Introduction to Combinatorics — Lecture 2

Eulerian Circuits and Hamilton Cycles

Before we get to the proof of Euler's result on Eulerian circuits, we need more background.

Definition 2.1. A graph G is connected if and only if for any two vertices u and v in $V(G)$, there exists a path connecting u and v.

Remark. If G is connected, then $|G| \leq \|G\| + 1$.

Definition 2.2. G_i is a component of G if G_i is a maximal connected subgraph of G. The number of components of G is denoted by $\omega(G)$.

Definition 2.3. G is a forest if G contains no cycles (G is acyclic), and G is a tree if G is connected and acyclic.

Theorem 2.1. *The following statements are equivalent.*

- *G is a tree.*
- *G is acyclic and $\|G\| = |G| - 1$.*
- *G is connected and $\|G\| = |G| - 1$.*
- *Any two vertices of G are connected with a unique path.*

Proof. We prove (1) $\Rightarrow$ (2) and leave the others for the readers to verify.

(1) $\Rightarrow$ (2) By definition, G is a tree implies that G is connected and acyclic. The proof is by induction on $|G|$ and it is true for $|G| = 1$ and 2.
Since G is connected, there exist two vertices u and v which are of maximum distance (diameter). Then, v must be of degree 1. For otherwise, v is either adjacent to some vertex on the path from u to v or v is adjacent to a new vertex. Both of them are not possible.
Hence, $deg_G(v) = 1$. Now, consider $G - v$. $G - v$ is connected and acyclic. By induction hypothesis,

$$\|G - v\| = |G - v| - 1 \implies \|G\| - 1 = |G| - 1 - 1 \implies \|G\| = |G| - 1.$$

□

Definition 2.4. An Eulerian circuit of a graph G is a circuit passing all the edges of G.

Theorem 2.2. *G has an Eulerian circuit if and only if G is connected and each vertex of G is even.*

Proof.
($\Rightarrow$) Since G has a walk passes all vertices, G is connected. If a circuit passes a vertex x h times, then $deg_G(x) = 2h$.

($\Leftarrow$) By induction on $\|G\|$. Since $\|G\| \geq 1$, $\delta(G) \geq 2$ (G is not a tree!) and thus G contains a cycle. Let Z be a circuit in G with the maximum number of edges. If Z is an Eulerian circuit, then we are done. Suppose not.
Let H be a nontrivial component of $G - E(Z)$. Since G is connected, $V(H) \cap V(Z) \neq \emptyset$. Let $x \in V(H) \cap V(Z)$. (Figure 2.1) Now, H is nontrivial connected graph (even graph). Hence, H contains an Eulerian circuit Y. By using x, we can attach Z and Y together to obtain a larger circuit. This contradicts to the maximality of $|E(Z)|$. Hence, Z must be an Eulerian circuit in G.

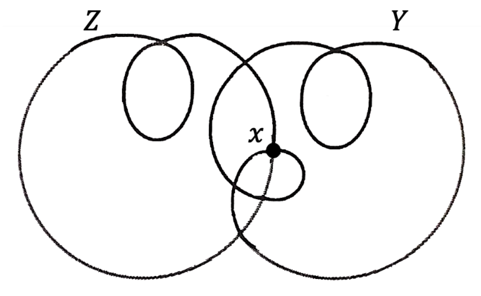


Figure 2.1

□

Open Problem. Find the number of distinct Eulerian circuits of an Eulerian graph G.

Remark.

- The Euler's theorem on circuits is also true for multi-graph, in which we have 2-cycle .
- In a digraph $D = (V, A)$, we use $N_D^+(v)$ (resp. $N_D^-(v)$) to denote the out neighbor (resp. in-neighbor) where $N_D^+(v) = \{u \in V | (v, u) \in A\}$ (resp. $N_D^-(v) = \{u \in V | (u, v) \in A)$. $|N_D^+(v)| = deg_D^+(v)$ and $|N_D^-(v)| = deg_D^-(v)$.

Definition 2.5. A digraph $D = (V, A)$ is connected if for each ordered pair (a, b), $a, b \in V$, there exists a directed path from a to b, i.e., there exists a sequence $< a = a_1, a_2, ..., a_t = b >$ where $(a_i, a_{i+1}) \in A$ for $i = 1, 2, ..., t - 1$.

Theorem 2.3. *A connected digraph $D = (V, A)$ has a directed Eulerian circuit if and only if for each $v\ inV$, $deg_D^+(v) = deg_D^-(v)$.*

Proof. By a similar argument as that of Theorem 2.2. □

Surprisingly, if D has a directed Eulerian circuit, then we can find all distinct directed Eulerian circuits. This is different from the case on un-directed graphs.

