Introduction to Combinatorics Lecture 4

# Topological Graph Theory

**Definition 4.1** (Proper drawing)**.** A proper drawing on a surface of a graph $G$ with $p$ vertices and $q$ edges follows the rules:

1. There are $p$ points on the surface which corresponds to the set of vertices in $G$.
2. There are $q$ curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

**Definition 4.2** (2-manifold)**.** A connected topological space in which every point has a neighborhood homomorphic to the open unit disk defined on $\mathbb{R}^2$.

**Definition 4.3** (Bound subspace)**.** A subspace $M$ of $\mathbb{R}^3$ is bounded if $\exists K \in \mathbb{R}^+$ such that $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 \leq K\}$.

**Definition 4.4** (Closed)**.** $M$ is closed if its boundary $\partial M$ coincides with $M$.

**Definition 4.5** (Orientable)**.** $M$ is orientable if for every simple closed curve $C$ on $M$, a clockwise sense of rotation is preserved once around $C$. Otherwise, $M$ is non-orientable.

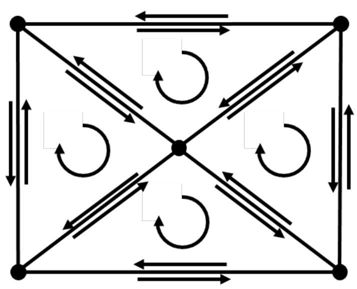
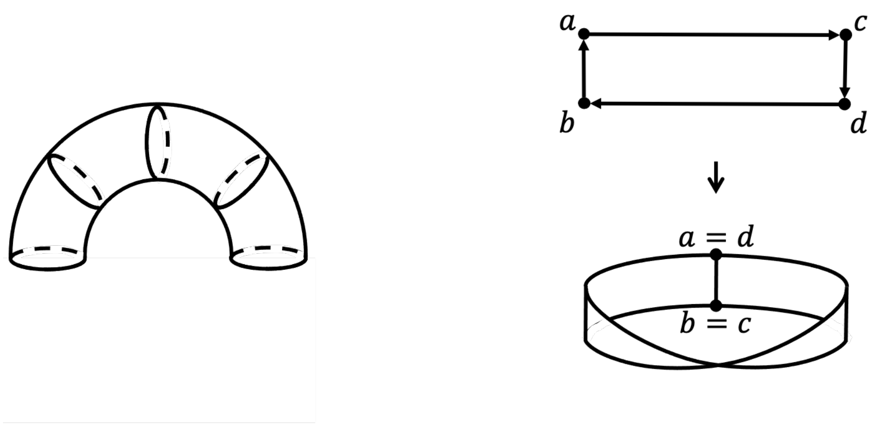


Figure 4.1: Orientable

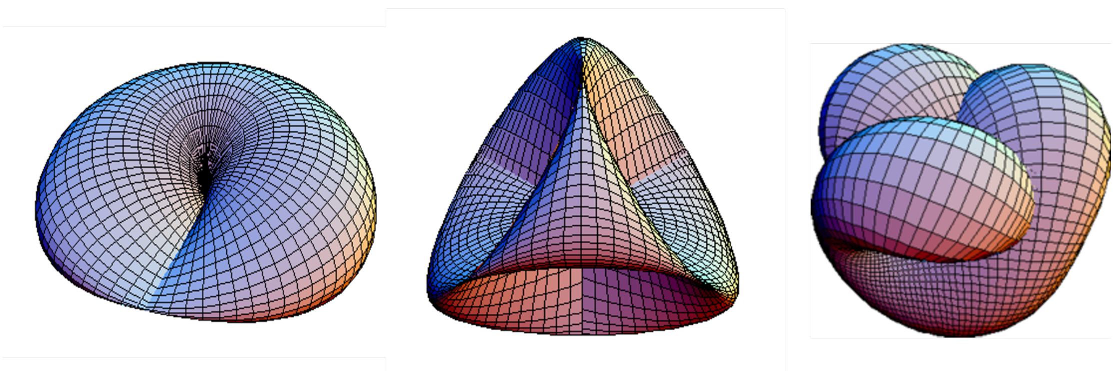
**Definition 4.6** (Orientable surface)**.** A surface $S_k$ is a compact 'orientable' 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number $k$ of 'handles' (holes).

**Definition 4.7** (Non-orientable surface)**.** A surface obtained by adding $k$ cross-caps to a sphere ($S_0$) is a non-orientable surface $N_k$. (Adding a cross-cap: attach the boundary of a Möbius band to a cycle on $S_0$.)




