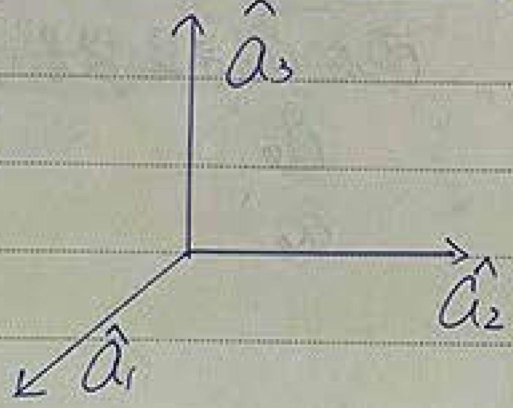
Ch2 Vector Analysis

§2-4 Orthogonal coordinate system

1. Base vector → 指向 A_1, A_2, A_3 增加的方向



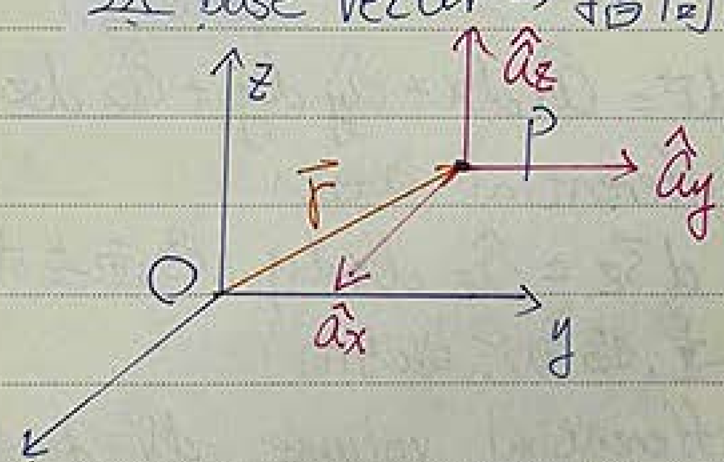
$$|\hat{a}_i|^2 = 1$$
$$\hat{a}_i \cdot \hat{a}_j = \delta_{ij}$$
$$\hat{a}_i \times \hat{a}_j = \hat{a}_k$$
$$\vec{A} = \underbrace{\hat{a}_1 A_1}_{\vec{A}\text{在 }\hat{a}_1\text{ 上的投影量}} + \hat{a}_2 A_2 + \hat{a}_3 A_3$$
$$= \hat{a}_1^{(1)} A_1^{(1)} + \hat{a}_2^{(1)} A_2^{(1)} + \hat{a}_3^{(1)} A_3^{(1)}$$

$$\Rightarrow |\vec{A}|^2 = \vec{A} \cdot \vec{A} = A_1^2 + A_2^2 + A_3^2 = A_1^{(1)2} + A_2^{(1)2} + A_3^{(1)2}$$

2. 令 $P = (u_1, u_2, u_3)$ 坐標定義在 P 点上

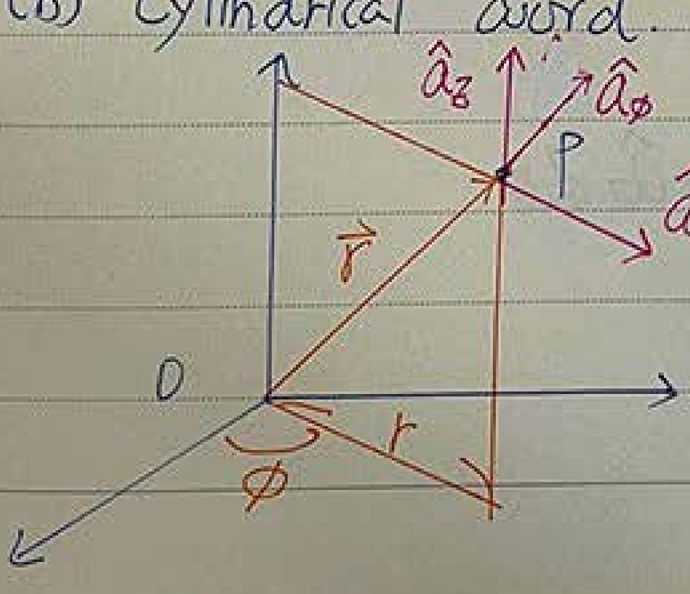
(a) Cartesian coord. : $P(u_1, u_2, u_3) = (x, y, z)$

选 base vector → 指向 u_1, u_2, u_3 增加的方向



$$\overline{OP} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z$$

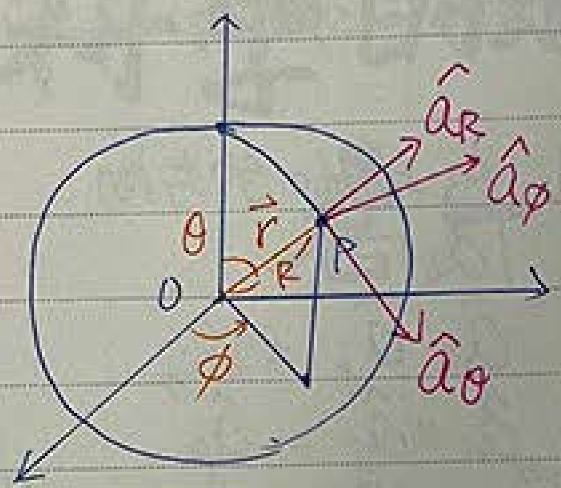
(b) Cylindrical coord. : $P(u_1, u_2, u_3) = (r, \phi, z)$



$$\left.\begin{array}{l}\hat{a}_1 = \hat{a}_r \\ \hat{a}_2 = \hat{a}_\phi \\ \hat{a}_3 = \hat{a}_z\end{array}\right\} \text{与位置有关}$$

$$\vec{r} = \hat{a}_r \underbrace{r}_{A_r} + \hat{a}_\phi \cdot \underbrace{0}_{A_\phi} + \hat{a}_z \underbrace{z}_{A_z}$$

(c) Spherical coord. : $P(u_1, u_2, u_3) = (r, \theta, \phi)$



$\hat{a}_R$ 是 ϕ, θ 之函數

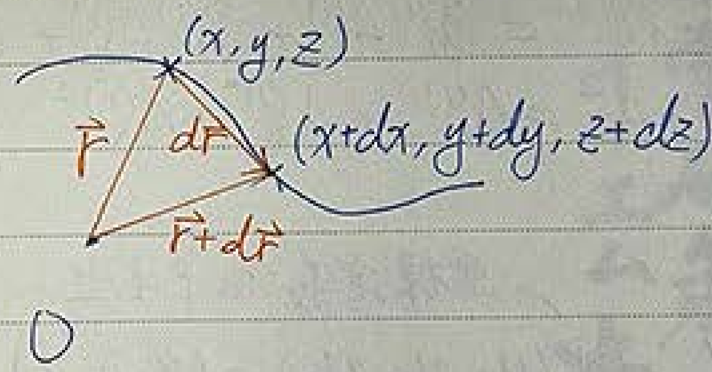
$\vec{r} = \hat{a}_R R + \hat{a}_\theta\, 0 + \hat{a}_\phi\, 0$

$\vec{r}$在 $\hat{a}_R$ 上之投影量

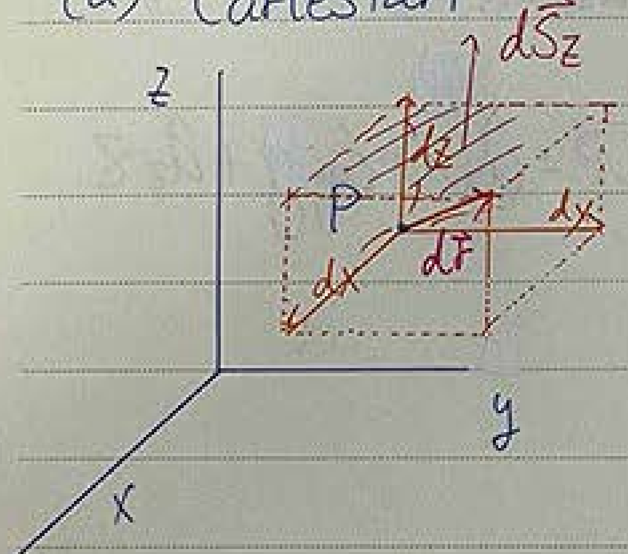
" $\hat{a}_\theta$ "

" $\hat{a}_\phi$ "

3. differential element



(a) Cartesian



$d\vec{r} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$

differential area:

$d\vec{S}_z = \hat{a}_z\, dx dy$ (將面當成一向量,指向法線方向)

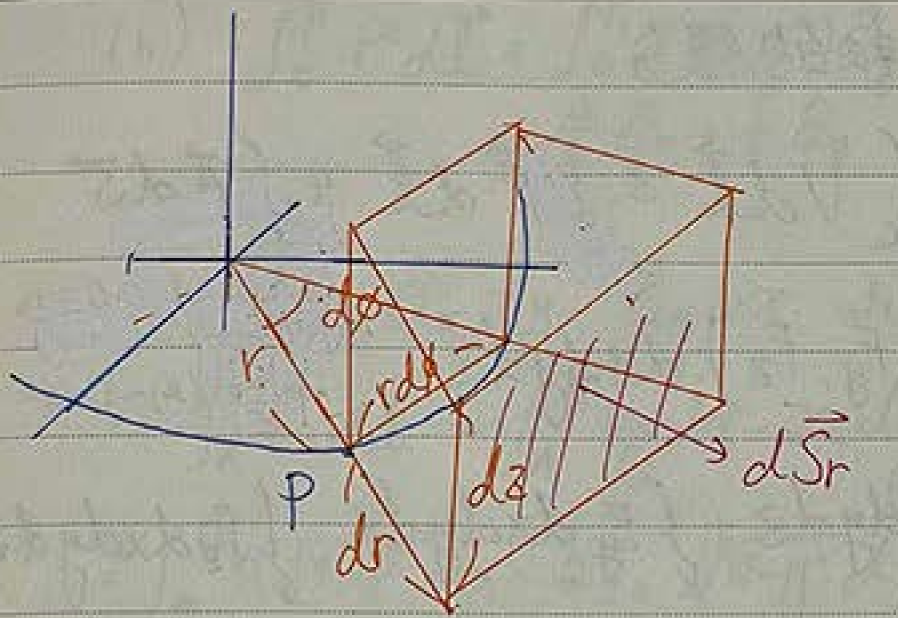
differential volume $dV = dxdydz$

(b) Cylindrical coord.

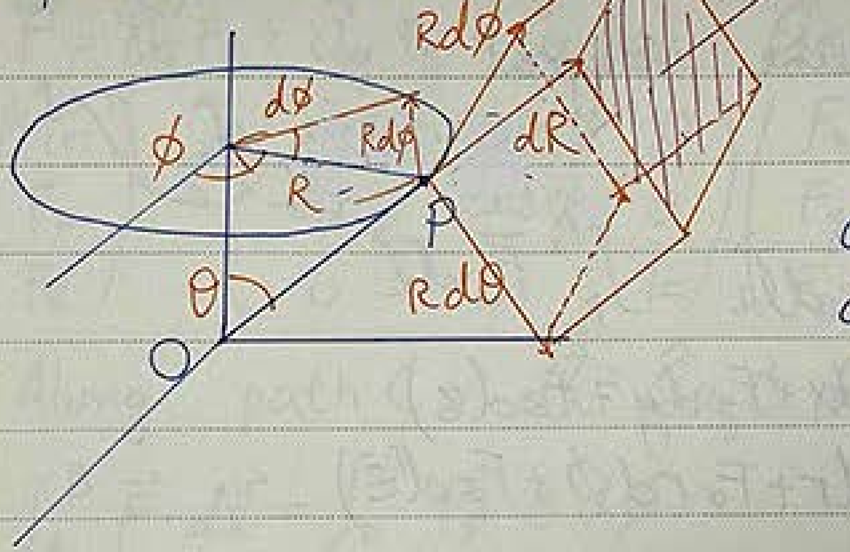
$d\vec{r} = \hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz$

$d\vec{S}_r = \hat{a}_r r\, d\phi\, dz$

$dV = r dr d\phi dz$



(c) Spherical coord



$$d\vec{r} = \hat{a}_R\, dR + \hat{a}_\theta\, R\,d\theta + \hat{a}_\phi\, R\sin\theta\, d\phi$$

$$d\vec{S}_R = R^2 \sin\theta\, d\theta\, d\phi$$

$$dV = R^2 \sin\theta\, dR\, d\theta\, d\phi$$

ex2-12. $\rho = -3\times10^{-8}$

$$Q = \int_V \rho\, dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{5} -3\times10^{-8}\,\frac{\cos^2\phi}{R^4}\,R^2\sin\theta\; dR\,d\theta\,d\phi$$

$$= -1.8\,\mu C$$ #

§2-5 包含向量函數的積分

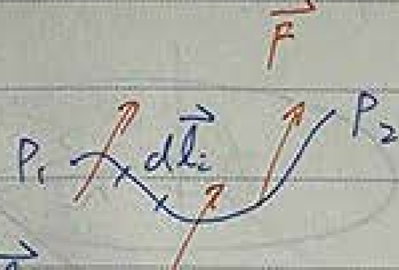
1. $\int_V \vec{F}\, dV$ 2. $\int_C V d\vec{l}$ 3. $\int_C \vec{F}\, d\vec{l}$ 4. $\int_S \vec{A}\, d\vec{S}$

1. $\int_V \vec{F}\, dV = \int_V (F_x, F_y, F_z)\, dx dy dz$

$= \left(\int_V F_x\, dxdydz, \int_V F_y\, dxdydz, \int_V F_z\, dxdydz\right)$

(少見，也不難∵只是純量積分做3次)

2. $\int_C \vec{F} d\vec{l} \rightarrow$ line integral

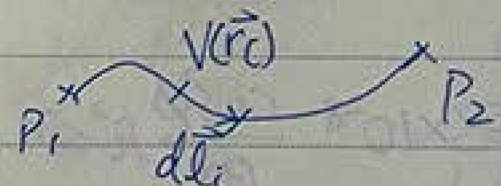


$\sum_{P_1 \to P_2} \Delta W_i = \sum \vec{F}_i \cdot d\vec{l}_i = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$

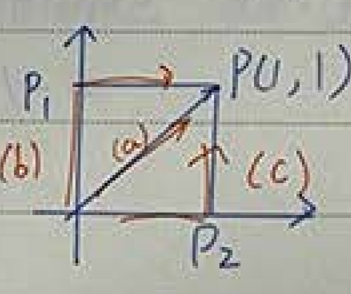
$= \int (F_x dx + F_y dy + F_z dz)$

$(= \int F_r dr + F_\phi r d\phi + F_z dz)$

3. $\int V d\vec{l} = \sum V(\vec{r}_i)\, d\vec{l}_i$



ex 2-13:



(a) $\int_0^P r^2 d\vec{l} = \int_0^{\sqrt{2}} r^2 \hat{a}_r\, dr$

$= \hat{a}_r \cdot \frac{2\sqrt{2}}{3}$

$= \left(\hat{a}_x \frac{\sqrt{2}}{2} + \hat{a}_y \frac{\sqrt{2}}{2}\right) \frac{2\sqrt{2}}{3}$

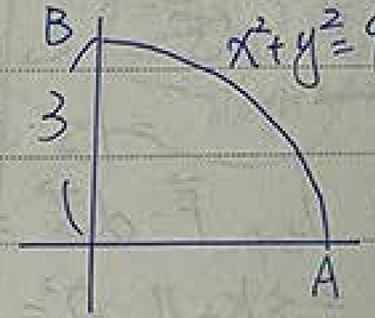
$= \frac{2}{3}\hat{a}_x + \frac{2}{3}\hat{a}_y$

(b) $\int_0^P r^2 d\vec{l} = \int_0^{P_1} (\cancel{x^2}^{\,0}+y^2)\hat{a}_y\, dy + \int_{P_1}^{P} (x^2+\cancel{y^2})\hat{a}_x\, dx$

$= \frac{4}{3}\hat{a}_x + \frac{1}{3}\hat{a}_y$

ex2-14. $\vec{F} = \hat{a}_x xy - \hat{a}_y 2x$ 求 $\int_A^B \vec{F}\cdot d\vec{l}$

-sol-(a) $\int_A^B \vec{F}\cdot d\vec{l} = \int_A^B xy\,dx - 2x\,dy$



$= \int_3^0 x\sqrt{9-x^2}\,dx - 2\int_0^3 \sqrt{9-y^2}\,dy$

$= \cdots = -9(1+\frac{\pi}{2})$

(b) Change to cylindrical coord.