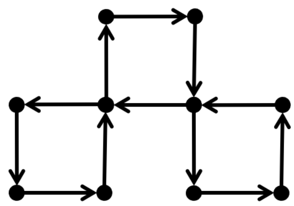


Figure 2.2: A directed eulerian circuit.

Theorem 2.4 (BEST)**.** *Let $t_i(D)$ be the number of spanning trees oriented toward v_i in D of order n. Then the number of distinct Eulerian circuits $s(D)$ is equal to*

$$t_i(D) \cdot \prod_{j=1}^{n} (deg_D^+(v_j) - 1)!.$$

- Note that in a directed eulerian graph $t_i(D) = t_j(D)$ for any two vertices v_i and v_j.
- This theorem was proved by two independent groups: deBruijn and van Aardenne-Ehrenfest, and Smith and Tutte.

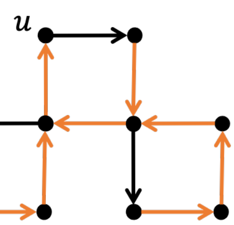


Figure 2.3: A spanning tree oriented toward u.

Definition 2.6. A cycle which contains all vertices of G is called a Hamilton cycle. G is called hamiltonian if G contains a Hamilton cycle.

As mentioned earlier, determining whether a graph is hamiltonian or not is a very difficult problem. In fact, determining whether G contains a cycle of length k is also difficult. So, the researchers are interested in finding good sufficient conditions for the existence of Hamilton cycles. The following theorem is a classical one.

Theorem 2.5 (Ore, 1960)**.** *If G is a graph of order $n \geq 3$ such that for all distinct non-adjacent vertices u and v, $deg(u) + deg(v) \geq n$, then G contains a Hamilton cycle.*

Proof. (By maximality argument)
Assume the assertion is false. Then, there exists a nonhamiltonian graph $\tilde{G}$ of order $n \geq 3$ which satisfies the hypothesis of the theorem. Therefore, for any two distinct vertices v_1 and v_2, $\tilde{G} + v_1v_2$ contains a Hamilton cycle. Furthermore, every Hamilton cycle, if any, of $\tilde{G} + v_1v_2$ contains the edge v_1v_2.

Now, let u and v be two non-adjacent vertices of G. Since $G + uv$ contains a Hamilton cycle, G contains a Hamilton path $< u = v_1, v_2, ..., v_n = v >$. (Figure 2.4)

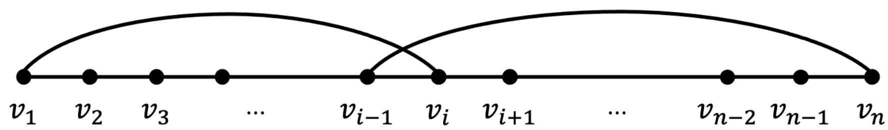
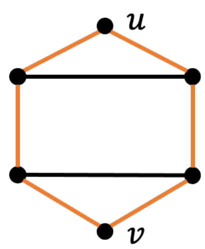


Figure 2.4

By observation, if $v_1v_i \in E(G)$, $2 \leq i \leq n$, then $v_{i-1}v_n \notin E(G)$. (?) For otherwise, we have a Hamilton cycle $(v_1, v_i, v_{i+1}, ..., v_n, v_{i-1}v_{i-2}, ..., v_1)$.
This implies that if $deg(v_i) = t$, $deg(v_n) \leq (n-1) - t$. Hence, $deg(u) + deg(v) \leq t + (n-1) - t = n - 1$, a contradiction. We conclude that G contains a Hamilton cycle. □

Remark. There are sufficient conditions (Quite a few!) for the existence of Hamilton cycles in a graph, but so far, none of them is also necessary. For example, the condition in above theorem is not necessary:



$deg(u) + deg(v) = 4 < 6.$

Weighted Graphs

Definition 2.7 (Weighted graphs)**.** A graph G is weighted if each edge is assigned a weight by using a weighted function $w : E(G) \to \mathbb{R}$.

Traveling Salesman Problem (TSP)

In a weighted complete graph G, find a minimum Hamilton cycle, i.e., the sum of all weights in the cycle is minimum comparing the sums of all the weighted of the other Hamilton cycles.

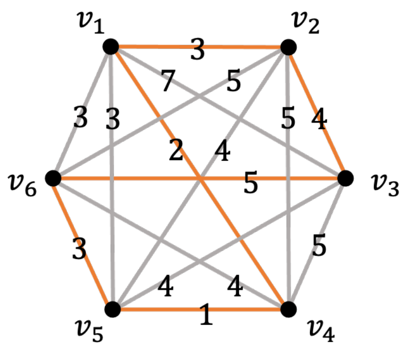


Figure 2.5: A minimum Hamilton cycle.

Remark.

- If each weight is a finite number, then a greedy algorithm can provide an answer. (May not be minimum.)
- If we are looking for minimum spanning trees, then it is an easier problem.