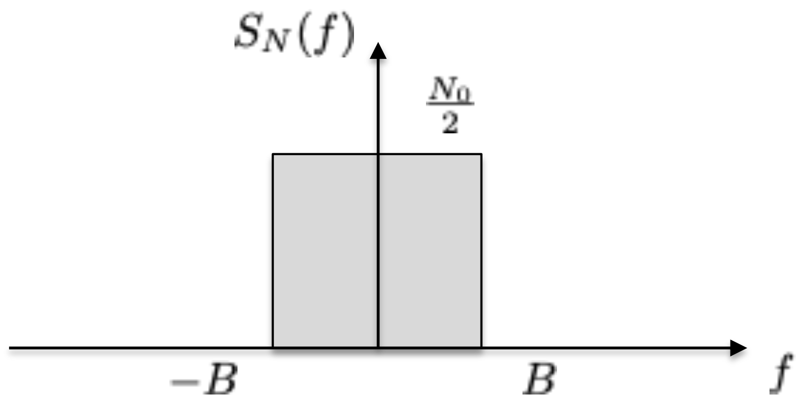
Narrowband Noise





Definitions of Bandwidth

- The bandwidth is the width of the frequency range outside which the power is essentially negligible.
  - E.g., the bandwidth of a (strictly) band-limited signal shown below is B.



Null-to-Null Bandwidth

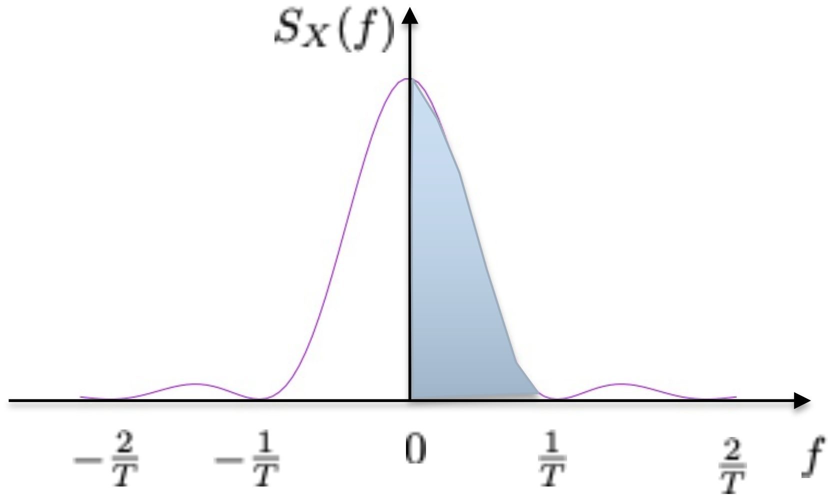
- Most signals of practical interest are not *strictly* band-limited.
  - Therefore, there may not be a universally accepted definition of bandwidth for such signals.
  - In such case, people may use ***null-to-null bandwidth***.
    - **Definition.** The width of the main spectral lobe that lies inside the positive frequency region ($f > 0$).

Null-to-Null Bandwidth

$X(t)=\sum_{n=-\infty}^{\infty} A\cdot I_n \cdot p(t-nT-t_d)$, where $p(t)$ is a rectangular pulse of duration T and amplitude A.

$$\Rightarrow S_X(f)=A^2T\text{sinc}^2(fT)$$

The null-to-null bandwidth is 1/*T*.