$\vec{F} = \hat{a}_x F_x + \hat{a}_y F_y + \hat{a}_z F_z = \hat{a}_r F_r + \hat{a}_\phi F_\phi + \hat{a}_z F_z$

$$\begin{pmatrix} F_r \\ F_\phi \\ F_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

Along path C : $d\vec{l} = \hat{a}_\phi \cdot 3\, d\phi$, $dr = dz = 0$

$\int_A^B \vec{F}\cdot d\vec{l} = \int_0^{\frac{\pi}{2}} F_\phi \cdot 3\, d\phi$

$= \int_0^{\frac{\pi}{2}} (xy(-\sin\phi) - 2x\cos\phi)\, 3\, d\phi$

$= \int_0^{\frac{\pi}{2}} 3\cos\phi\, 3\sin\phi(-\sin\phi) - 2\cdot 3\cos\phi \cdot 3\, d\phi$

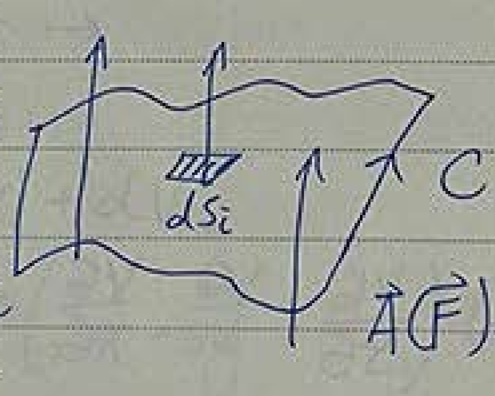
$= \cdots = -9(1+\frac{\pi}{2})$ #

4. Flux integral

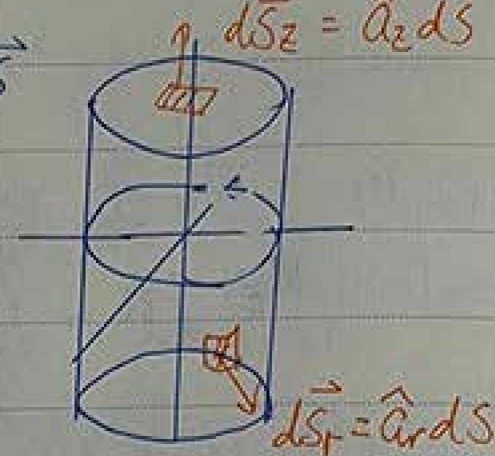
* Volume V $\leftrightarrow$ closed surface S
  $\Rightarrow$ outward direction for $d\vec{S}$

  open surface S $\leftrightarrow$ closed path C

  $d\vec{S}$: right hand rule for C



ex 2-15 $\vec{F} = \hat{a}_r \frac{k_1}{r} + \hat{a}_z k_2 z$，求 $\oint_s \vec{F}\, d\vec{S} = ?$



-sol- $\oint_S \vec{F}\cdot dS = \left(\int_{top} + \int_{bottom} + \int_{side}\right) \vec{F}\cdot d\vec{S}$

(a) top: $d\vec{S} = \hat{a}_z dS$

$\Rightarrow \vec{F}\cdot d\vec{S} = k_2 z\, dS\big|_{top}$

$= k_2 \cdot 3\, dS$

$\Rightarrow \int_{top} \vec{F}\cdot d\vec{S} = \int_{top} 3k_2\, dS$

$= 3k_2 \pi \cdot 2^2 = 12 k_2 \pi$

(b) Bottom: $d\vec{S} = -\hat{a}_z dS$

$\vec{F}\cdot d\vec{S} = -k_2 z\, dS\big|_{bottom}$

$\int_{bottom} \vec{F}\cdot d\vec{S} = 12\pi k_2$

(c) side: $d\vec{S} = \hat{a}_r dS$

$\vec{F}\cdot d\vec{S} = \frac{k_1}{r} dS\big|_{side} = \frac{k_1}{2} dS$

$\int_{side} \vec{F}\cdot d\vec{S} = \frac{k_1}{2}\cdot 6 \cdot 2\pi \cdot 2 = 12\pi k_1$

$\Rightarrow \oint_S \vec{F}\cdot d\vec{S} = 24\pi k_2 + 12\pi k_1$ #

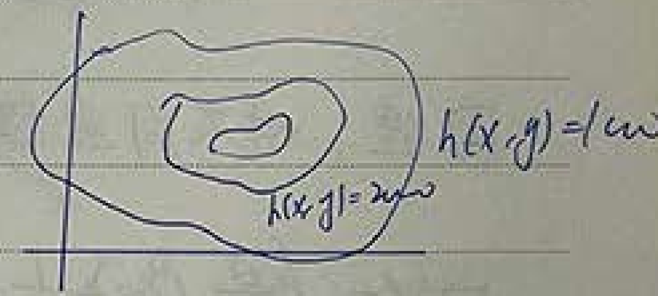
note: 若 $\int$ 內無法全部提出，則需用 $dS = r\,dr\,d\phi$, ... 等進一步計算.

§2-6 Gradient.

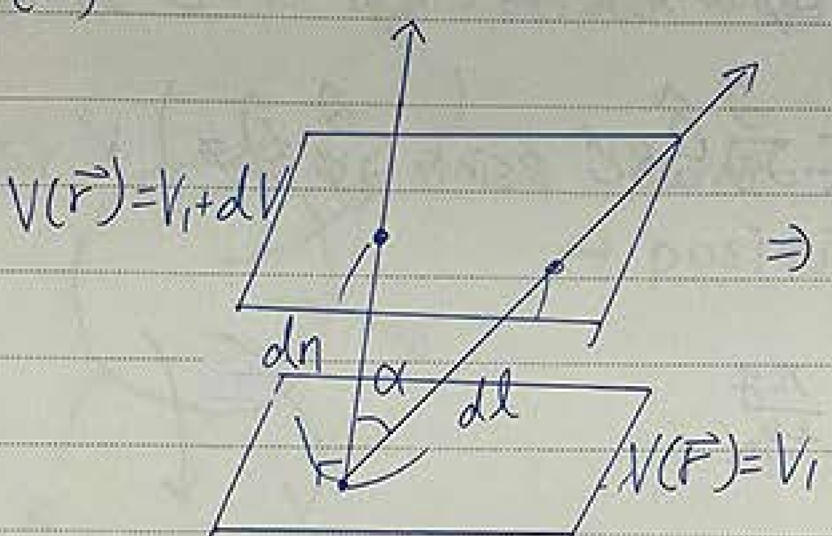
1. $\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$

2. Contour $h(x,y) = h(\vec{r})$

→ 從几何定義出grad.



3. Gradient of scalar field $V(\vec{r})$



$$\Rightarrow \frac{dV}{dl} \le \frac{dV}{dn}$$

定義V之梯度為V在$\hat{n}$方向的增加率

$$\text{grad } V = \hat{a}_n \frac{dV}{dn} = \vec{\nabla} V$$

☆ 延 $\vec{l}$ 方向變化

$$\frac{dV}{dl} = \frac{dV}{dn/\cos\alpha} = \frac{dV}{dn}(\hat{a}_n \cdot \hat{a}_l) = \vec{\nabla} V \cdot \hat{a}_l$$

$$(\rightarrow dV = \vec{\nabla} V \cdot \vec{a}_l \, dl = \vec{\nabla} V \cdot d\vec{l})$$

4. $\vec{\nabla}$ 可為一算子. 在 Cartesian 上 →

令 $V(\vec{r}) = V(x,y,z)$,

且 $V(x+dx, y+dy, z+dz) = V(x,y,z) + dV$,

$\because dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \cdot (dx, dy, dz)$ 且 $d\vec{l} = (dx, dy, dz)$

與 $dV = \vec{\nabla} V \cdot d\vec{l}$ 比較

$$\therefore \vec{\nabla} V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$
$$= \hat{a}_x \frac{\partial V}{\partial x} + \hat{a}_y \frac{\partial V}{\partial y} + \hat{a}_z \frac{\partial V}{\partial z}$$
$$= \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) V$$

同理，对不同坐標系下之$d\ell$做修改，可得

$$\nabla V = \left(\frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z\right) V$$

$$\nabla V = \left(\frac{\partial}{\partial R}\hat{a}_R + \frac{1}{R}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi\right) V$$

ex2-16. $\vec{E} = \vec{\nabla} V$

(a) $V = V_0 e^{-\frac{x}{x_0}} \sin\frac{\pi y}{4 y_0}$

(b) $V = E_0 R \cos\theta$

-sol-.

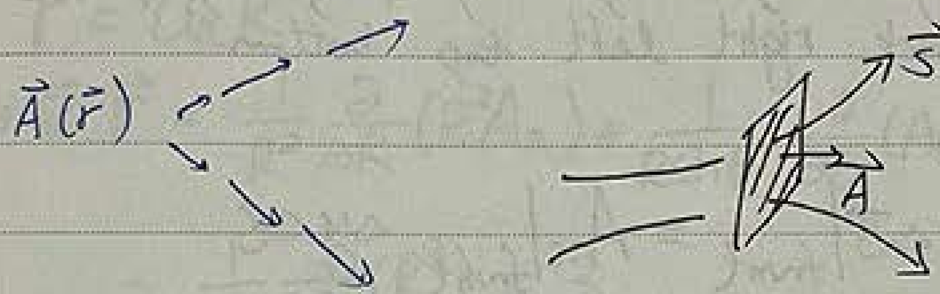
(a) $$\vec{E} = \vec{\nabla} V = -\left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y}\right) V$$
$$= \left(-\hat{a}_x V_0 \left(-\frac{1}{x_0}\right) e^{-\frac{x}{x_0}} \sin\frac{\pi y}{4y_0} + \hat{a}_y V_0 e^{-\frac{x}{x_0}} \left(\frac{\pi}{4y_0}\right) \cos\frac{\pi y}{4 y_0}\right)$$
$$= E_0 e^{-x}\left(\hat{a}_x \sin\frac{\pi y}{4} - \hat{a}_y \frac{\pi}{4}\cos\frac{\pi y}{4}\right)$$

(b) $$\vec{E} = \vec{\nabla} V = -\left(\hat{a}_R \frac{\partial}{\partial R} + \hat{a}_\theta \frac{\partial}{R\partial\theta} + \hat{a}_\phi \frac{\partial}{R\sin\theta\partial\phi}\right) E_0 R\cos\theta$$

$$= -E_0\left(\hat{a}_R \cos\theta + \hat{a}_\theta \frac{1}{R}\cdot R\sin\theta + \hat{a}_\phi \frac{0}{R\sin\theta}\right)$$
$$= -E_0\left(\hat{a}_R\cos\theta + \hat{a}_\theta \sin\theta\right)$$
$$= -E_0 \hat{a}_z \quad \#$$

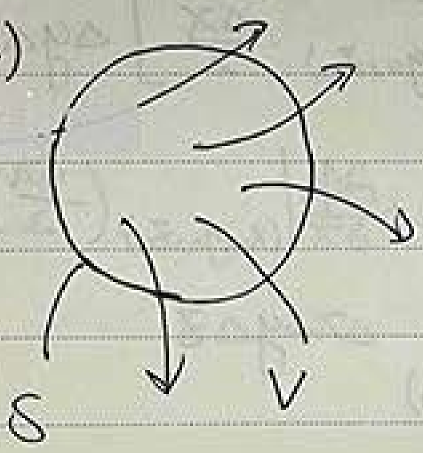
§2-7 Divergence $\vec{A}(\vec{r})$

1. flux line / field line / stream line



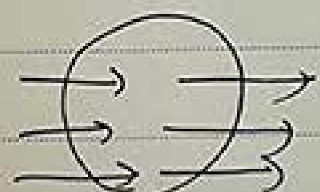
2. $\Phi \triangleq \vec{A} \cdot \vec{S}$ or $d\Phi = \vec{A} \cdot d\vec{S} \Rightarrow \Phi = \int \vec{A} \cdot d\vec{S}$

3. (a) Source → net outward flux
→ positive divergence



(b) net inward flux
→ negative divergence

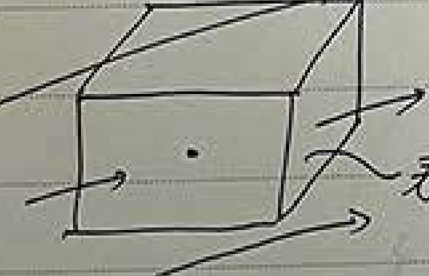
(c) No source, no sink → net zero flux
→ zero divergence



divergence 之数学建模

4. $\vec{A} = (A_x, A_y, A_z)$

$\operatorname{div} \vec{A} = \nabla \cdot \vec{A}$



box 之
表面積 ΔS

$$\triangleq \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\Delta S} \vec{A} \cdot d\vec{S}$$

5. 化簡

$$\oint_{\Delta S} \vec{A}\cdot d\vec{S} = \left(\int_{front} + \int_{back} + \int_{right} + \int_{left} + \int_{top} + \int_{bottom}\right) \vec{A}\cdot d\vec{S}$$

$$\int_{front} \vec{A}\cdot d\vec{S} = \vec{A}\cdot\Delta\vec{S}\Big|_{front} = A_x\Big|_{front}\,\Delta y\Delta z$$

$$= A_x\left(x_0+\frac{\Delta x}{2}, y_0, z_0\right)\Delta y\Delta z$$

By formula $f(x_0+\Delta x) \doteqdot f(x_0) + \frac{df}{dx}\Big|_{x_0}\Delta x$

$$\rightarrow \quad = \left[A_x(x_0,y_0,z_0) + \frac{\partial A_x}{\partial x}\Big|_{(x_0,y_0,z_0)}\cdot\frac{\Delta x}{2}\right]\Delta y\Delta z$$

Similarly, $\int_{back} \vec{A}\cdot d\vec{S} \doteqdot \left[A_x(x_0,y_0,z_0) + \frac{\partial A_x}{\partial x}\Big|_{(x_0,y_0,z_0)}\left(-\frac{\Delta x}{2}\right)\right]\Delta y\Delta z$

$$\Rightarrow \left(\int_{front} + \int_{back}\right)\vec{A}\cdot d\vec{S} = \frac{\partial A_x}{\partial x}\Big|_{(x_0,y_0,z_0)}\Delta x\Delta y\Delta z$$

$$\Rightarrow \oint_{\Delta S}\vec{A}\cdot d\vec{S} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x\Delta y\Delta z$$

$$\Rightarrow \mathrm{div}\vec{A} = \lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{1}{\Delta x\Delta y\Delta z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Big|_{(x_0,y_0,z_0)}\Delta x\Delta y\Delta z$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\left(= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(A_x, A_y, A_z) = \vec{\nabla}\cdot\vec{A} \right)$$

ex 2-17. 試求 $\vec{\nabla}\cdot\vec{r} = ?$ $\vec{r} = \overrightarrow{OP}$

-sol-

(a) $\vec{r} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z$

$$\rightarrow \vec{\nabla}\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

(b) Spherical coord.

$$\vec{F} = \hat{a}_R R$$

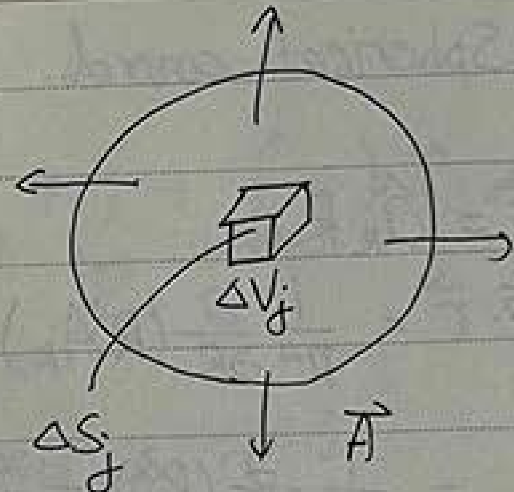
$$\rightarrow \vec{\nabla}\cdot\vec{r} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^3) = 3 \ \#$$

§2.8 Divergence Thm.

