Introduction to Combinatorics

Hung-Lin Fu February 23, 2021

Combinatorics is an area of mathematics primarily concerned with **counting** and **certain properties** of **finite structures**.

- The study of Combinatorics: Combinatorial Theory
- The scope of Combinatorics: Discrete Mathematics and beyond
- The inside of Combinatorics: Applied Algebra (mainly)
- Topics on Combinatorics:
  1. Graph Theory
  2. Design Theory (Combinatorial Designs)
  3. Enumerative Combinatorics
  4. Algebraic Combinatorics
  5. Additive Combinatorics
  6. Combinatorial Geometry
  7. Combinatorial Optimization
  8. Combinatorial Number Theory
  9. Applications of Combinatorics

0.1 Finite Structures

$\mathbb{X}$: A finite non-empty set
$\mathbb{B}$: A collection of subsets of $\mathbb{X}$ (repeated subsets are allowed)
Incidence matrix A of a finite structure $(\mathbb{X}, \mathbb{B})$:
Let $\mathbb{X} = \{x_1, x_2, ..., x_v\}$ and $\mathbb{B} = \{B_1, B_2, ..., B_b\}$,

$$A_{\mathbb{X},\mathbb{B}} = \begin{array}{c} \\ x_1 \\ x_1 \\ \vdots \\ x_v \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_b \end{array} \\ \begin{pmatrix} & & & \\ & \boldsymbol{a_{i,j}} & & \\ & & & \end{pmatrix} \\ v \times b \end{array}, \text{ where } a_{i,j} = \begin{cases} 1 & \text{if} x_i \in B_j, \\ 0 & \text{otherwise.} \end{cases}$$

$(\mathbb{X}, \mathbb{B})$ can be viewed as:

1. a graph where $\mathbb{X}$ is the vertex set and $\mathbb{B}$ is the edge set;
2. a design where $\mathbb{X}$ is the set of varieties (elements) and $\mathbb{B}$ is the set of blocks (experiments);
3. a code of length v where each column of $A_{\mathbb{X},\mathbb{B}}$ is a codeword;
4. pooling design with v test and b items; and more.

0.2 Graphs

- Simple graph : $\mathbb{B} \subseteq \binom{\mathbb{X}}{2}$ (a set not a multi-set)
- Multi-graph : $\mathbb{B} \subseteq \binom{\mathbb{X}}{2}$ (with possible repeated elements)
- Hypergraph : $\mathbb{B} \subseteq 2^{\mathbb{X}}$ (repeated elements are possible)
- Random graph : The existence of an edge is of probability $0 \leq p \leq 1$.
  (Revised version: $\forall x, y \in \mathbb{X}, \exists p(x, y)$, i.e., each edge has its own probability.)
- Directed graph : $\mathbb{B} \subseteq \mathbb{X} \times \mathbb{X} = \mathbb{X}^2$

0.3 Graph Theory

- Study the structure of graphs and its applications.
- Topics on Graph Theory:
  1. Subgraphs
     What kind of subgraphs does a graph contain?
  2. Chromatic Theory: Graph colorings.
  3. Topological Graph Theory: Graph embeddings, proper drawing of a graph on surface.
  4. Extremal Graph Theory
     The graph of maximum size which forbids a given graph.
     If both G and $\bar{G}$ (or $G_1, G_2, ..., G_t$ with $\bigcup_{i=1}^{t} E(G_i) = E(G)$) are concerned, we have Ramsey Theory.
  5. Random Graph Theory
     Study the structure of random graphs.

6. Algebraic Graph Theory
   Use the adjacency matrices or Laplacians of a graph G.
7. Graph Labelings
   Label the graph (either the vertices or edges) to satisfy given constraints. (Many topics/ problems can be converted into labeling graph problems.)
8. Digraphs
   The most popular topic is Network.
9. Algorithmic Graph Theory
   Graph algorithms are applied to solve graph optimization problems.

0.4 Plan of Lectures

1. Cover chapters 1, 2, 3, 4, 5, 32, 33 (in the textbook) for Graph Theory.
2. Cover chapters 10, 13, 14 for Enumerative Combinatorics.
3. Cover chapters 5, 6, 17, 19 for Design Theory.
4. Introduce Additive Combinatorics; Sum sets.
5. Introduce some topics of Combinatorial Optimization.

Textbook: A course in Combinatorics, Van Lint and Wilson.
Three term papers will cover 1, 2+4, 3+5, respectively.