Supplement: Hilbert Transform & Sinc Effect

5. From Slide 3-12, we know that

$$g(t) = \mathrm{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}.$$

Prove that

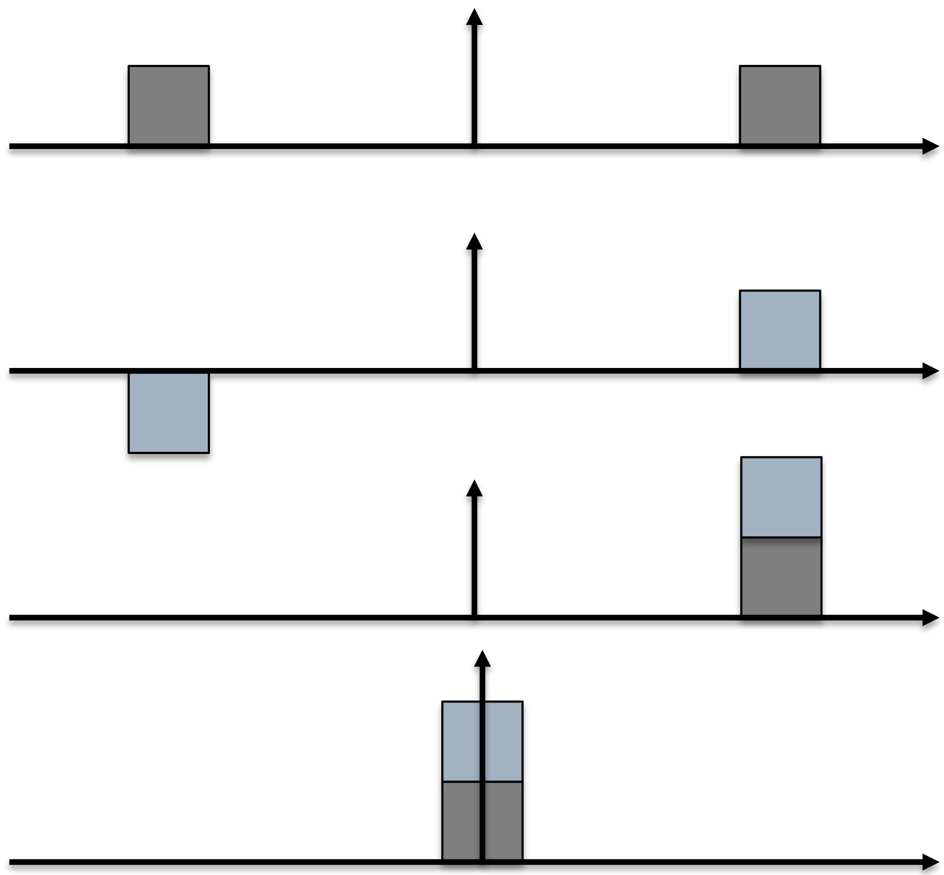
$$G(f) = \frac{1}{2}(\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c)).$$

Note: Here, $\tilde{G}(f)$ may be complex-valued.

Solution.

$$\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \mathrm{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}\, e^{-j2\pi f t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(\tilde{g}(t)e^{j2\pi f_c t} + (\tilde{g}(t)e^{j2\pi f_c t})^*\right)e^{-j2\pi f t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(\tilde{g}(t)e^{j2\pi f_c t} + \tilde{g}^*(t)e^{-j2\pi f_c t}\right)e^{-j2\pi f t}\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \tilde{g}(t)e^{-j2\pi(f-f_c)t}\mathrm{d}t + \frac{1}{2}\left(\int_{-\infty}^{\infty} \tilde{g}(t)e^{-j2\pi(-f-f_c)\tau}\mathrm{d}t\right)^* \\
&= \frac{1}{2}(\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c))
\end{aligned}$$