By def. of $\vec{\nabla}\cdot\vec{A}$, net flux of Δv_j

$$=(\vec{\nabla}\cdot\vec{A})_j\,\Delta v_j = \oint_{S_j}\vec{A}\cdot d\vec{s}$$

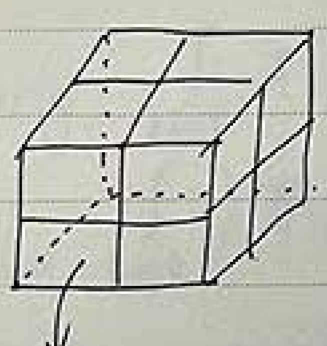


$\Rightarrow$ $\sum$ net flux of Δv_j:

$$\sum_j^N(\vec{\nabla}\cdot\vec{A})_j\,\Delta v_j = \sum_j^N\oint_{S_j}\vec{A}\cdot d\vec{s}$$

① When $N\to\infty$, $\Delta v_j\to 0$, $LHS=\int_V\vec{\nabla}\cdot\vec{A}\,dv$

②



$$\Rightarrow\begin{cases}\Delta\vec{S}_j\text{ left} = \Delta\vec{S}_{j+1}\text{ right}\\ \vec{A}_j\text{ left} = \vec{A}_{j+1}\text{ right}\end{cases}$$

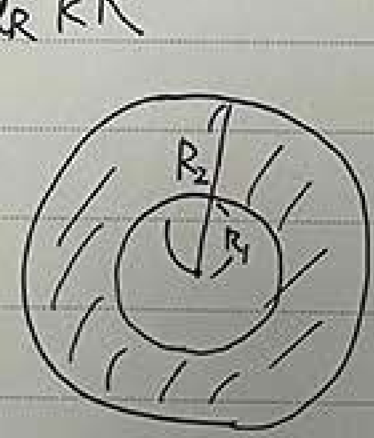
每塊之左面/右 = 相鄰塊之右面/左

$$\Rightarrow RHS = \sum_{\substack{\text{external}\\ \text{area}}}\vec{A}\cdot d\vec{s} = \int_S\vec{A}\cdot d\vec{s} = \int_V\vec{\nabla}\cdot\vec{A}\,dv$$

$$\Rightarrow\oint_S\vec{A}\cdot d\vec{s} = \int_V\vec{\nabla}\cdot\vec{A}\,dv$$ (Gauss' divergence Thm)

ex3-20. 驗證 div. Thm. $\vec{F}=\hat{a}_R kR$

(1) $\oint_S\vec{F}\cdot ds = \left(\oint_{S_1}+\oint_{S_2}\right)\vec{F}\cdot d\vec{s}$



$$\oint_{S_1}\vec{F}\cdot d\vec{s} = \oint(\hat{a}_R kR_1)(-\hat{a}_R)\,ds$$
$$= -kR_1\cdot 4\pi R_1^2$$
$$= -4k\pi R_1^3$$

$$\oint_{S_2} \vec{F}\cdot d\vec{S} = \oint_{S_2} (\hat{a}_R k R_2)\cdot \hat{a}_R \, dS_2$$

$$= 4k\pi R_2^3$$

$$\Rightarrow \oint_S \vec{F}\cdot d\vec{S} = 4\pi k(R_2^3 - R_1^3)$$

(2) Volumn integral

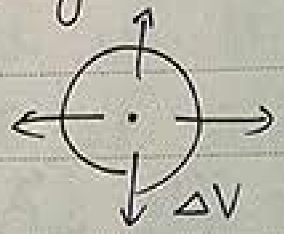
$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(k\cdot\underbrace{(\hat{a}_R R)}_{=\vec{r}}) = 3k$$

$$\Rightarrow \int_V \vec{\nabla}\cdot\vec{F}\, dv = 3k\cdot\frac{4}{3}\pi(R_2^3 - R_1^3) = \oint_S \vec{F}\cdot d\vec{S} \#$$

MINIATURE LIFE EXHIBITION

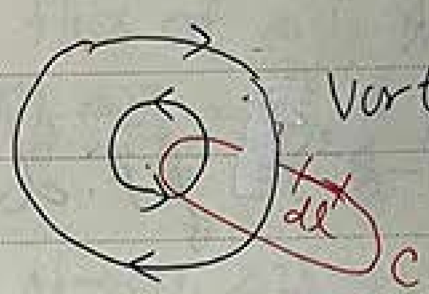
§ 2.9. Curl

1. divergence $\leftrightarrow$ net flow / flux



flow source

2. curl / circulation



Vortex source

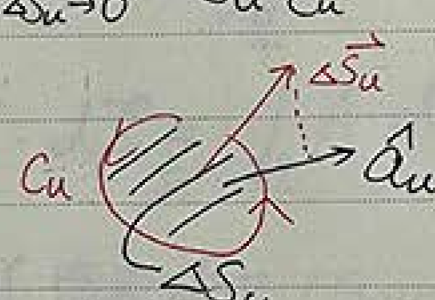
3. circulation of $\vec{A}$ around C $\triangleq \oint_C \vec{A}\cdot d\vec{l}$

4. curl $\vec{A}$ 旋度：不直接定義，而是定義這个向量 (curl $\vec{A}$) 投影在一方向上的量：

$$(\text{curl}\vec{A})_u = (\text{curl}\vec{A})\cdot\hat{a}_u$$

$$\triangleq \lim_{C_u\to 0,\ \Delta S_u\to 0} \frac{1}{\Delta S_u}\oint_{C_u}\vec{A}\cdot d\vec{l},$$

$C_u,\ \Delta\vec{S}_u \perp \hat{a}_u$

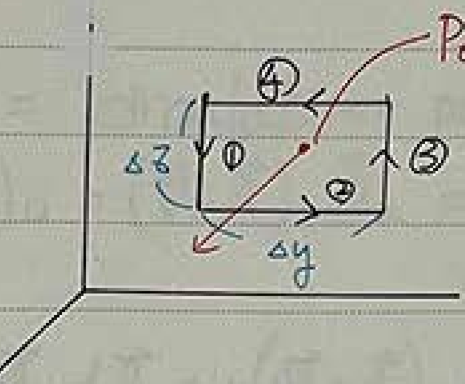


$*\ (\text{curl}\vec{A})_u \le |\text{curl}\vec{A}|$

$$\Rightarrow \text{curl}\vec{A} \triangleq \nabla\times\vec{A} \triangleq \lim_{\Delta S\to 0}\max_{\hat{a}_n}\left[\frac{\hat{a}_n\oint_C \vec{A}\cdot d\vec{l}}{\Delta S}\right]$$

5. Cartesian coord.

x component of $\vec{\nabla}\times\vec{A}$:



$\rightarrow \hat{a}_u = \hat{a}_x$, $\Delta S_u = \Delta S_x = \Delta y \Delta z$, C_u : ① → ② → ③ → ④

$$(\vec{\nabla} \times \vec{A})_x = \lim_{\Delta y, \Delta z \to 0} \frac{1}{\Delta y \Delta z} \oint_{C_u} \vec{A} \cdot d\vec{l}$$

$$\oint_{C_u} \vec{A} \cdot d\vec{l} = \left(\int_{①} + \int_{②} + \int_{③} + \int_{④} \right) \vec{A} \cdot d\vec{l}$$

$$\int_{①} \vec{A} \cdot d\vec{l} = \int_{①} \vec{A} \cdot (-\hat{a}_z)\, dz = A_z \Big|_{(x_0, y_0 - \frac{\Delta y}{2}, z_0)} \cdot \Delta z$$

$$\doteq \left(A_z(x_0, y_0, z_0) - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y} \Big|_{(x_0, y_0, z_0)} \right) \Delta z$$

$$\int_{③} \vec{A} \cdot d\vec{l} = \int_{③} \vec{A} \cdot \hat{a}_z\, dz = A_z \Big|_{(x_0, y_0 + \frac{\Delta y}{2}, z_0)} \cdot \Delta z$$

$$\doteq \left(A_z(x_0, y_0, z_0) + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y} \Big|_{(x_0, y_0, z_0)} \right) \Delta z$$

$$\Rightarrow \left(\int_{①} + \int_{③} \right) \vec{A} \cdot d\vec{l} = \cdots$$

$$\Rightarrow (\vec{\nabla} \times \vec{A})_x = \lim_{\Delta y, \Delta z \to 0} \frac{1}{\Delta y \Delta z} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \Delta y \Delta z$$

$$= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

由 $x \overset{换}{\rightarrow} y \rightarrow z \rightarrow x$ ，可得

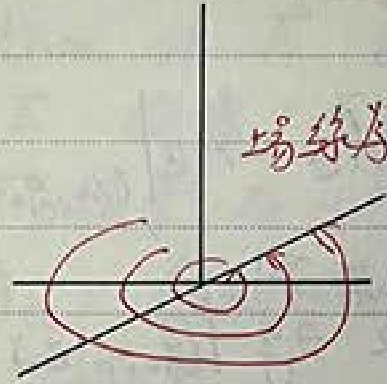
$$\vec{\nabla} \times \vec{A} = \hat{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

ex. 1. $\vec{A} = \hat{a}_\phi \frac{k}{r}$

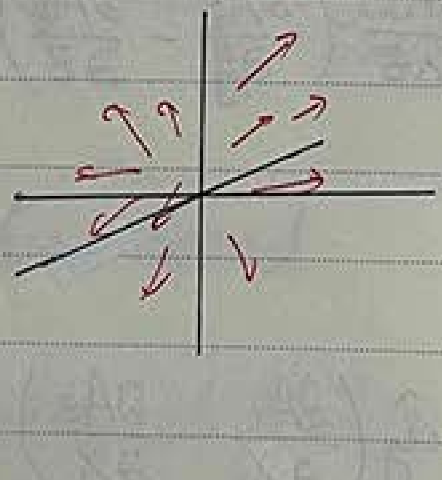
$$\text{curl } A = \frac{1}{r} \begin{vmatrix} \hat{a}_r & \hat{a}_\phi r & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \underset{0}{\underset{\parallel}{A_r}} & \underbrace{rA_\phi}_{\substack{\parallel \\ \frac{k}{r}}} & \underset{0}{\underset{\parallel}{A_z}} \end{vmatrix} = 0$$

∵k为const
∴怎么微都是0



2. $\vec{A} = \hat{a}_R f(R)$

$$\nabla \times \vec{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{a}_R & \hat{a}_\theta R & \hat{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \underset{f(R)}{\underset{\parallel}{A_R}} & \underset{0}{\underset{\parallel}{R\cancel{A_\theta}}} & \underset{0}{\underset{\parallel}{R\sin\theta \cancel{A_\phi}}} \end{vmatrix} = 0.$$



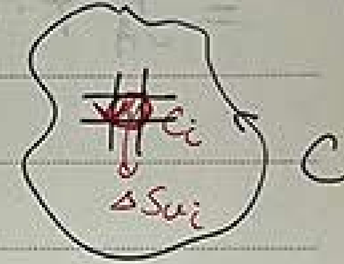
§2-10 Stoke Thm.

1. $\vec{\nabla}\times\vec{A}$ = circulation per unit Area

$$(\vec{\nabla}\times\vec{A})_u = (\vec{\nabla}\times\vec{A})\cdot\hat{a}_u = \lim_{\Delta S_u\to 0}\frac{1}{\Delta S_u}\oint_{C_u}\vec{A}\cdot d\vec{l}$$

$$\Rightarrow \oint_{C_u}\vec{A}\cdot d\vec{l} \sim (\vec{\nabla}\times\vec{A})\cdot\hat{a}_u\,\Delta S_u = (\vec{\nabla}\times\vec{A})\cdot\Delta\vec{S}_u$$

$$\Rightarrow \sum_i\oint_{C_i}\vec{A}\cdot d\vec{l} = \sum_i(\vec{\nabla}\times\vec{A})\cdot\underbrace{\Delta\vec{S}_i}_{\to 0}$$



$$\text{RHS} = \int_S(\vec{\nabla}\times\vec{A})\,d\vec{S}$$

$$\text{LHS} = \sum_i\sum_{j=1}^{4}\vec{A}\cdot d\vec{l}_j = \oint_C\vec{A}\cdot d\vec{l}$$

$d\vec{l}_{i_3} = -d\vec{l}_{i+1,}$

→互相抵消

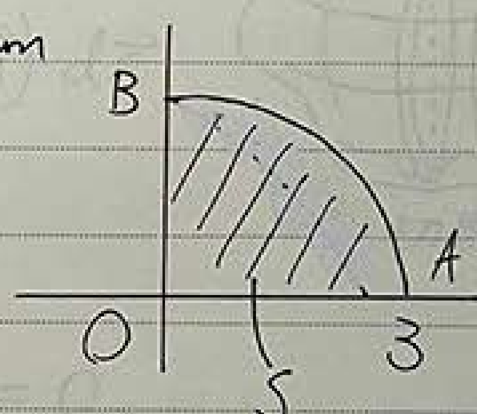
→只要在内部都会被抵消

$$\Rightarrow \oint_C\vec{A}\cdot d\vec{l} = \int_S\vec{\nabla}\times\vec{A}\,dS$$

(Stoke's Thm).

ex 2-22 驗證 stoke's Thm

$\vec{F} = \hat{a}_x xy - \hat{a}_y 2x$.



surface integral:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & -2x & 0\end{vmatrix}$$

$$= \dots = -\hat{a}_z(2+x)$$

$$\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \int(-\hat{a}_z)(2+x)\cdot\hat{a}_z\,dx\,dy$$

$$= \int_0^3\int_0^{\sqrt{9-y^2}} -(2+x)\,dx\,dy$$

Path Integral

$$\int_{0\to A\to B\to 0} \vec{A}\cdot d\vec{l} = \int_{0\to A} + \int_{B\to 0} + \int_{A\to B}$$

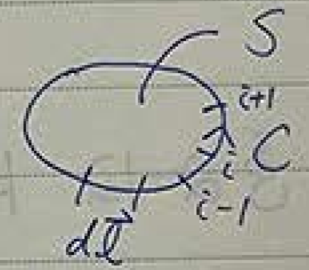
$\because y=0$ $\therefore$ 為 0 ; $\because x=0$ $\therefore$ 為 0

$$= \int_A^B \vec{F}\cdot d\vec{l} \underset{ex2\text{-}14}{=} -9\left(1+\frac{\pi}{2}\right)$$

#

§ 2.11 Null Identity.