(a) (b)

Figure 4.2: (a) A handle. (b) A Möbius band.



(a) (b) (c)

Figure 4.3: (a) The cross-cap. (b) The Roman surface. (c) The Boy's surface.

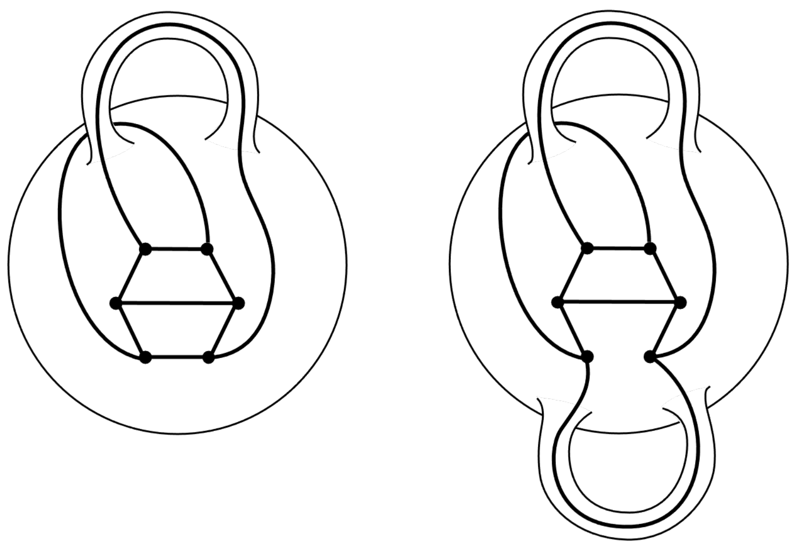
**Definition 4.8** (Embeddable)**.** A $(p, q)$-graph $G$ is said to be embeddable on a surface if it is possible to draw $G$ *properly (drawing without crossings) on the surface.*

**Definition 4.9** (Planar graph)**.** A graph is planar if it can be embedded in the plane, equivalently, embedded on the sphere.

**Definition 4.10** (2-cell embedding)**.** A region is called a 2-cell if any simple closed curve in that region can be continuously deformed or contracted in that region to a single point, equivalently, a 2-cell is topologically homeomorphic to $\mathbb{R}^2$. An embedding of $G$ on a surface is a 2-cell embedding of $G$ if all the regions determined are 2-cells.

*Remark.*

- $S_0$: Sphere
- $S_k$: a surface obtained by attaching $k$ handles to $S_0$.
- $N_0 \simeq S_0$ (Homeomorphic)
- $N_h$: attach $h$ cross-caps to $N_0(S_0)$.



(a) (b)

Figure 4.4: Embeddings of $K_{3,3}$ on surfaces $S_1$ and $S_2$ respectively.

**Definition 4.11** (Genus)**.** The number of handles (resp. cross-caps) (on a surface) is referred to as genus of the orientable surface (resp. non-orientable surface). We use $\gamma(G)$ (resp. $\tilde{r}(G)$) to denote the smallest genus of all orientable surfaces (resp. non-orientable surfaces) on which $G$ can be embedded.

*Remark.*

- If $G$ is a planar graph, then $\gamma(G)$ (so is $\tilde{r}(G)$) is equal to zero. But, $G$ can also be embedded on a surface with genus larger than '0'.
- Given a graph $G$, determining $\gamma(G)$ is a difficult problem.

**Theorem 4.1** (Euler's formula)**.** *Let $G$ be a connected planar graph with $p$ vertices, $q$ edges and $f$ faces (regions). Then, $p - q + f = 2$.*

*Proof.* By induction on $q$. Since $G$ is connected, $G$ has at least $p-1$ edges. (?) If $G$ has $p-1$ edges and $G$ is connected, then $G$ is a tree which contains no cycles. This implies that $f = 1$ and thus $p - (p-1) + 1 = 2$. The assertion is true for 'minimal' graphs. Assume the hypothesis is true for $k = ||G|| \geq p - 1$. Now, consider $G$ with $k+1$ edges. Clearly, $G$ contains a cycle. Let $e$ be a cycle edge. Since $G$ is connected planar graph (with $q$ faces), $G - e$ is also a connected planar graph. Moreover, $G - e$ has $k$ edges and $q-1$ faces. By induction, $p - k + (q-1) = 2$ and thus $p - (k+1) + q = 2$. This concludes the proof. □

