Vertex Coloring

Motivated by the well-known 4-color theorem, this topic attracts many researchers to work on. Nowadays, the study of colorings either on vertices, edges or regions was known as the chromatic theory. Besides of its original problem on map colorings, there are quite a few different versions of colorings. We start here with the original coloring which is on vertices of a graph. How many colors do we need to color the vertices of the following graph?

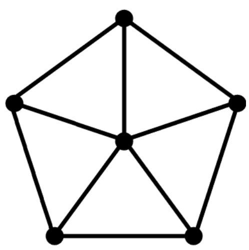

Figure 5.1: Wheel

Definition 5.1.

- k-coloring (proper): $\varphi : V(G) \to \{1, 2, 3, ..., k\}$ s.t. $uv \in E(G) \;\Rightarrow\; \varphi(u) \neq \varphi(v)$.
- $\chi(K) = \min\{k|G \text{ has a } k \text{ coloring}\}$ (Chromatic number of G)
- G is n-critical (chromatically) if $\chi(G - v) < \chi(G)$ for each $v \in V(G)$.

Remark.

- Every graph G has an n-critical induced subgraph H.
- Let $w(G)$ denote the order of a maximum clique, i.e., the order of complete subgraphs is maximum. So, $\chi(G) \geq \omega(G)$. When does the equality holds?

Definition 5.2. A graph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. Clearly, not every graph is perfect. $\chi(H) - \omega(H)$ can be very large!

Theorem 5.1 (Mycielski). *For every integer n, there exists a triangle-free graph G such that $\chi(G) = n$. ($\chi(G) - \omega(G) = n - 2$.)*

Proof. By induction on n and K_1, K_2, C_5 do have the property respectively for $n = 1, 2$, and 3. Now, assume that H is a triangle-free k-chromatic graph, i.e., $\mathcal{H} = k$. We construct a graph G based on H such that G is a triangle-free $(k+1)$-chromatic graph.

Let $V(H) = \{v_1, v_2, ..., v_p\}$ and $V(G) = V(H) \cup \{u_1, u_2, ..., u_p, u_0\}$. Let $E(G) = E(H) \cup \{u_0 u_i | i = 1, 2, ..., p\} \cup \{u_i v_j | v_j \in N_H(v_i)\}$. See Figure 5.2 for an example when $k = 3$.

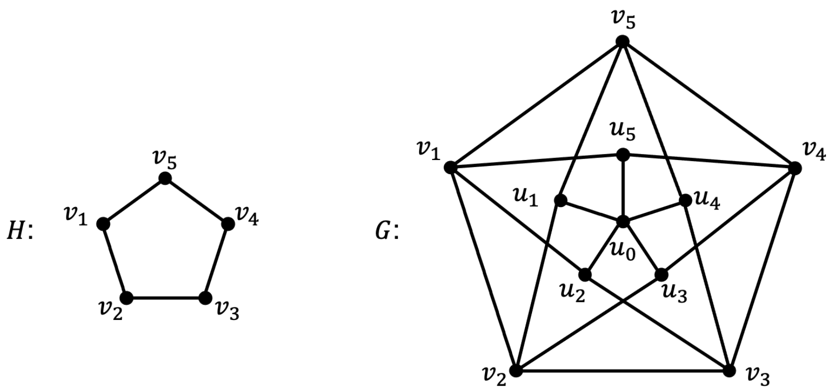

Figure 5.2: Grötzsch graph

Since $\langle\{u_1, u_2, ..., u_p\}\rangle_G$ contains no edges, u_0 is not in any triangle. Bt assumption, $H \not\geq K_3$. So, the only possibility will be a triangle consists of u_i, v_j and v_k where $u_i v_j$ and $u_i v_k$ are edges of G. If they form a triangle, then $\langle\{v_i, v_j, v_k\}\rangle_H$ is a triangle in H. Hence, G is triangle-free.