1. $\vec{\nabla}\times\vec{\nabla}V \equiv 0 \ , \ \forall V.$



pf: $\displaystyle\int_S \vec{\nabla}\times(\vec{\nabla}V)\, d\vec{s} \overset{\text{stoke's Thm}}{=} \oint_C (\vec{\nabla}V)\cdot d\vec{\ell} = \int dV$

$= \sum (V_{i+1} - V_i) = 0.$

$\because$ C and S are arbitrary

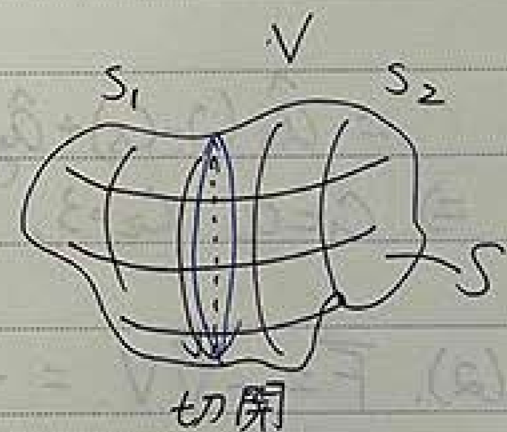
$\therefore \vec{\nabla}\times\vec{\nabla}V = 0.$

Q.E.D.

∗ 若 $\vec{\nabla}\times\vec{E}=0$，則 $\exists V$ s.t. $\nabla V = \vec{E}$

若 $\vec{E}$ 為 static electric field，則 $\exists V$ s.t. $\vec{E} = -\vec{\nabla}V$，V 稱為 electric potential.

2. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0$



pf: $\displaystyle\int_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\, dv \overset{\text{div Thm}}{=} \oint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s}$

$\displaystyle\overset{\text{切一刀}}{=} \left(\int_{S_1} + \int_{S_2}\right)(\vec{\nabla}\times\vec{A})\cdot d\vec{s}$

$\displaystyle\overset{\text{stoke Thm.}}{=} \oint_{C_1}\vec{A}\cdot d\vec{\ell} + \oint_{C_2}\vec{A}\cdot d\vec{\ell} = 0.$

$\Rightarrow \int \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\, dv \equiv 0$

$\because$ V is arbitrary

$\therefore \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0.$

Q.E.D.

* 若 $\nabla \cdot \vec{B} = 0$，則 $\exists \vec{A}$ s.t. $\vec{B} = \nabla \times \vec{A}$

magnetic field　　magnatic vector potential

§ 2-12 Helmholtz Thm.

$\nabla \cdot \vec{F} \leftrightarrow$ flow source

$\nabla \times \vec{F} \leftrightarrow$ vortex source

ex. $\vec{F} = \hat{a}_x(3y - C_1 z) + \hat{a}_y(C_2 x - 2z) - \hat{a}_z(C_3 y + z)$,

(1) 若 $\nabla \times \vec{F} = 0$，求 $C_1, C_2, C_3 = ?$

(2) 求 V s.t. $-\nabla V = \vec{F}$.

-sol-. (1). $\nabla \times \vec{F} = 0 = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y - C_1 z & C_2 x - 2z & -C_3 y - z \end{vmatrix}$

$= \hat{a}_x(2 - C_3) + \hat{a}_y(-C_1) + \hat{a}_z(C_2 - 3)$

$\Rightarrow C_1 = 0,\ C_2 = 3,\ C_3 = 2.$

(2). $\vec{F} = -\nabla V. = -\hat{a}_x \frac{\partial V}{\partial x} - \hat{a}_y \frac{\partial V}{\partial y} - \hat{a}_z \frac{\partial V}{\partial z} =$

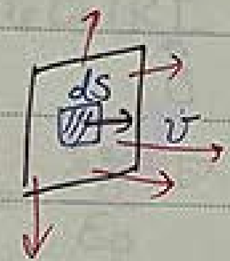
$$\Rightarrow \begin{cases} \frac{\partial V}{\partial x} = -3y \\ \frac{\partial V}{\partial y} = -3x + 2z \\ \frac{\partial V}{\partial z} = 2y + z \end{cases} \Rightarrow \begin{cases} V = -3xy + f_1(y, z) \\ V = -3xy + 2yz + f_2(x, z) \\ V = 2yz + \frac{z^2}{2} + f_3(x, y) \end{cases}$$

Compare $\Rightarrow$ $V(x, y, z) = -3xy + 2yz + \frac{z^2}{2} + k$ (k is const).

Ch3. 静电場

*前言

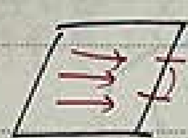
電荷密度 $\rho(\vec{r})=\dfrac{dq}{dV}$

current: 

$I=\dfrac{dq}{dt}$ (A, C/s)

体積 current density $\vec{J}=\lim\limits_{\Delta s\to 0}\dfrac{\Delta I}{\Delta s}\hat{a}_{\Delta s}$ (A/m^2)

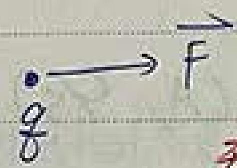
(表面)电流密度 $\vec{J}_s=\lim\limits_{\Delta l\to 0}\dfrac{\Delta I}{\Delta l}\hat{a}_v$



取与载子運动方向垂直的一小段長度

§3.2. 真空中静电学的基本假說.

1. 先不考慮 E 是怎麼来的，先当成空間中一特性.



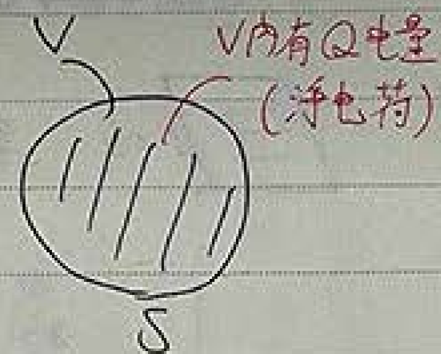
$$\vec{E} = \lim_{q \to 0} \frac{F}{q}$$

希望測試电荷对电場不要造成影响.

2. Differential form

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0 & (3.4) \\ \vec{\nabla} \times \vec{E} = 0 & (3.5) \end{cases}$$

(a) Integral form of (3.4)

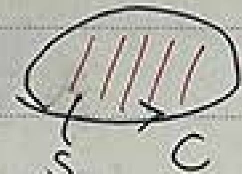


$$\int_V (\vec{\nabla} \cdot \vec{E}) \, dv = \int_V \frac{\rho}{\varepsilon_0} dv$$

$$\overset{\text{div. Thm}}{\Longrightarrow} \oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0} \quad \text{(Gauss law)}$$

(b) " (3.5)

$$\int_S (\vec{\nabla} \times \vec{E}) \, d\vec{s} = 0 \xrightarrow[\text{stoke's Thm.}]{} \oint_C \vec{E} \cdot d\vec{l} = 0 \Rightarrow \vec{E} \text{為保守場}$$

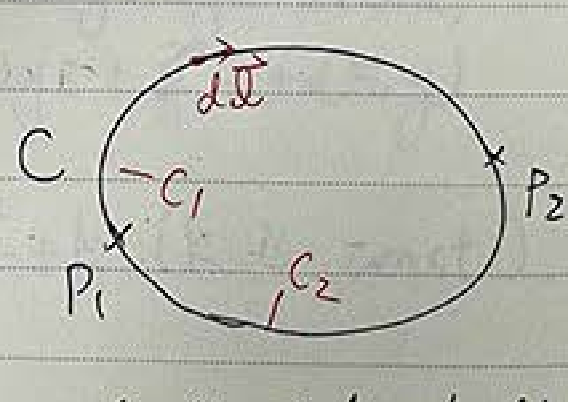


(Kirchhoff's voltage law)

3.
$$\oint_C \vec{E} \cdot d\vec{l} = \left(\int_{P_1}^{P_2} + \int_{P_2}^{P_1}\right) \vec{E} \cdot d\vec{l}$$

$$= \left(\int_{P_1}^{P_2} - \int_{P_1}^{P_2}\right) \vec{E} \cdot d\vec{l} = 0$$

$$\Rightarrow \int_{C_1} \vec{E} \cdot d\vec{l} = \int_{C_2} \vec{E} \cdot d\vec{l}$$

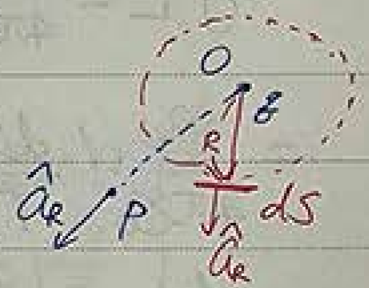


(path-independent)

⇒ 靜电場对單位电荷做的功对不同路徑是一樣的！

§3-3 Columb's law

1. Spherical symmetry ⇒ 點电荷形成电場，各方向应对称、
   ⇒ 可令 $\vec{E} = \hat{a}_R E_R(R)$ → 不知 $E_R(R)$ 是誰？
   → 試求 E_R.

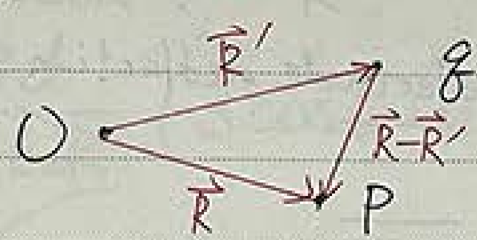


$$\oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \oint_S \hat{a}_R E_R(R) \cdot (\hat{a}_R\, ds) = E_R(R) \cdot 4\pi R^2 = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \vec{E} = \hat{a}_R E_R(R) = \hat{a}_R \frac{q}{4\pi\varepsilon_0 R^2} = \vec{R}\frac{q}{4\pi\varepsilon_0 R^3}$$

2. 若 q 不在原點，如何表示 $\vec{E}$ 呢？

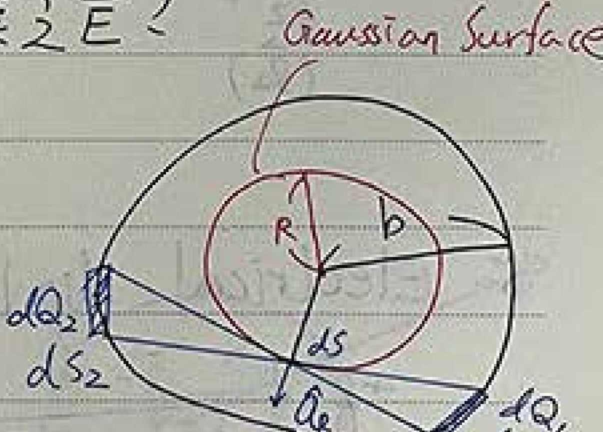


(source 之位置以 $\vec{R}'$ 表示)

$$\rightarrow \vec{E}\Big|_P = (\vec{R}-\vec{R}')\frac{Q}{4\pi\varepsilon_0 |\vec{R}-\vec{R}'|}$$

3. $\vec{F}_{12} = q_2 \cdot \vec{E}_{12} = \hat{a}_R \dfrac{q_1 q_2}{4\pi\varepsilon_0 \underline{R_{12}^2}}$ (Columb's law)
   由 q_1 指向 q_2 之方向.

ex. 球殼上帶电量 Q，求內部一點之 $\vec{E}$？



-sol- 〈法 I〉.

1. From spherical symmetry，可
   令 $\vec{E} = \hat{a}_R E_R$

$$\oint_S \vec{E} \cdot d\vec{S} = E_R(R) \cdot 4\pi R^2 = \frac{0}{\varepsilon_0}$$

$$\Rightarrow E_R = 0.$$

〈法Ⅱ〉. $\because \rho_s = \dfrac{Q}{4\pi b^2}$

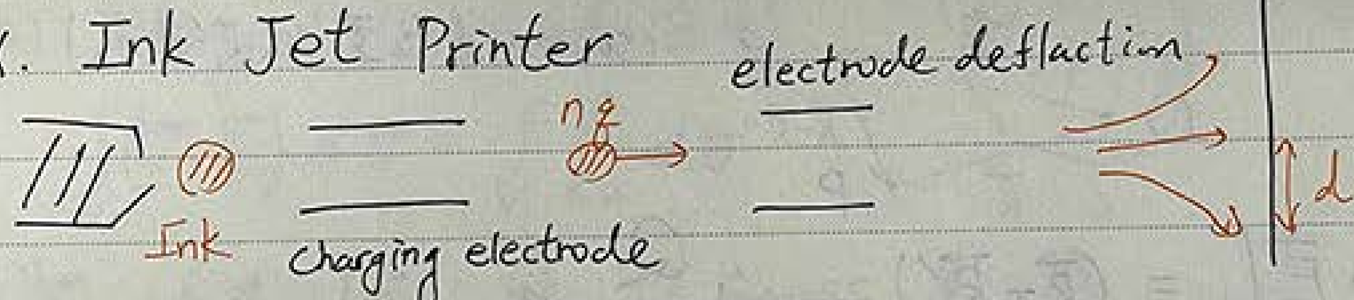
$\therefore \quad dE = \dfrac{1}{4\pi\varepsilon_0}\dfrac{dq}{r^2} \Rightarrow dE_1 + dE_2 = \left(\dfrac{\rho_s dS_1}{r_1^2} - \dfrac{\rho_s dS_2}{r_2^2}\right)$

$= \dfrac{\rho_s}{4\pi\varepsilon_0}\left(\dfrac{dS_1}{r_1^2} - \dfrac{dS_2}{r_2^2}\right)$ —(3-21)

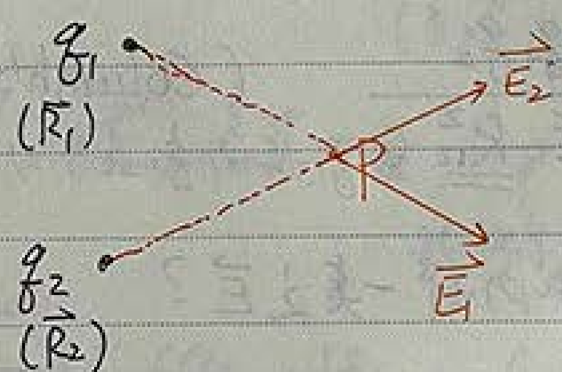
$\because \dfrac{dS_1}{dS_2} = \dfrac{r_1^2}{r_2^2}$ —(3-20')

$\therefore \quad E = \int dE = 0$ #

ex. Ink Jet Printer

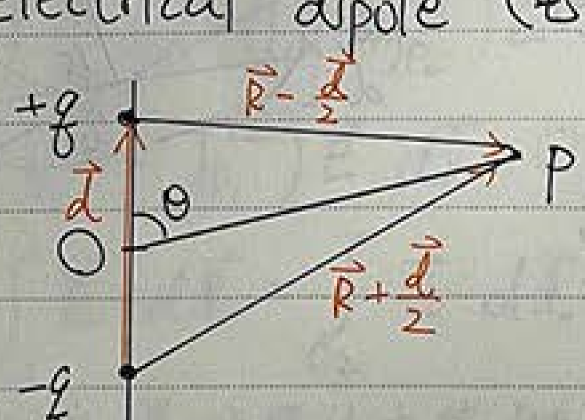


4.



$\vec{E}_k = \dfrac{q_k}{4\pi\varepsilon_0}\dfrac{\vec{R} - \vec{R}_k'}{|\vec{R} - \vec{R}_k|^3}$

5. Electrical dipole (电偶極)($|\vec{R}| >> d$)



求P点之电場 $\vec{E}$?