Applying Hilbert Transform



$$G(f)$$

$$j\hat{G}(f) = j(-j\,\mathrm{sgn}(f))G(f) = \mathrm{sgn}(f)G(f)$$

$$G_+(f) = G(f) + j\hat{G}(f)$$

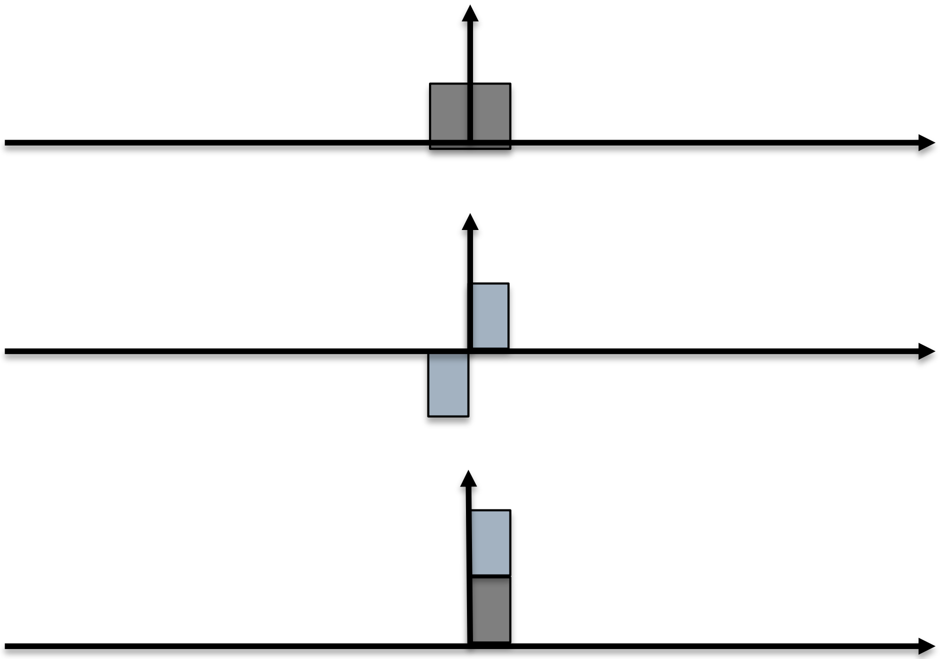
$$\tilde{G}(f) = G_+(f + f_c)$$

Applying Hilbert Transform

$g(t)$	$G(f)$
$j\hat{g}(t) = \frac{j}{\pi t} \star g(t)$ $\left(\hat{g}(t) = \frac{1}{\pi t} \star g(t)\right)$	$j\hat{G}(f) = j(-j\text{sgn}(f))G(f)$ $= \text{sgn}(f)G(f)$
$g_+(t) = g(t) + j\hat{g}(t)$ $(g(t) = \text{Re}\{g_+(t)\})$	$G_+(f) = G(f) + j\hat{G}(f)$
$\tilde{g}(t) = g_+(t)e^{-j2\pi f_c t}$	$\tilde{G}(f) = G_+(f + f_c)$

$$g(t) = \text{Re}\{g_+(t)\} = \text{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}$$

Applying Hilbert Transform



$$M(f)$$

$$j\hat{M}(f) = \text{sgn}(f)M(f)$$

$$M_+(f) = M(f) + j\hat{M}(f)$$

Single Side Band (SSB)

$$\tilde{G}(f) = M_+(f)$$

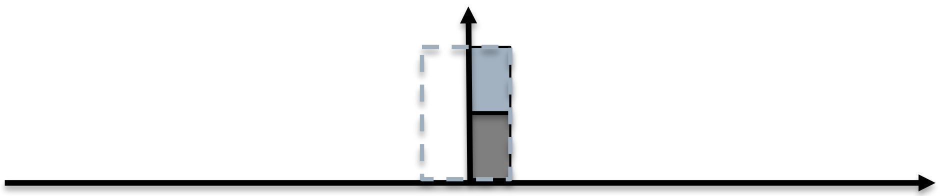
Applying Hilbert Transform

$$g(t) = \mathrm{Re}\{\tilde{g}(t)e^{j2\pi f_c t}\}$$

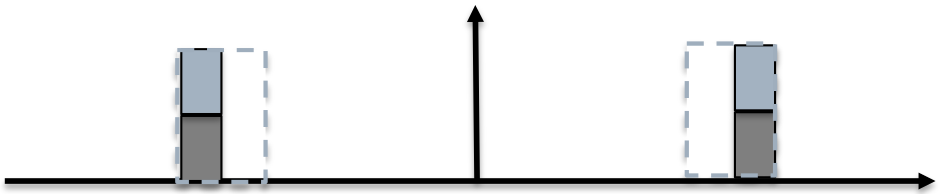
$$\Longleftrightarrow \quad G(f) = \frac{1}{2}\left(\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c)\right)$$

$$g(t) = \mathrm{Re}\{m_+(t)e^{j2\pi f_c t}\}$$

$$\Longleftrightarrow \quad G(f) = \frac{1}{2}\left(\tilde{M}(f - f_c) + \tilde{M}^*(-f - f_c)\right)$$