Root-Mean-Square Bandwidth

- rms bandwidth

$$B_{rms} = \left(\frac{\int_{-\infty}^{\infty} f^2 S_X(f)df}{\int_{-\infty}^{\infty} S_X(f)df} \right)^{1/2}$$

  - Disadvantage: Sometimes,

$$\int_{-\infty}^{\infty} f^2 S_X(f)df = \infty$$

even if

$$\int_{-\infty}^{\infty} S_X(f)df < \infty.$$

Bandwidth of Deterministic Signals

- The previous definition can also be applied to Deterministic Signals, where PSD is replaced by ESD.
  - For example, a deterministic signal with spectrum $G(f)$ has rms bandwidth:

$$B_{rms} = \left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \, df}{\int_{-\infty}^{\infty} |G(f)|^2 \, df} \right)^{1/2}$$

Noise Equivalent Bandwidth

- An important consideration in communication systems is the *noise power* at a linear filter output due to a white noise input.
  - We can characterize the *noise-resistant ability* of this filter by its *noise equivalent bandwidth*.
  - **Definition.** Noise equivalent bandwidth = The bandwidth of an *ideal low-pass filter* through which the same noise power at the filter output is resulted.

Noise Equivalent Bandwidth

- Output noise power for a *general linear filter*

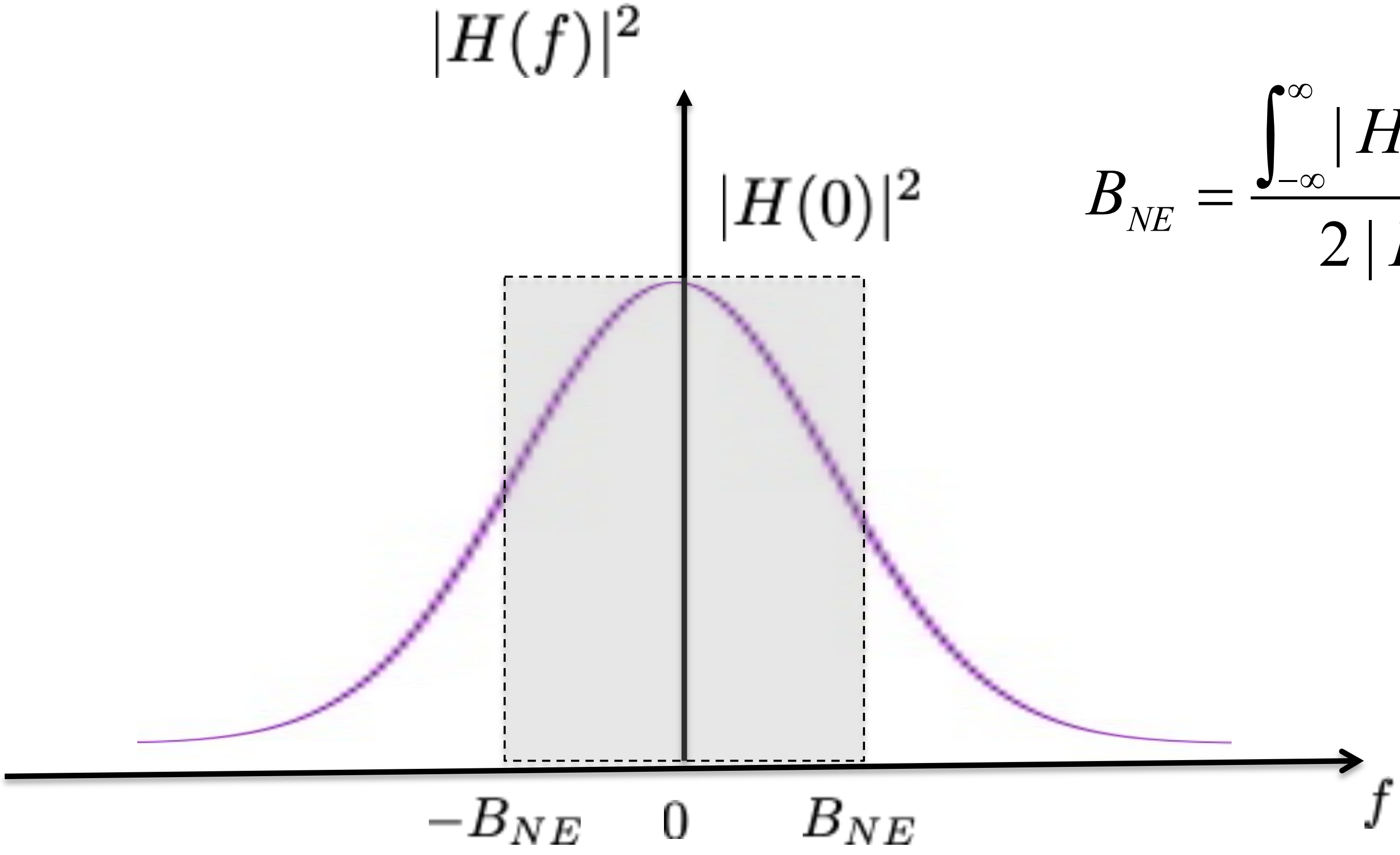
$$\int_{-\infty}^{\infty} S_W(f)\,|H(f)|^2\,df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df$$