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{R}-\frac{\vec{d}}{2}}{R_+^3} - \frac{\vec{R}+\frac{\vec{d}}{2}}{R_-^3}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[(\vec{R}-\frac{\vec{d}}{2})(R^2+(\frac{d}{2})^2-2R\cdot\frac{d}{2}\cos\theta)^{-\frac{3}{2}}\right.$$
$$\left.-(\vec{R}+\frac{\vec{d}}{2})\cdot(R^2+(\frac{d}{2})^2+2R\cdot\frac{d}{2}\cos\theta)^{-\frac{3}{2}}\right]$$

提出 $\frac{1}{R^3}$, Taylar expansion.

$$= \frac{q}{4\pi\varepsilon_0 R^3}\left((\vec{R}-\frac{\vec{d}}{2})(1+\frac{3}{2}\frac{d\cos\theta}{R}) - (\vec{R}+\frac{\vec{d}}{2})(1-\frac{3}{2}\frac{d\cos\theta}{R})\right)$$

$$= \frac{q}{4\pi\varepsilon_0 R^3}\left(\frac{3d\cos\theta}{R}\vec{R} - \vec{d}\right)$$

dipole之 $\vec{E}$ 大小与R的3次成反比!

$$= \frac{qd}{4\pi\varepsilon_0 R^3}(3\cos\theta\,\hat{a}_R - \hat{a}_z)$$

Define dipole moment (偶極矩) $\vec{p} = q\vec{d} = \hat{a}_z q d$.

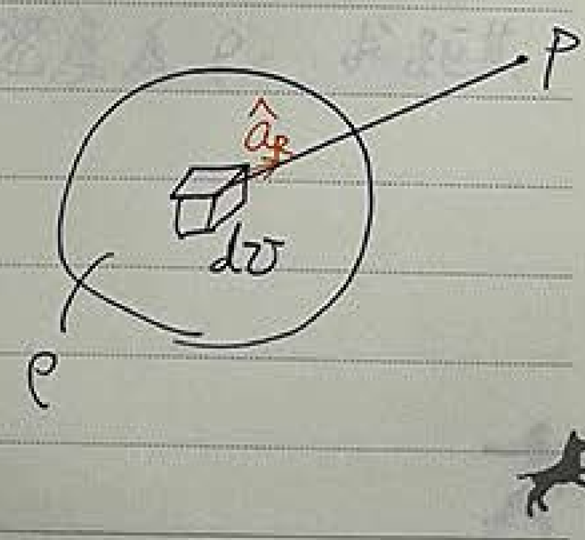
$$\Rightarrow \vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}(3\cos\theta\,\hat{a}_R - \hat{a}_z)$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}(\hat{a}_R 2\cos\theta + a_\theta\sin\theta)$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}\left(\frac{3\vec{R}\cdot\vec{p}}{R^2}\vec{R} - \vec{p}\right) \#$$

6. 求P点电場 $\vec{E}$

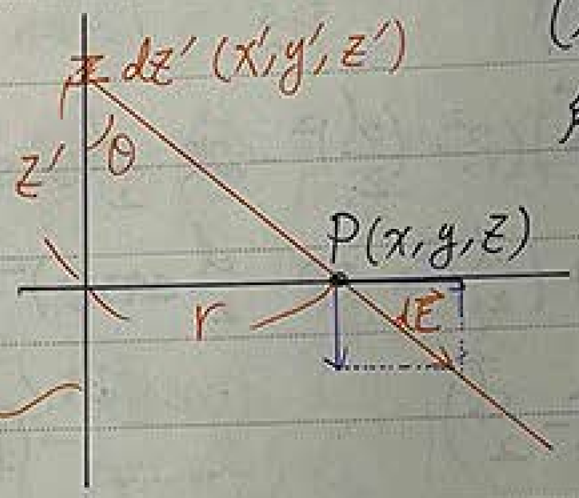
$$\Rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dv}{R^2}\cdot\hat{a}_R$$



$$\Rightarrow \vec{E} = \int d\vec{E} = \cdots$$

ex 3-4. 導線∞長，电荷密度$=\rho_l$，求距導線r處之电場?

(有加" ' "表source所在之位置)



-sol-

$$dE_r = dE\sin\theta$$

$$= \frac{dq}{4\pi\varepsilon_0 R^2}\cdot\frac{r}{R}$$

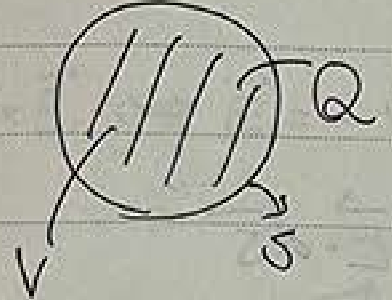
$$= \frac{\rho_l dz'}{4\pi\varepsilon_0}\cdot\frac{r}{(r^2+z'^2)^{\frac{3}{2}}}$$

$$\Rightarrow \vec{E} = \left(\int dE_r\right)\hat{a}_r$$

$$= \hat{a}_r\frac{\rho_l r}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dz'}{(r^2+z'^2)^{\frac{3}{2}}}$$

$$= \hat{a}_r\frac{\rho_l}{2\pi\varepsilon_0 r}$$ #

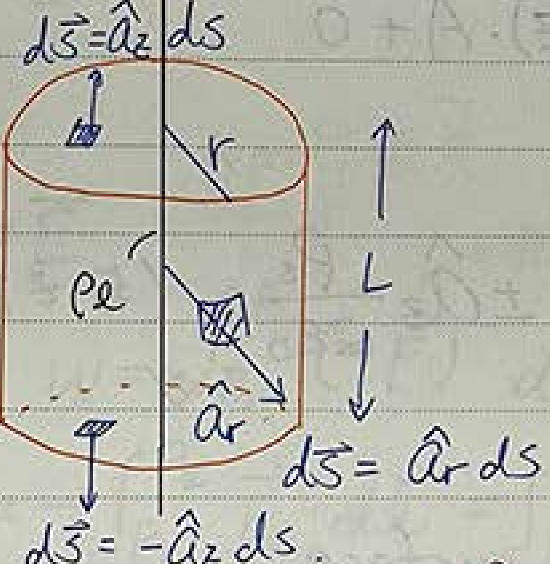
§3-4. Gauss' law



$$\oint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

ex 3-5. 導線∞長，电荷密度ρ_ℓ，求距導線r處之电場？

-sol-. 取一园柱 Gauss surface 如右



$$\oint_S \vec{E}\cdot d\vec{s} = \Big(\int_{side} + \cancel{\int_{top}}^{\,0} + \cancel{\int_{bottom}}^{\,0}\Big)\,\vec{E}\cdot d\vec{s}$$

$$= \int_{side} E_r(r)\,ds$$

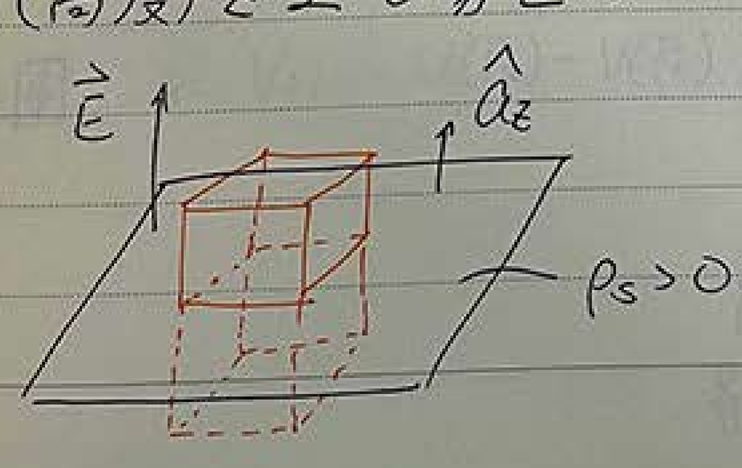
$$= E_r(r)\cdot 2\pi r L$$

By Gauss Law, $E_r(r)\,2\pi r L = \dfrac{\rho_\ell \cdot L}{\varepsilon_0}$

$$\Rightarrow \vec{E} = \hat{a}_r E_r(r) = \hat{a}_r \frac{\rho_\ell}{2\pi\varepsilon_0 r}$$

\#

ex. 3-6. 一∞大之平板其上电荷密度為ρ_s，求距其(高度)z之电場$\vec{E}$？



—sol—

$\vec{E} = \pm \hat{a}_z E_z |z|, \ z \neq 0.$

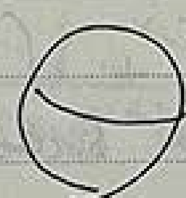
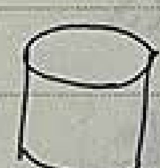
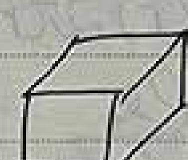
$$\oint_S \vec{E} \cdot d\vec{s} = \left(\int_{top} + \int_{bottom} + \int_{side} \right) \vec{E} \cdot d\vec{s}$$

$$= E_z(z) \cdot A + E_z(z) \cdot A + 0$$

$$= \frac{Q}{\varepsilon_0} = \frac{A\rho_s}{\varepsilon_0}$$

$$\Rightarrow E_z(z) = \frac{\rho_s}{2\varepsilon_0} \ \Rightarrow \vec{E} = \pm \hat{a}_z \frac{\rho_s}{2\varepsilon_0}$$ (与z無关!) #

△结論:

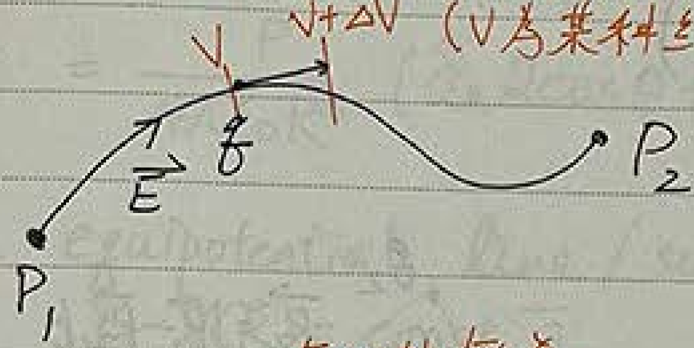
type	Gauss surface	$\propto$
点电荷		$\frac{1}{r^2}$
線"		$\frac{1}{r}$
面"		1

→ 若某种电荷分佈有对称性 and (面積好算) 就用 Gauss law

使 $\oint \vec{E} \cdot d\vec{s} = \square \int ds$

§3.5 Electric Potential.

1. $\vec\nabla\times\vec E \Longleftrightarrow \vec E=-\vec\nabla V$

2. 

(V為某种純量)

與path無关

抵抗$\vec E$ field

$$\frac{\vec F}{q}=\vec E$$

作功 $W=\int_{P_1}^{P_2}(-\vec F)\cdot d\vec l = -q\int_{P_1}^{P_2}\vec E\cdot d\vec l$ (2-88)

$$\Rightarrow \frac{W}{q} = -\int_{P_1}^{P_2}\vec E\cdot d\vec l = \int_{P_1}^{P_2}\vec\nabla V\cdot d\vec l \overset{\downarrow}{=} \int_{P_1}^{P_2} dV$$

$$= \underbrace{V|_{P_2} - V|_{P_1}}_{\text{電位差=單位電荷具有之位能}} \left(= V(\vec r)-V(\vec r_0)\right)$$ → 參考點

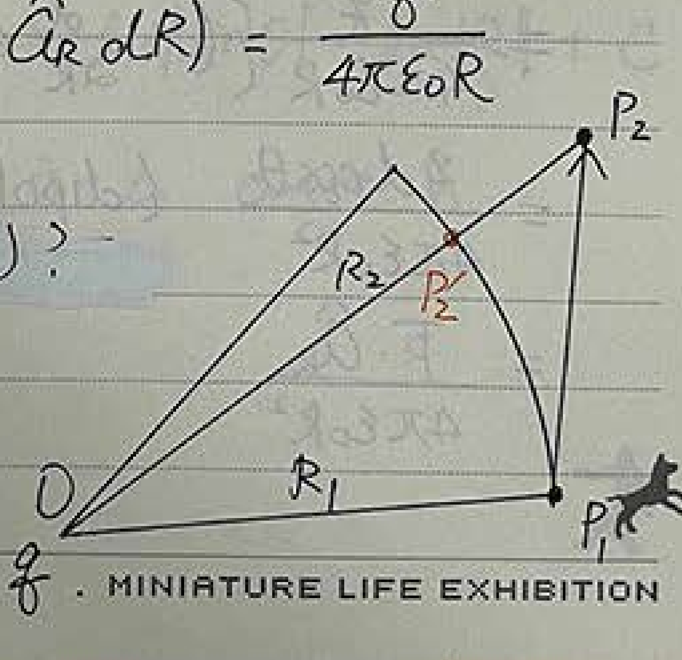
3. Point Charge 造成之電位

$$V(P_2) = -\int_{P_1}^{P_2}\vec E\cdot d\vec l = -\int_{\infty}^{P_2}\hat a_R\frac{q}{4\pi\varepsilon_0 R^2}\,d\vec l$$

可取 $\vec R \parallel \hat a_R \parallel d\vec l$

$$= -\int_{\infty}^{R}\hat a_R\frac{q}{4\pi\varepsilon_0 R^2}\left(-\hat a_R\,dR\right) = \frac{q}{4\pi\varepsilon_0 R}$$

4. 如圖，求 $V_{21}=V(P_2)-V(P_1)$？

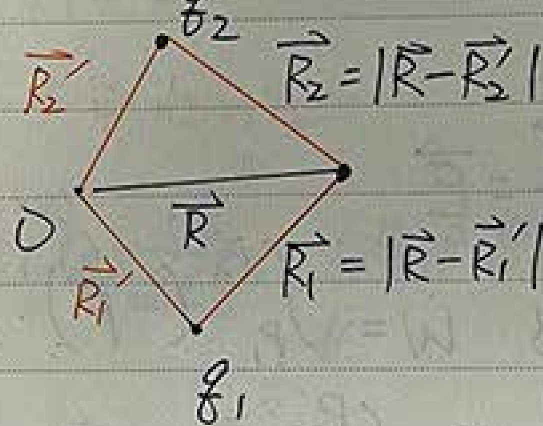


$$V_{21} = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} \overset{\text{path indep.}}{=} -\left(\cancelto{0}{\int_{P_1}^{P_2'}} + \int_{P_2'}^{P}\right)$$

$$= V(P_2) - V(P_2')$$

$$= \frac{q}{4\pi\varepsilon}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

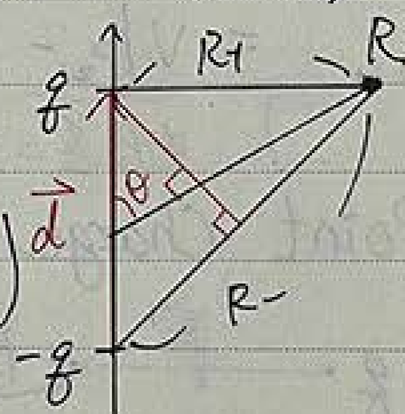
5. 求 $V(\vec{R})$?



$$V(\vec{R}) = \underbrace{V_1(\vec{R})}_{q_1\text{所造成之电位}} + \underbrace{V_2(\vec{R})}_{q_2\text{''}}$$

$$= \frac{q_1}{4\pi\varepsilon_0 R_1} + \frac{q_2}{4\pi\varepsilon_0 R_2}$$

6. Dipole (1) 求 $V(\vec{R})$, (2) 求 $E(\vec{R})$. $(d \ll R)$



(1) $$V = \frac{q}{4\pi\varepsilon_0}\frac{1}{R_+} + \frac{-q}{4\pi\varepsilon_0}\frac{1}{R_-}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R - \frac{d}{2}\cos\theta} - \frac{1}{R + \frac{d}{2}\cos\theta}\right)$$