**Theorem 4.2.** *If $G$ is a planar graph with largest size, then $||G|| = 3|G| - 6$.*

*Proof.* By observation, if $G$ has maximum size, then each region of $G$ is a triangle. Since each edge of $G$ is in the boundary of exact two regions, $3f = 2q$ where $f$ is the number of regions and $q$ is the size of $G$, i.e., $q = ||G||$. Now, by Euler's formula, $p - q + f = 2$, equivalently, $|G| - ||G|| + \frac{2}{3}||G|| = 2$ and thus $3|G| - 6 = ||G||$. ($G$ is a *maximal planar graph!)* □

**Corollary 4.3.** *If $G$ is a planar graph, then $||G|| \leq 3|G| - 6$.*

**Corollary 4.4.** *In any planar graph, there exists at least one vertex of degree smaller than 6. (This corollary is very useful.)*

**Corollary 4.5.** *The degree sum of a planar graph is at most* $6|G| - 12$.

We can give a more accurate estimation of the above corollary:

**Theorem 4.6.** *Let* $G$ *be a maximal planar graph (triangulated) of order* $p$*, and let* $p_i$ *denote the number of vertices of degree* $i$ *in* $G$ *for* $i = 3, 4, ..., \Delta(G) = d$*. Then,*

$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \cdots + (d-6)p_d + 12.$$

*Proof.* Since $p = \sum_{i=3}^{d} p_i$ and $2q = \sum_{i=3}^{d} i \cdot p_i$, we have $\sum_{i=3}^{d} i \cdot p_i = 2(3p-6) = 6 \cdot \sum_{i=3}^{d} p_i - 12$. This implies the conclusion. □

**Theorem 4.7.** *There are exactly five regular polyhedra.*

*Proof.* Notice that a regular polyhedron is a polyhedron whose faces (regions) are bounded by congruent (全等) regular polygons and whose polyhedral angles are congruent.
First, we convert a polyhedron into a regular planar graph. (See Figure 4.5 for examples.) Let the number of vertices, edges and faces be $p, q$ and $f$ respectively. By Euler's formula, $p - q + f = 2$. Hence,

$$\begin{aligned}
-8 &= 4q - 4p - 4f \\
&= 2q + 2q - 4p - 4f \\
&= \sum_{i\geq 3} i \cdot f_i + \sum_{i\geq 3} i \cdot p_i - 4\sum_{i\geq 3} p_i - 4\sum_{i\geq 3} f_i (f_i : \#\text{ of } i\text{-face}) \\
&= \sum_{i\geq 3}(i-4)f_i + \sum_{i\geq 3}(i-4)p_i.
\end{aligned}$$

Since the polyhedron is regular, all degrees and face sizes are the same, let them be $k$ and $h$ respectively. Therefore,

$$-8 = (h-4)f_h + (k-4)p_k.$$

By the fact that every planar graph contains a vertex of degree less than six, we only have nine cases to consider: $3 \leq h \leq 5$ and $3 \leq k \leq 5$. From direct checking, only 5 cases are possible, namely,

(1) $f_3 = p_3 = 4$ (Tetrahedron, 四面體)

(2) $f_3 = 8$ and $p_4 = 6$ (Octahedron, 八面體)

(3) $f_3 = 20$ and $p_5 = 12$ (Icosahedron, 二十面體)

(4) $f_4 = 6$ and $p_3 = 8$ (Cube, 六面體)

(5) $f_5 = 12$ and $p_3 = 8$ (Dodecahedron, 十二面體)