Now, we claim $\chi(G) = k+1$. Let φ be a k-coloring of H. Let $\tilde{\varphi} : V(G) \to \{1, 2, ..., k+1\}$ by letting $\tilde{\varphi}(u_i) = \varphi(v_i)$ and $\tilde{\varphi}(u_0) = k+1$. Hence, we have a $(k+1)$-coloring of G, thus $\chi(G) \leq k+1$. On the other hand, we show that $\chi(G) \geq k+1$. Suppose not. Let φ' be a k-coloring of G and the colors used are $1, 2, ..., k$. First, we assign u_0 the color k, i.e., $\varphi'(u_0) = k$. So, the colors used for $u_1, u_2, ..., u_p$ must be in $\{1, 2, ..., k-1\}$. Since $\chi(H) = k$, k occurs somewhere in H, say v_i. (May have more vertices.) Now, we recolor v_i by using $\varphi'(u_i)$. Since u_i is adjacent to every vertex of $N_H(v_i)$, $\varphi'(u_i) \neq \varphi'(v)$ for each $v \in N_H(v_i)$ and thus we have a proper coloring of H using at most $k-1$ colors. (?)
A contradiction.

□

Remark. This result was generalized later to the graph G with given girth g and $\chi(G)$ can be any larger $n \in \mathbb{N}$ by P. Erdös.

Now, we consider the original problem of map coloring. So, we would like to show that if G is planar, then $\chi(G) \leq 4$. Clearly, so far, all proofs include the aid of computer checking. But, the weaker version of showing $\chi(G) \leq 5$ was obtained around 1890.

Theorem 5.2. *If G is a connected planar graph, then $\chi(G) \leq 5$.*

Proof. By induction on $|G|$. For $\delta(G) = 1, 2, 3$ and 4, the proof can be obtained easily. (?) Hence, it suffices to consider a planar graph H whose minimum degree is 5.

Let $v \in V(H)$ such that $deg_H(v) = 5$. By induction, $\chi(H - v) \leq 5$. Let φ be a 5-coloring of H and we consider the colors assigned on $N_H(v)$. Let them be $\varphi(v_1), \varphi(v_2), ..., \varphi(v_5)$. Clearly, if any two of them are of the same color, them there is a color for v such that we have a proper 5-coloring of H. So assume that $\varphi(v_i),\ i = 1, 2, 3, 4, 5$, the vertices are in clockwise order, and join consecutive vertices if they are missing. See Figure 5.3.

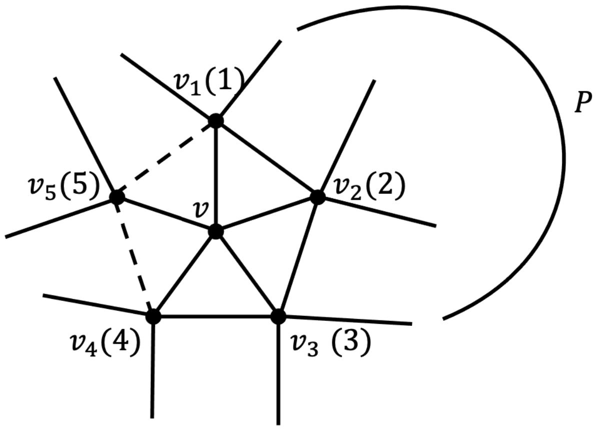

Figure 5.3

Now, consider the induced subgraph $H_{1,3} = \langle \varphi^{-1}(1) \cup \varphi^{-1}(3) \rangle_H$. If v_1 and v_3 are in distinct components, then by changing the colors 1 and 3 in the $\varphi(v_1) = 3$ and $\varphi(v_3) = 3$. Hence, 3 is available for v.

On the other hand, there exists a path P connecting v_1 and v_3. Hence, $v - v_1 - P - v_3 - v$ is a cycle such that v_2 and v_4 are in different regions. By a similar argument, we may change the color of v_2 to 4. Then 2 is available for v.

□

Theorem 5.3 (4CT). *Every planar graph is 4-colorable.*

- The most recent proof was obtained by N. Robertson, D.P. Sanders, P.D. Seymour and R. Thomas (1996): A new proof of the 4CT, Electron. Res. Announc. A.M.S. 2,17-25.
- This first proof was obtained in 1976-1977, by K. Appel and W. Haken.

Open Problem. Characterize the planar graphs which are 3-colorable.