$(\because$ for $R \gg d$, $R_+ \doteqdot R - \frac{d}{2}\cos\theta$,

$R_- \doteqdot R + \frac{d}{2}\cos\theta)$

$$= \frac{q}{4\pi\varepsilon_0 R}\left[\left(1 - \frac{d}{2R}\cos\theta\right)^{-1} - \left(1 + \frac{d}{2R}\cos\theta\right)^{-1}\right]$$

$$\doteqdot \frac{q}{4\pi\varepsilon_0 R}\left[\left(1 + \frac{d}{2R}\cos\theta\right) - \left(1 - \frac{d}{2R}\cos\theta\right)\right]$$

$$= \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} \quad (\text{dipole moment } \vec{p} = q\vec{d})$$

$$= \frac{\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}$$

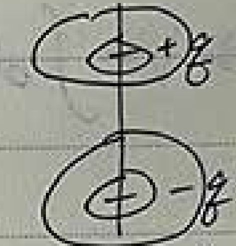
(2). $\vec{E} = \vec{\nabla} V$

$$= -\left(\hat{a}_R \frac{\partial V}{\partial R} + \hat{a}_\theta \frac{\partial V}{R\partial\theta} + \hat{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\right)$$

$$= \cdots = \frac{P}{4\pi\varepsilon_0 R^3}\left(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta\right)$$

(3). equipotential line / surface : find $\vec{R}(R,\theta,\phi)$

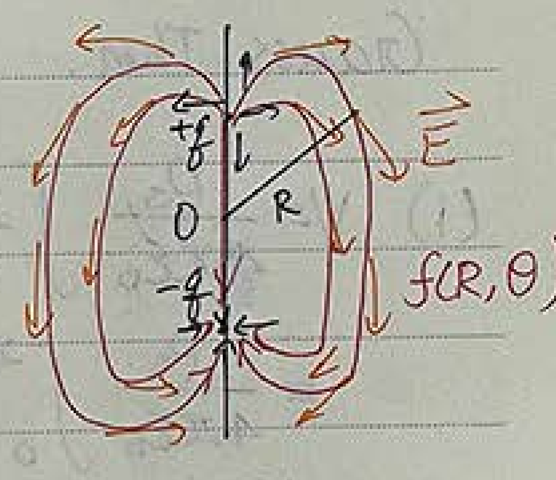
s.t. $V(\vec{R}) = \text{const.}$



令 $V = \dfrac{q d\cos\theta}{4\pi\varepsilon_0 R^2} = \text{const}$

$\Rightarrow \dfrac{\cos\theta}{R^2} = \text{const} \triangleq C_V \Rightarrow R = C_V\sqrt{\cos\theta}$ 即為等位面方程式

(4) $\vec{E}$ field line : $d\vec{l} \to 0$ 時, 即為該点 $\vec{E}$ 之切線方向 (亦為該点 $\vec{E}$ 之方向). $\Rightarrow d\vec{l} \parallel \vec{E}$



$\because d\vec{l} = \hat{a}_R dR + \hat{a}_\theta (R d\theta)$ and

$\vec{E} \propto \hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta$

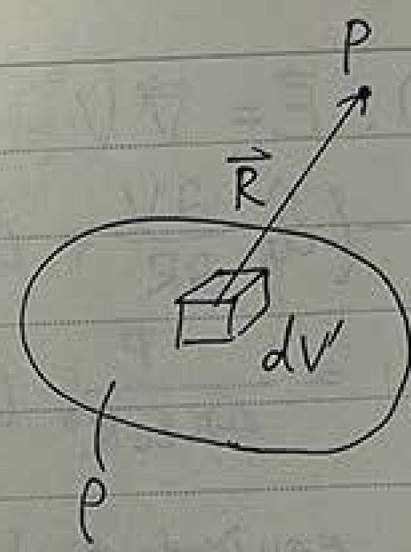
$\therefore$ 对应座標成比例,

$$\frac{2\cos\theta}{dR} = \frac{\sin\theta}{R d\theta}$$

$$\Rightarrow \int \frac{2\cos\theta}{\sin\theta} d\theta = \int \frac{dR}{R} \Rightarrow \ln|\sin\theta| = \ln|R| + C$$

$\Rightarrow R = \sin^2\theta \cdot C_E$ (C_E is a const.)

7. Volume charge



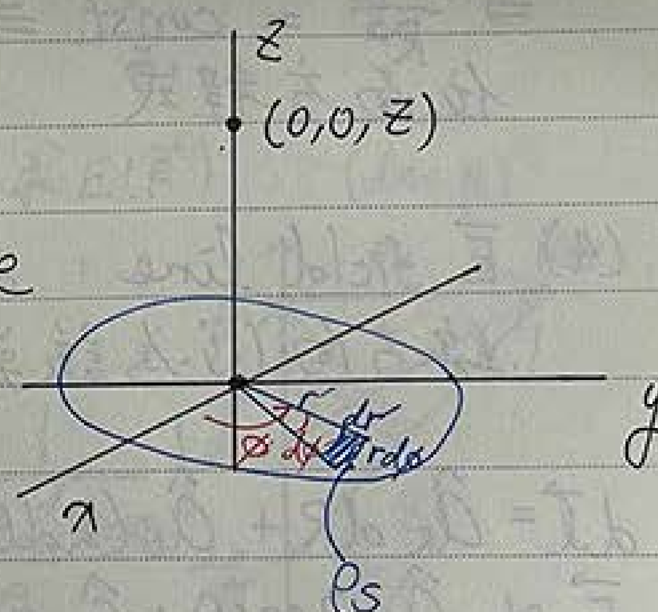
note that $dq = \rho\, dv'$

volume charge

$$dV = \frac{dq}{4\pi\varepsilon_0 R}$$

$$\Rightarrow V = \int dV = \int \frac{dq}{4\pi\varepsilon_0 R}$$

ex3-9. 求 $V(0,0,z)$?



-sol- ∵非∞大之板,不知 $|\vec{E}| = \text{const}$ 之 Gauss surface 如何选！故無法用 Gauss Thm.

(1) $$V = \frac{\rho_s}{4\pi\varepsilon_0} \int \frac{ds'}{R}$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{b} \frac{r'dr'd\phi'}{(z^2+(r')^2)^{\frac{1}{2}}}$$

$= \cdots$

$$= \frac{\rho_s}{2\varepsilon_0} \left(\sqrt{z^2+b^2} - |z| \right)$$

(2) 由 $\vec{E} = \vec{\nabla} V$ 求 z 軸上 $\vec{E}$

$$\Rightarrow \vec{E}_z = (-\vec{\nabla} V)_z = -\hat{a}_z \frac{\partial V}{\partial z}$$

$$= \hat{a}_z \frac{\rho_s}{2\varepsilon_0} \left(1 \pm 1 - z(z^2+b^2)^{-\frac{1}{2}} \right), \text{ for } z \neq 0.$$

#

note: ① 当 $z\to\infty$ (远離圆盤), V, $\vec{E}$ 就如同是
由 pt charge 造成的一樣.
② 接下来"物体"才要出現 < 導体
绝缘体

§3.6 Conductor.

1. Inside conductor:

If $Q, \rho \neq 0 \Rightarrow \vec{E} \neq 0$

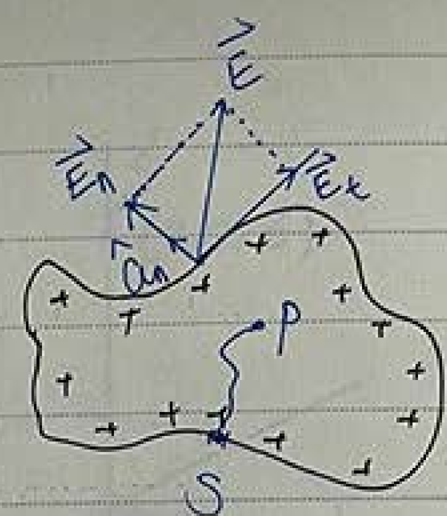
$\Rightarrow \vec{J}, I \neq 0$ ($\vec{E}$造成 charge移动)

$\Rightarrow$ non-static until charge redistribute到表面

$\Rightarrow$ 直到 $\vec{E}=0$, $Q, \rho=0$.

$\Rightarrow$ 導体内部 $\begin{cases} \rho=0 & \text{(符合 Gauss law!)} \\ \vec{E}=0 \end{cases}$

2. On conductor surface



If $E_t \neq 0 \Rightarrow$ surface current

$\Rightarrow$ non static

⋮

$\Rightarrow \begin{cases} \text{surface current} = 0 \\ E_t = 0 \end{cases}$

又 $\because \vec{E} = -\vec{\nabla} V$, $\therefore$ conductor 之表面為一等位面!

3. 求内部一点P之电位V?

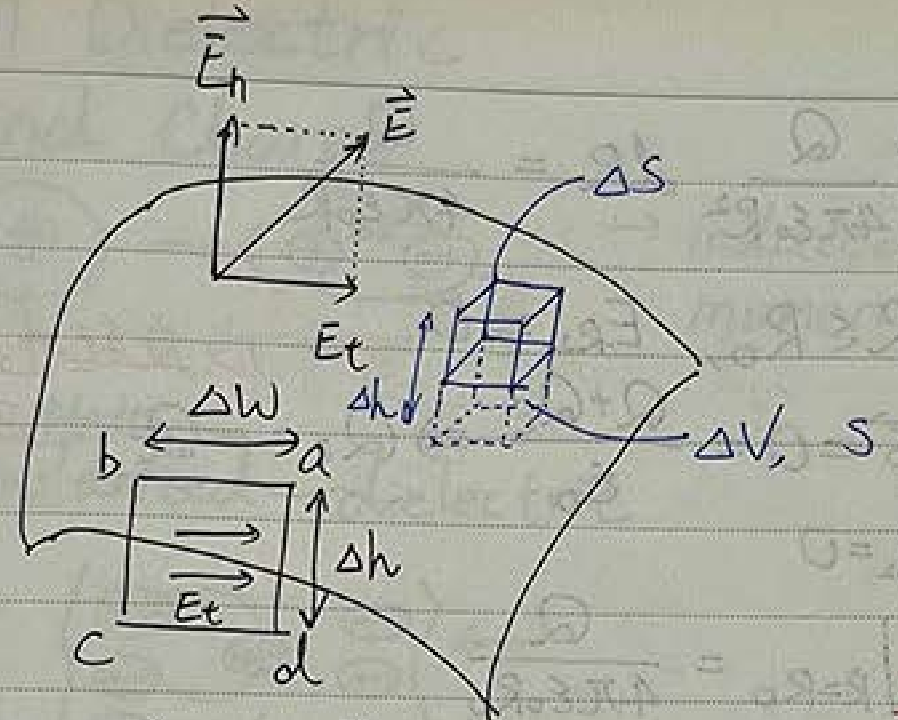
$\rightarrow V(P) - V(S) = -\int_S^P \overset{\text{内部}\vec{E}=0}{\vec{E}} \cdot d\vec{l} = 0. \Rightarrow V(P) = V(S)!$

4. Boundary Condition (真空&介質交界)

$\rightarrow$ 因為空間連續性不同，可能会不連續!

(a) Tangential $\Rightarrow$ surface上下之 E_t 一樣否?

(or, why $E_t|_{\text{surface}} = 0$?)



$$\because \oint_{abcda,\Delta h\to 0} \vec{E}\cdot d\vec{l} = E_t\Big|_{\text{surface}}^{(ab)} \cdot \Delta W + E_n\Big|_{bc} \cancel{\Delta h}^{\to 0} - \cancel{E_t}\Big|_{\text{inside}}^{0} \cdot \Delta W$$

$$- E_n\Big|_{da} \Delta h$$

$$= E_t\Big|_{\text{surface}} \Delta W = 0$$

$$\Rightarrow E_t\Big|_{\text{surface}} = 0$$

(b) Normal component

$$\oint_S \vec{E}\cdot d\vec{S} = \Big(\int_{\text{top}} + \cancel{\int_{\text{bottom}}}^{0} + \cancel{\int_{\text{side}}}^{0}\Big)\vec{E}\cdot d\vec{S}$$

$$= \int_{\text{top}} \vec{E}_n\, d\vec{S} = E_n \Delta S = \frac{\rho_s \Delta S}{\varepsilon_0}$$

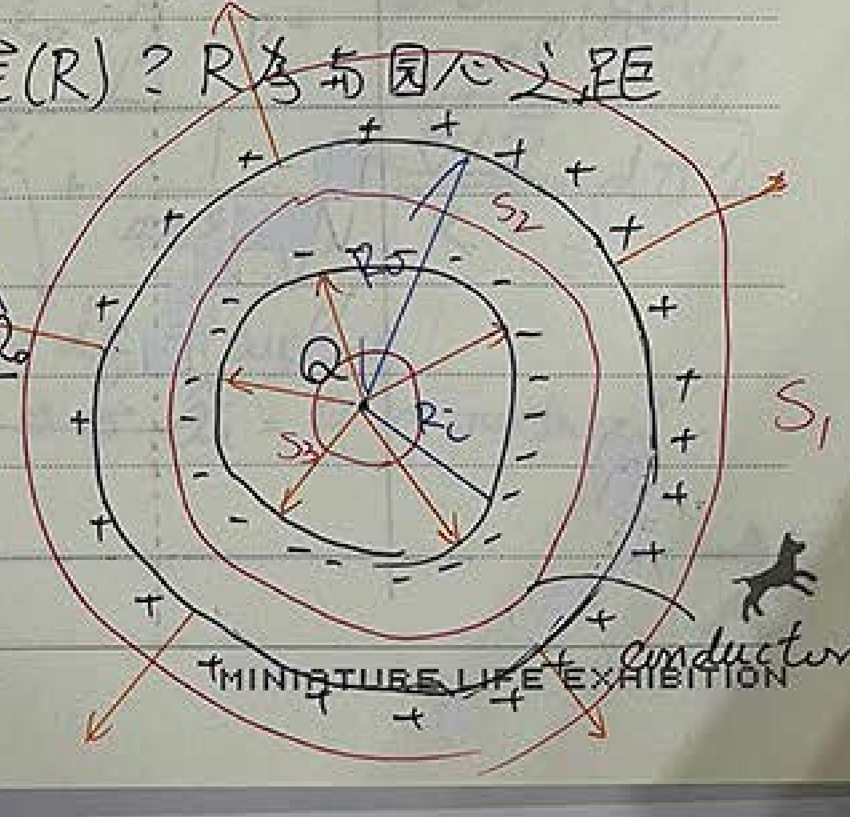
$$\Rightarrow E_n = \frac{\rho_s}{\varepsilon_0} \quad \text{(on the surface)}$$

ex 3-11. 同心摆一Q，求$\vec{E}(R)$？R为与圆心之距

(a) $R > R_o$, Let $\vec{E} = \vec{E}_{R_1}$

$$\oint_{S_1} \vec{E}\cdot d\vec{S} = E_{R_1}\cdot 4\pi R^2 = \frac{Q+Q_i+Q_o}{\varepsilon_0}$$

$$\Rightarrow E_{R_1} = \frac{Q}{4\pi\varepsilon_0 R^2}$$ (如同 field due to pt charge)



$$V_{R_1} = -\int_{\infty}^{R} \frac{Q}{4\pi\varepsilon_0 R^2}\, dR = \frac{Q}{4\pi\varepsilon_0 R}$$

(b) $R_i \le R \le R_0$, $E_{R_2} = 0$.