See Figure 4.5 for regular polyhedra. □

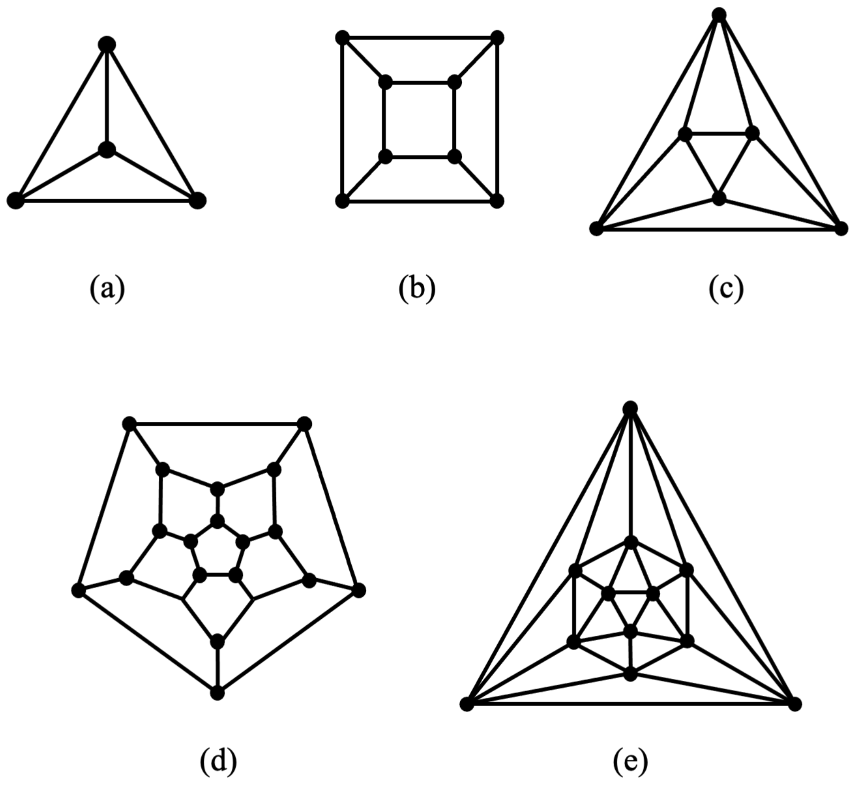



Figure 4.5: (a) Tetrahedron. (b) Cube. (c) Octahedron. (d) Dodecahedron. (e) Icosahedron.

**Theorem 4.8** (Fáry (1948), Wagner (1936)). *A planar graph $G$ can be embedded in the plane so that each edge is a straight line segment.*

*Proof.* The proof is by induction on the order of $G$. It suffices to prove that case when $G$ is a connected maximal planar graph. Clearly, it is true for small orders. Assume the hypothesis is true for order $k$ and let $G$ be a connected maximal planar graph of order $k+1$. Since $G$ is maximal $3 \leq \delta(G) \leq 5$.

Case 1. $\delta(G) = 3$.

Let $v_0 \in V(G)$ such that $deg_G(v_0) = 3$ and $v$ is adjacent to $v_1, v_2$ and $v_3$. Since $G$ is maximal, $\langle\{v_1, v_2, v_3\}\rangle_G \cong K_3$. This implies that $G - v_0$ is also a maximal planar graph. By induction $G - v_0$ has a straight line segment embedding. Now, put $v_0$ back to the graph $G - v_0$ such that $v_0$ is inside the region bounded by $\langle\{v_1, v_2, v_3\}\rangle_G$ and connect $v_0$ to the three vertices by straight line segment. This concludes the proof this case.

Case 2. $\delta(G) = 4$.

The proof follows by a similar process as above by letting $N(v_0) = \{v_1, v_2, v_3, v_4\}$. Now, $G - v_0 + v_1v_3$ is a maximal planar graph and this it has a straight line segment embedding. The proof follows by placing $v_0$ back to $G - v_0 + v_1v_3 - v_1v_3$. By consider the drawing of the embedding (Figure 4.6 (a)), we are able to put $v_0$ back and connected $v_0$ to its neighbors in $G$ by straight line segment.

Case 3. $\delta(G) = 5$.

Again, we use the same technique and the drawing can be seen in Figure 4.6 (b). □