- Output noise power for an ideal low-pass filter of bandwidth B and the same amplitude as the *general linear filter* at $f = 0$.

$$\int_{-\infty}^{\infty} S_W(f)\,|H(f)|^2\,df = \frac{N_0}{2}\int_{-B}^{B} |H(0)|^2\,df = BN_0\,|H(0)|^2$$

$$B_{NE} = \frac{\int_{-\infty}^{\infty} |H(f)|^2\,df}{2\,|H(0)|^2}$$

Noise Equivalent Bandwidth



$$B_{NE} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 \, df}{2|H(0)|^2}$$

Time-Bandwidth Product

- Time-Scaling Property of Fourier Transform
  - Reducing the time-scale by a factor of a extends the bandwidth by a factor of a.

$$g(t) \overset{Fourier}{\rightarrow} G(f) \Leftrightarrow g(at) \overset{Fourier}{\rightarrow} \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

  - This hints that the product of *time-* and *frequency-parameters* should remain constant, which is named the *time-bandwidth product* or *bandwidth-duration product*.

Time-Bandwidth Product

- Since there are various definitions of time-parameter (e.g., duration of a signal) and frequency-parameter (e.g., bandwidth), the *time-bandwidth product* constant may change for different definitions.
- E.g., *rms duration* and *rms bandwidth* of a pulse $g(t)$

$$T_{rms} = \left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt}{\int_{-\infty}^{\infty} |g(t)|^2 \, dt} \right)^{1/2} \quad B_{rms} = \left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \, df}{\int_{-\infty}^{\infty} |G(f)|^2 \, df} \right)^{1/2}$$

Then $T_{rms} B_{rms} \geq \frac{1}{4\pi} = 0.07957\ldots$

Time-Bandwidth Product

Example: $g(t) = \exp(-\pi t^2)$. Then $G(f) = \exp(-\pi f^2)$.

$$T_{rms} = B_{rms} = \left(\frac{\int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt}{\int_{-\infty}^{\infty} e^{-2\pi t^2} dt} \right)^{1/2} = \frac{1}{2\sqrt{\pi}}. \quad \text{Then } T_{rms} B_{rms} = \frac{1}{4\pi}.$$

Example: $g(t) = \exp(-|t|)$. Then $G(f) = 2/(1+4\pi^2 f^2)$.

$$T_{rms} B_{rms} = \left(\frac{\int_{-\infty}^{\infty} t^2 e^{-2|t|} dt}{\int_{-\infty}^{\infty} e^{-2|t|} dt} \right)^{1/2} \left(\frac{\int_{-\infty}^{\infty} \frac{f^2}{(1+4\pi^2 f^2)^2} df}{\int_{-\infty}^{\infty} \frac{1}{(1+4\pi^2 f^2)^2} df} \right)^{1/2} = \frac{1}{\sqrt{2}} \times \frac{1}{2\pi} \geq \frac{1}{4\pi}.$$

Summary

- Fourier transform
  - Dirichlet's condition and Dirac delta function
  - Fourier series and its relation to Fourier transform
- PSD and ESD
- Stable LTI filter
  - Linearity and convolution
- Narrowband process
- Bandwidth
  - Null to null, rms, noise-equivalent
  - Time-bandwidth product
- White Noise