感应起电 ⇒ 电量相同, 正负相反

$$\oint_{S_2} \underbrace{\vec{E}}_{\hookrightarrow E_{R_2}=0} \cdot d\vec{s} = 0 = \frac{Q+Q_i}{\varepsilon_0} \Rightarrow Q_i = -Q$$

$$V_{R_2} = V_{R_1}\Big|_{R=R_0} = \frac{Q}{4\pi\varepsilon_0 R_0}$$

(c) $R \le R_i$. Let $\vec{E} = \vec{E}_{R_3}$

$$\oint_{S_3} \vec{E}\cdot d\vec{s} = E_{R_3}\cdot 4\pi R^2 = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E_{R_3} = \frac{Q}{4\pi\varepsilon_0 R^2}$$

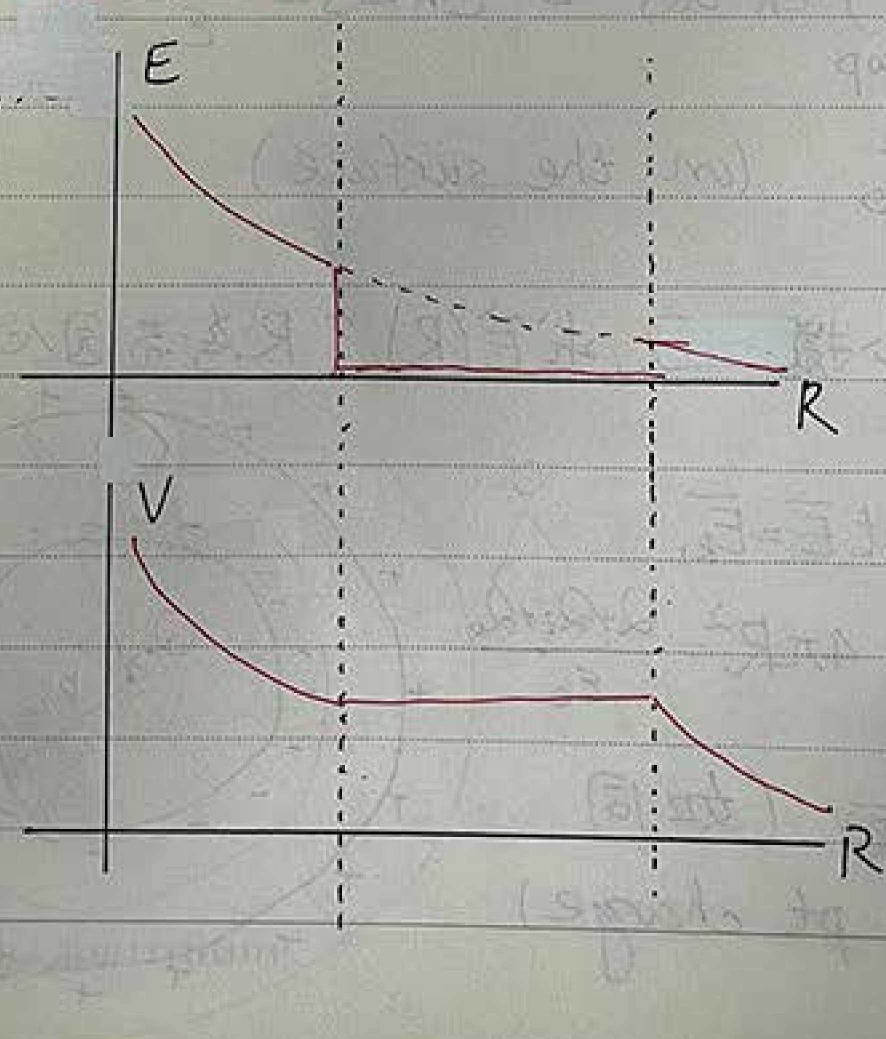
(∵ E 不连续, 分開算)

$$\Rightarrow V_3 = V_{R_2}\Big|_{R=R_i} - \int_{R_i}^{R} \frac{Q}{4\pi\varepsilon_0 R^2}\, dR$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R} + \frac{1}{R_0} - \frac{1}{R_i}\right)$$

以 R_i 为参考位

以∞遠處为参考位



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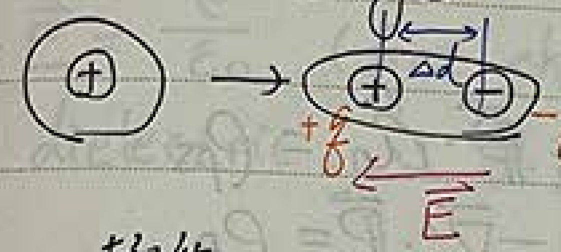
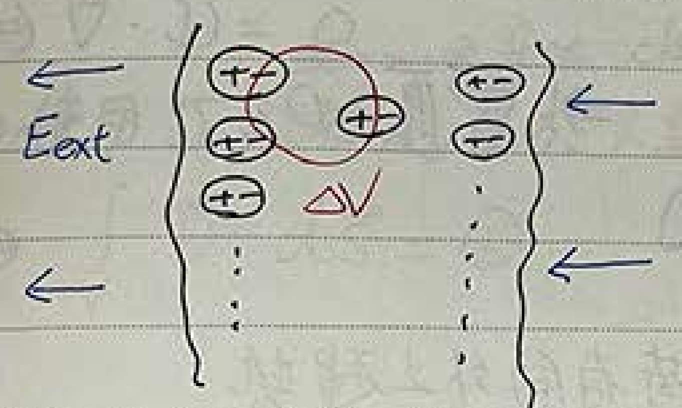
§3.7 Dielectric

1. Bound Charge

① 

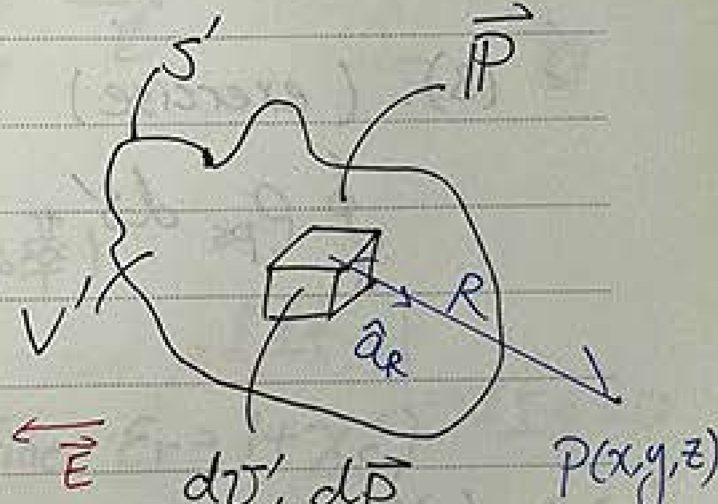
→ induced dipole moment $\vec{p} = q\Delta\vec{d}$

② polarized dielectric (極化)



define polarization vector $\vec{\mathbb{P}} = \lim\limits_{\Delta V\to 0}\dfrac{\sum_k \vec{p}_k}{\Delta V}$ (3-79)

〃 ⇒ dipole 之密度分佈



若 $\vec{\mathbb{P}}(x',y',z')$ 為已知，則

$$\begin{cases} d\vec{p} = \vec{\mathbb{P}}\, dv' \\ \underbrace{dV}_{\text{电位}} = \dfrac{d\vec{p}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2} \end{cases}$$

$$= \frac{\vec{\mathbb{P}}\cdot\hat{a}_R}{4\pi\varepsilon_0 R^2}\, dv'$$

$$\Rightarrow \text{Total } V = \int_{V'} dV = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\vec{\mathbb{P}}\cdot\hat{a}_R}{R^2}\, dv'$$

$$= \cdots = \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\vec{\mathbb{P}}\cdot d\vec{s}'}{R} + \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\overbrace{-\vec{\nabla}'\cdot\vec{\mathbb{P}}}^{\rho\,(\text{內部})}}{R}\, dv'$$

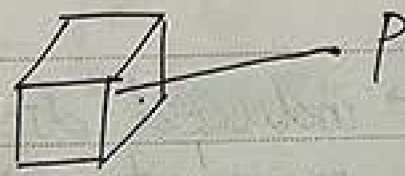
dq

= dq on surface!

⇒ $\vec{\mathbb{P}}\cdot\hat{a}_n = \rho$ on surface!

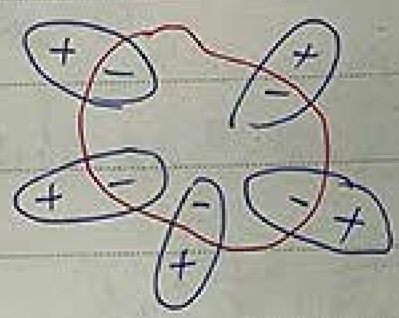
與电位之公式比較

(1)



$$dV=\frac{1}{4\pi\varepsilon_0}\frac{dq}{R} \Rightarrow \begin{cases}\vec{P}\cdot\hat{a}_n=\rho_{ps}\\ -\vec{\nabla}'\cdot\vec{P}=\rho_p\end{cases}$$

(2).



$\Rightarrow \vec{\nabla}\cdot\vec{P}>0$

→ $\vec{\nabla}'\cdot\vec{P}>0$ 表正电荷有向外之趨勢

→此時 $\rho_p<0$!

(3). (exercise)

$$\oint_S \rho_{ps}\,dS' + \int_{V'}\rho_p\,dv'=0$$

§3.8 电通密度

1. $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ (free charge density) ...

in dielectric, $\because \vec{\nabla}\cdot\vec{E} = \frac{\rho+\rho_p}{\varepsilon_0}$, $\rho_p = -\vec{\nabla}\cdot\vec{P}$

$\therefore \vec{\nabla}(\varepsilon_0\vec{E}+\vec{P}) = \rho$. We define $\vec{D} = \varepsilon_0\vec{E}+\vec{P}$

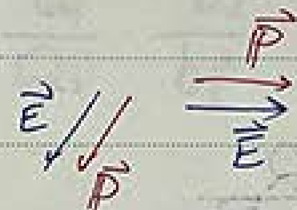
☆ $\Rightarrow \vec{\nabla}\cdot\vec{D} = \rho$ → free charge only!

→使用$\vec{D}$之好處在於只需考慮free charge density.

$\Leftrightarrow \int_V \vec{D}\cdot d\vec{s} = Q$

2. isotropic (等向性)

→ $\vec{E}$与$\vec{P}$恒同向,



故可寫 $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

const　与材料有关

χ_e = susceptibility (电極化率)

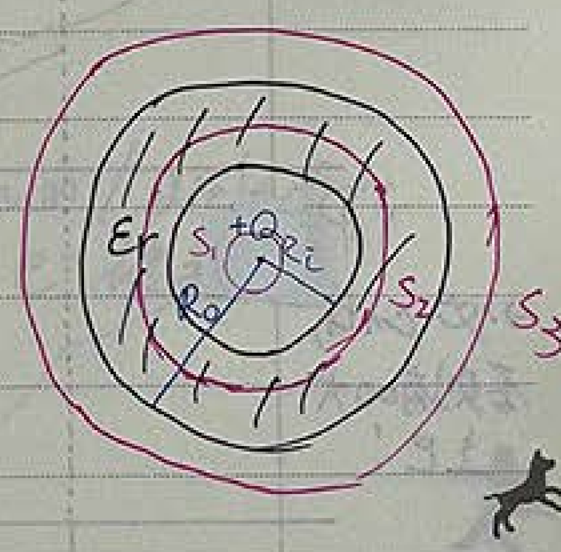
故 $\vec{D} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E}$

真空中=1

$= \varepsilon_0\varepsilon_r\vec{E}$　(define $\varepsilon_r = 1+\chi_e$) 真空中=0

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon} \Leftrightarrow$ $= \varepsilon\vec{E}$　(define $\varepsilon = \varepsilon_0(1+\chi_e) = \varepsilon_0\varepsilon_r$)

ex 3-12. 介电質球殼介电常数為 ε_r, 求 E、V、D、P 对R作图?



(a) $R > R_0$

$E = \frac{Q}{4\pi\varepsilon_0 R^2}$

$V_1 = \frac{Q}{4\pi\varepsilon_0 R}$

$$D_{R_1} = \varepsilon_0 E_{R_1} = \frac{Q}{4\pi R^2}$$

$$\mathbb{P}_{R_1} = \underbrace{\varepsilon_0 \chi_e}_{\text{在真空中}\to 0} E = 0 \quad (= D_R - \varepsilon_0 E_R)$$

(b) $R_i < R < R_o$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_r \varepsilon_0} \Rightarrow E_{R_2} = \frac{Q}{4\pi \varepsilon_r \varepsilon_0 R^2}$$

$$D_{R_2} = \varepsilon_0 \varepsilon_r E_{R_2} = \frac{Q}{4\pi R^2}$$

$$\mathbb{P}_{R_2} = D_{R_2} - \varepsilon_0 E_{R_2} = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\varepsilon_r}\right)$$

$$V_2(R) = V_1\big|_{R=R_o} - \int_{R_o}^{R} E_{R_2}\, dR$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} + \frac{1}{\varepsilon_r}\frac{1}{R}\right]$$

(c) $R < R_i$

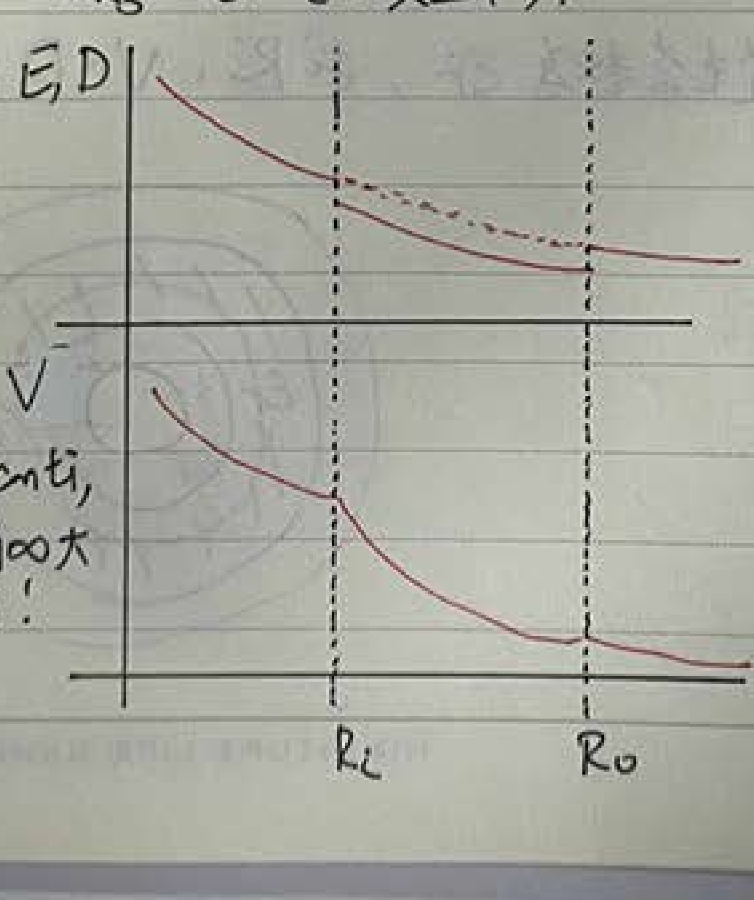
$$E_{R_3} = \frac{Q}{4\pi\varepsilon_0 R_3^2}$$

$$D_{R_3} = \varepsilon_0 \underbrace{\varepsilon_r}_{=1} E_{R_3} = \frac{Q}{4\pi R^2}$$

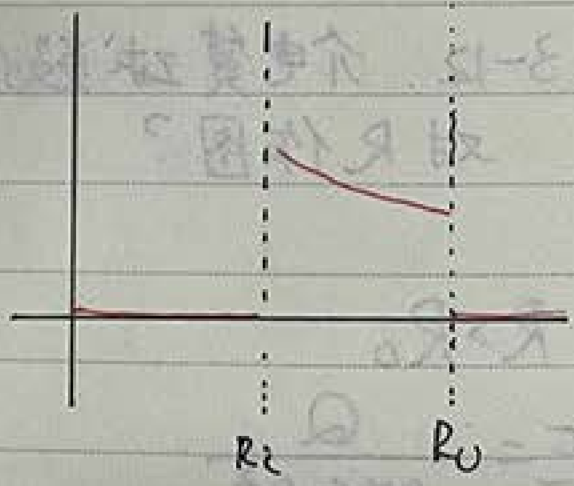
$$V_3(R) = V_2\big|_{R=R_i} - \int_{R_i}^{R} E_{R_3} \cdot dR$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\left(\frac{1}{R_o} - \frac{1}{R_i}\right) + \frac{1}{R}\right]$$