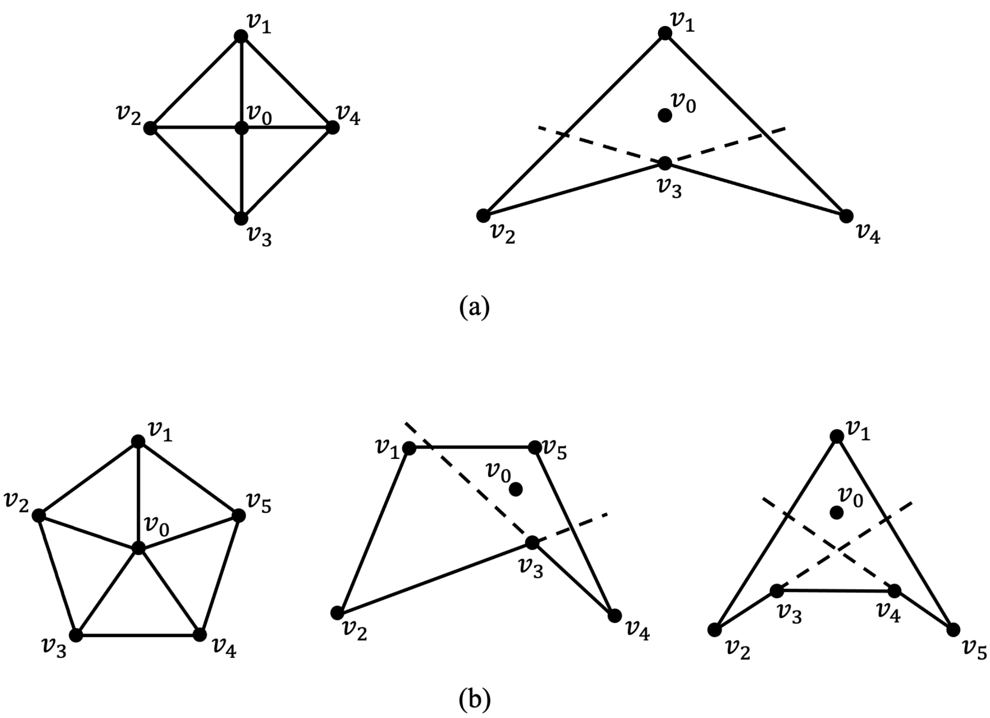



Figure 4.6: Location of $v_0$.

The following theorem considers pseudographs, i.e., loops and multiedges are allowed.

**Theorem 4.9.** *Let $G$ be a $(p,q)$-pseudograph which has a 2-cell embedding on $S_n$. Then, $p-q+f=2-2n$ where $f$ is the number of faces in the embedding.*

*Proof.* By induction on $n$ and it is true when $n=0$ (by Euler's planar graph formula). Assume that the assertion is true when $n=k\geq 0$ and $G$ is a $(p,q)$-pseudograph which has a 2-cell embedding on $S_{k+1}$. Since $k+1\geq 1$, there exists a handle in the embedding, see Figure 4.7 (a). It suffices to consider the embedding such that there exists at least one edge which passes through the handle (on the surface). Note that if we can pull back an edge without passing through the handle, then pull it back, see Figure 4.7 (b). Now, we apply the idea of 'cut and past' to obtain a 2-cell embedding of $\tilde{G}$ on $S_k$.
By using a circle around the handle, we can cut the handle through the circle and obtain $\tilde{G}$, see Figure 4.7 (c). As a consequence, the graph $\tilde{G}$ is embedded in $S_k$. If there are $t$ edges passing through the handle, then $|\tilde{G}|=p+2t$, $||\tilde{G}||=q+3t$ and the embedding in $S_k$ has $f+t+2$ faces. Hence, $(p+2t)-(q+3t)+(f+t+2)=2-2k$. This implies that $p-q+f=2-2(k+1)$. □

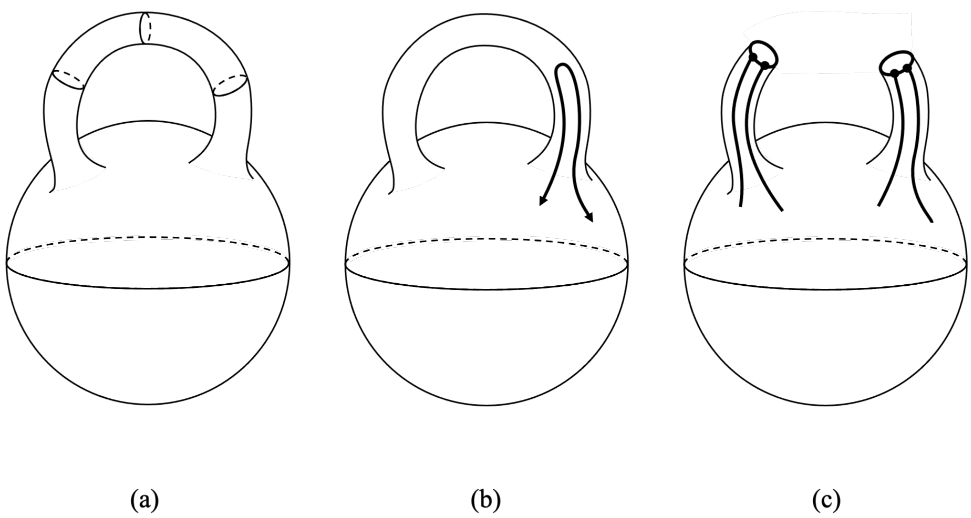



Figure 4.7