$\tilde{G}(f) = M_+(f)$



$G(f)$

$$
\begin{aligned}
g(t) &= \mathrm{Re}\{m_+(t)e^{j2\pi f_c t}\} \\
&= \mathrm{Re}\{(m(t) + j\hat{m}(t))e^{j2\pi f_c t}\} \\
&= m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)
\end{aligned}
$$

$$
\begin{aligned}
G(f) = \mathcal{F}\{g(t)\} &= \mathcal{F}\{m(t)\cos(2\pi f_c t)\} - \mathcal{F}\{\hat{m}(t)\sin(2\pi f_c t)\} \\
&= \mathcal{F}\{m(t)\} \star \mathcal{F}\{\cos(2\pi f_c t)\} - \mathcal{F}\{\hat{m}(t)\} \star \mathcal{F}\{\sin(2\pi f_c t)\} \\
&= M(f) \star \frac{\delta(f - f_c) + \delta(f + f_c)}{2} - \hat{M}(f) \star \frac{\delta(f - f_c) - \delta(f + f_c)}{2j} \\
&= M(f) \star \frac{\delta(f - f_c) + \delta(f + f_c)}{2} + j\hat{M}(f) \star \frac{\delta(f - f_c) - \delta(f + f_c)}{2} \\
&= \frac{M(f) + j\hat{M}(f)}{2} \star \delta(f - f_c) + \frac{M(f) - j\hat{M}(f)}{2} \star \delta(f + f_c) \\
&= \frac{M_+(f)}{2} \star \delta(f - f_c) + \frac{M_-(f)}{2} \star \delta(f + f_c) \\
&= \frac{M_+(f - f_c)}{2} + \frac{M_-(f + f_c)}{2} \\
&= \frac{M_+(f - f_c)}{2} + \frac{M_+^*(-f - f_c)}{2}
\end{aligned}
$$

Finite Duration Random Process = Zero Time-average PSD

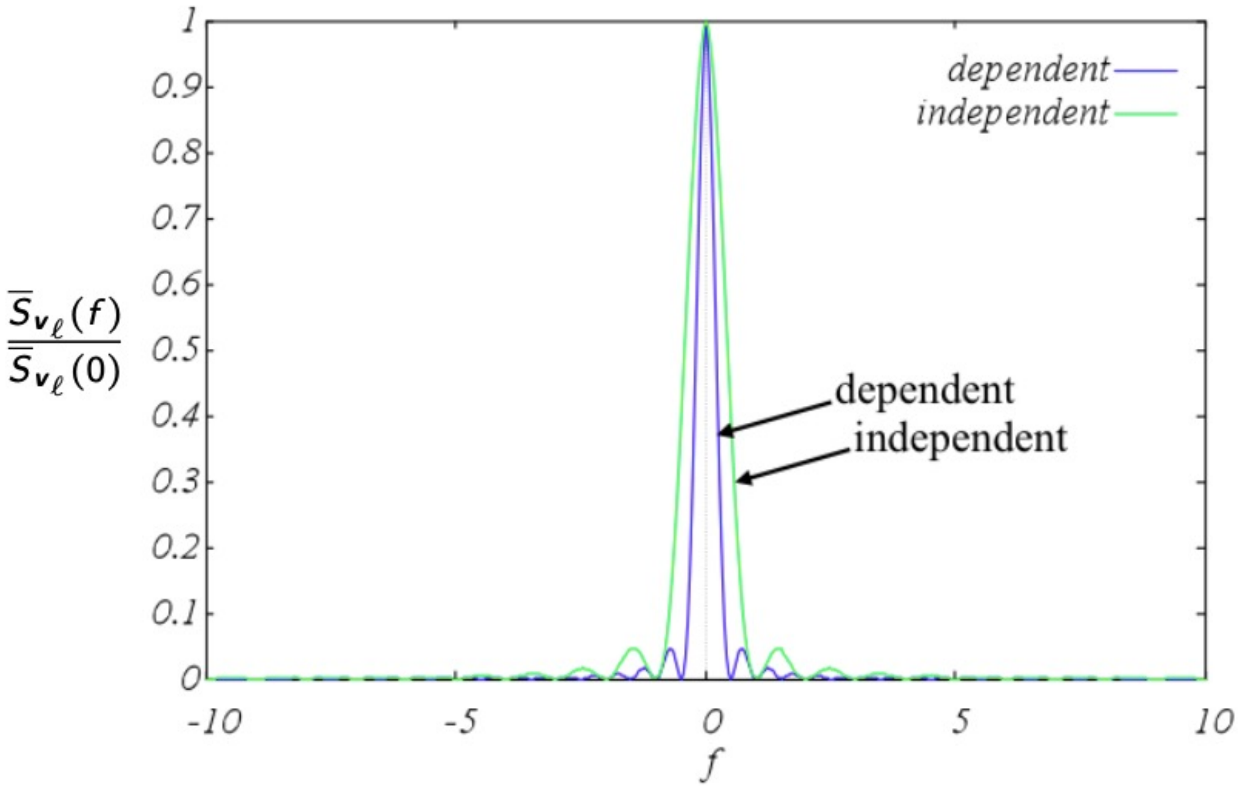
Time-average PSD

$$\bar{S}_X(f) = \lim_{T\to\infty} \frac{1}{2T} E\left[X(f)X_{2T}^*(f)\right]$$

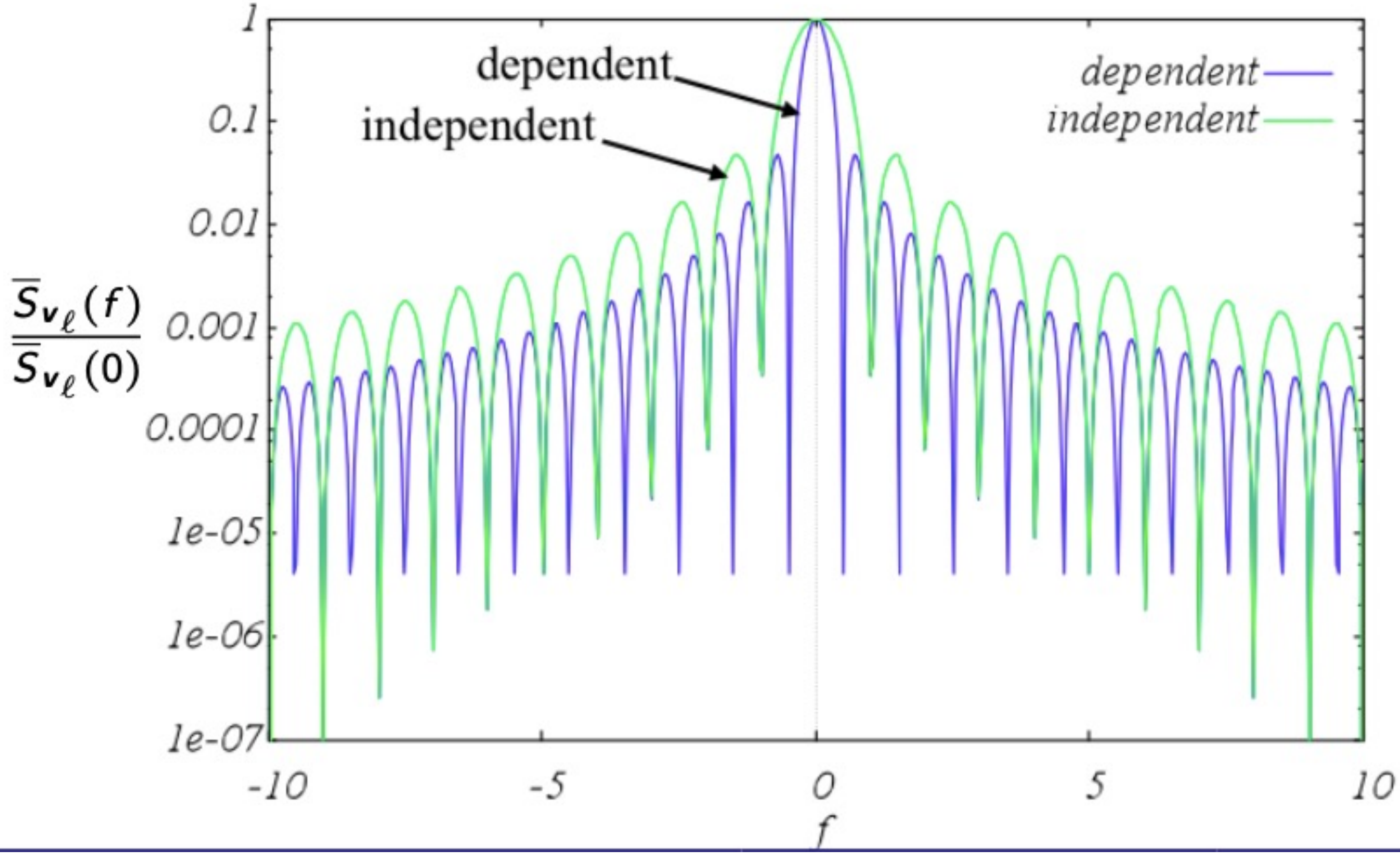
If $X_{2T}(f) = X(f)$ for $T > T_{\text{range}}$, and $E[|X(f)|^2] \le U$,

$$\begin{aligned} \bar{S}_X(f) &= \lim_{T\to\infty} \frac{1}{2T} E\left[X(f)X^*(f)\right] \\ &= \lim_{T\to\infty} \frac{1}{2T} E\left[|X(f)|^2\right] \le \lim_{T\to\infty} \frac{1}{2T} U = 0 \end{aligned}$$

Time-average PSD: Example



Time-average PSD: Example



Time-average PSD: Example