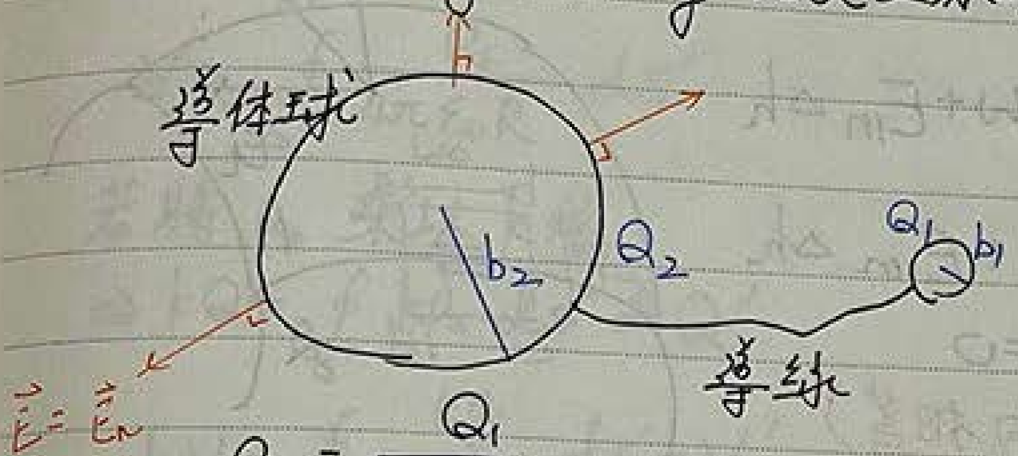
$$\mathbb{P}_{R_3} = 0 \quad (\because \text{真空中}).$$



→必 conti, 否则有∞大之E!



ex3-13 lightening rod 之原理.



$$\rho_{s1}=\frac{Q_1}{4\pi b_1} \qquad \rho_{s2}=\frac{Q_2}{4\pi b_2}$$

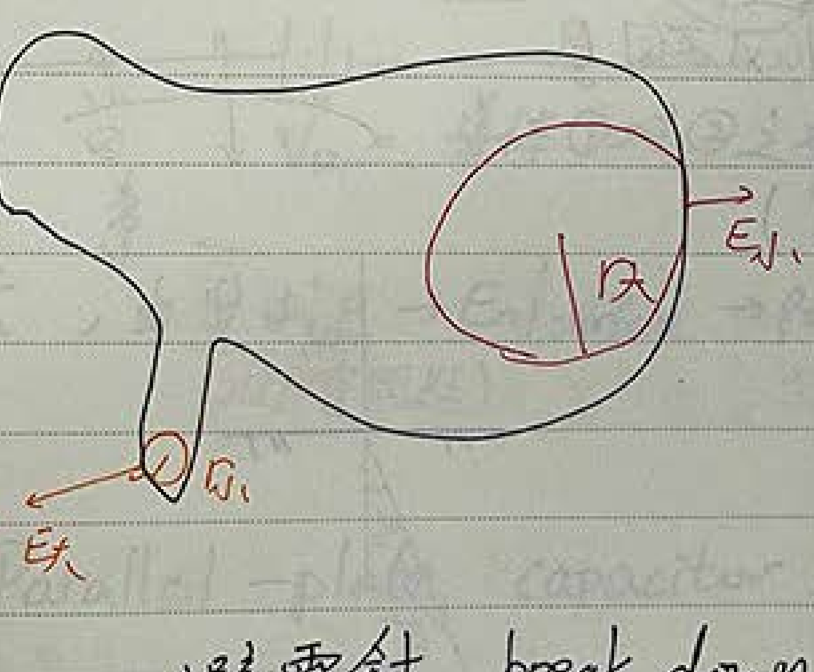
$\because V_1 = V_2$

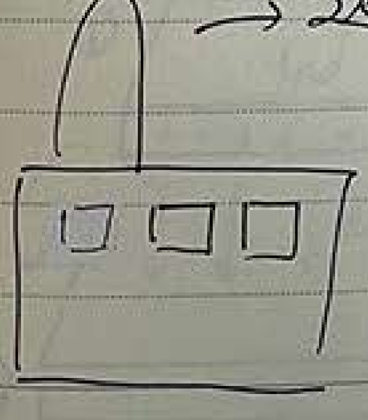
$$\therefore \frac{Q_1}{4\pi\varepsilon_0 b_1}=\frac{Q_2}{4\pi\varepsilon_0 b_2}$$

but $E_n \propto \frac{Q}{R^2} \Rightarrow \frac{E_1}{E_2}=\frac{Q_1}{Q_2}\cdot\frac{b_2^2}{b_1^2}$

$$\Rightarrow \frac{E_1}{E_2}=\frac{b_2}{b_1} \quad ※$$

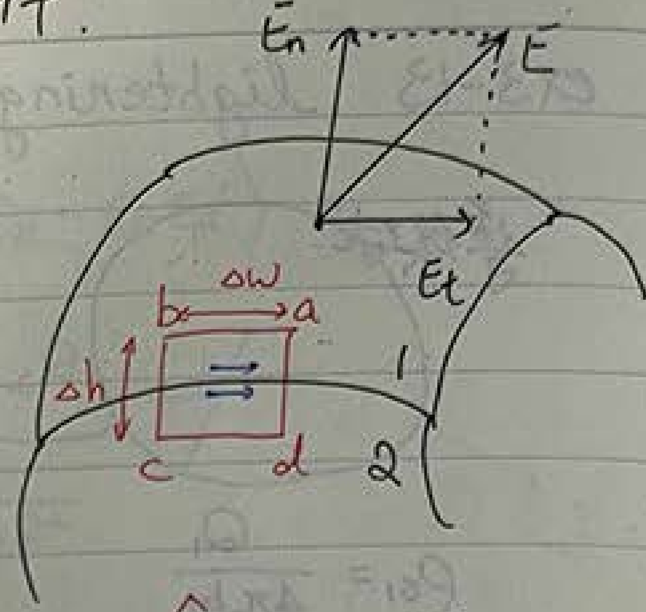
⇒ 曲率(半径)越大之處，E越大!





→ 避雷針 break down / dielectric strength ⇒ 束縛 e^- 被拉出.

§3-9 靜电場之边界條件.

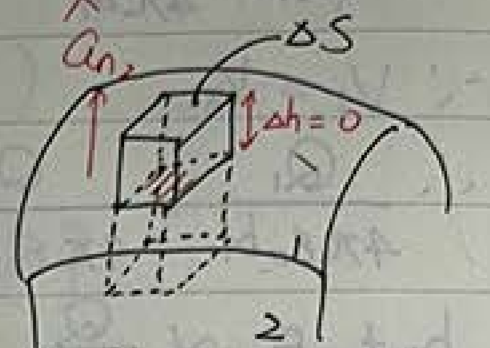


1. Tangent E_t

$$\oint_{\substack{abcda \\ \Delta h \to 0}} \vec{E} \cdot d\vec{l} = E_{1t}\Delta w + E_{1n}\Delta h - E_{2t}\Delta w + E_{1n}\Delta h$$

$$= (E_{1t} - E_{2t})\Delta w = 0$$

$\Rightarrow E_{1t} = E_{2t}$ (切向相等)



2. Normal E_n

$$\oint_{\substack{S \\ \Delta h \to 0}} \vec{D} \cdot d\vec{S} = Q = \rho \overset{\to 0}{\cancel{\Delta v}} + \rho_s \Delta S$$

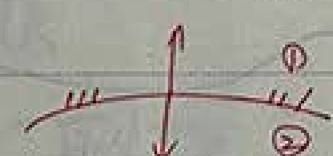
$$\Rightarrow \oint_S \vec{D} \cdot d\vec{S} = \vec{D}_1 \cdot \hat{a}_{n_2} \Delta S + \vec{D}_2 \cdot \hat{a}_{n_1} \cdot \Delta S \quad (\text{側} \to 0)$$

$$= \hat{a}_{n_2} (\vec{D}_1 - \vec{D}_2) \Delta S$$

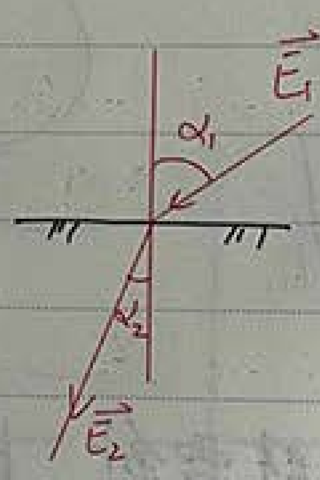
$$= \rho_s \Delta S$$

$\Rightarrow \underbrace{\vec{D}_{1n} - \vec{D}_{2n} = \rho_s}$ (or, $\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$)

向上/下之 flux 不同号



ex 3-15. (閱)



$$\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

§3-10. 电容.

1. $Q=\oint_{S'} \rho_s \, ds'$

$$V=\oint_{S'} \frac{\rho_s}{4\pi\varepsilon_0 R} \, ds'$$

若将 ρ_s 乘以 k倍

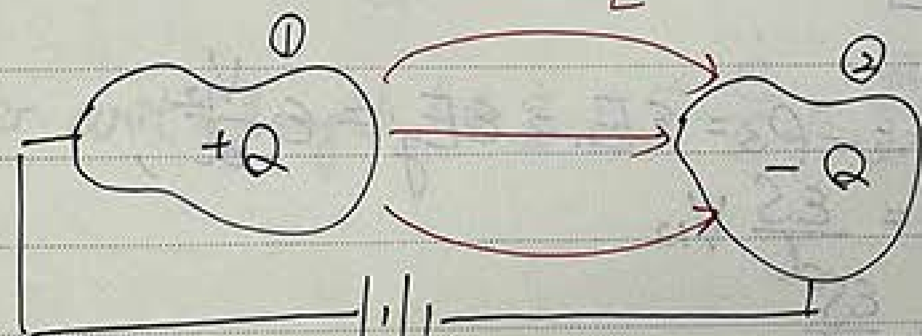
$\Rightarrow kQ = \oint_{S'} k\rho_s \, ds' \triangleq Q'$

$$kV = \oint_{S'} \frac{k\rho_s}{4\pi\varepsilon_0 R} \, ds' = V'$$

$\Rightarrow \dfrac{Q}{V} = \dfrac{Q'}{V'} = const \triangleq C$ (單位: F)

C 称电容, 表 增加一單位potential 所需之charge

2. 導体間之C.



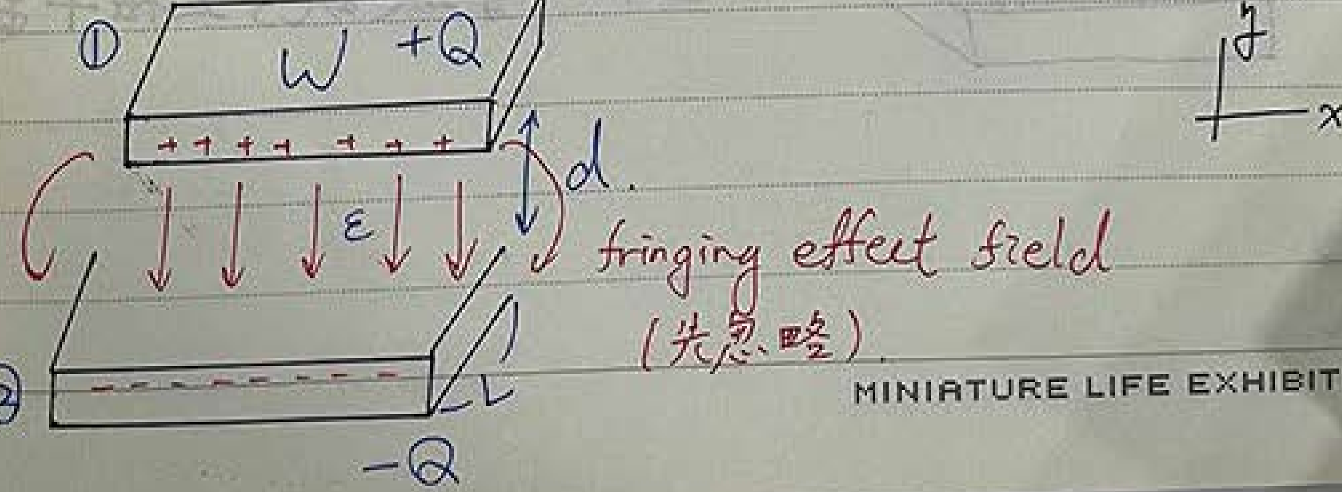
V_{12} = 導体①对②之相对potential

布

$\Rightarrow C=\dfrac{Q}{V_{12}}$, 由 V 求 $E \rightarrow E_n|_{surface} \rightarrow \rho_s \rightarrow Q$

(边界值問題)

ex 3-7. Parallel-plate capacitor.



(a) $\rho = \frac{Q}{S}$, $E_n = \frac{\rho_s}{\varepsilon}$ (看前面)

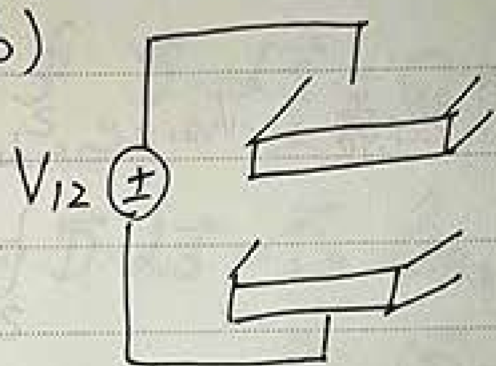
$\Rightarrow \vec{E} = -\hat{a}_y \cdot E_n = -\hat{a}_y \frac{\rho_s}{\varepsilon}$

$\Rightarrow V_{12} = -\int_{②}^{①} \vec{E} \cdot d\vec{l} = -\int_0^d -\hat{a}_y \frac{\rho_s}{\varepsilon} \cdot \hat{a}_y \, dy$

$= \frac{Q}{\varepsilon S} d$

$\Rightarrow C = \frac{Q}{V_{12}} = \frac{\varepsilon S}{d}$

(b)



相同之兩平板，通电压 V_{12}

$\vec{E} = -\hat{a}_y \frac{V_{12}}{d}$

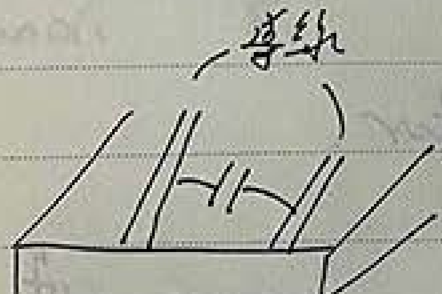
From B.C., $\rho_s = \varepsilon E_n = \varepsilon E_y = \varepsilon \frac{V_{12}}{d}$,

$Q = \rho_s \cdot S = \frac{\varepsilon S}{d} V_{12}$

$\Rightarrow C = \frac{Q}{V_{12}} = \frac{\varepsilon S}{d}$ #

＊ 如下图，∵ 兩導体間 若有电位差 必有电容值

∴ 兩導線間会有「寄生电容」



高頻時，∵ 电容阻抗 $= \frac{1}{sC}$ ↓

∴ 阻抗↓ 導致訊号互相干擾

ex. Cylinder capacitor (L>>ab, 忽略在兩端之影响, 如同∞長圆柱之一小段) 求 C?

才能用 Gauss' law

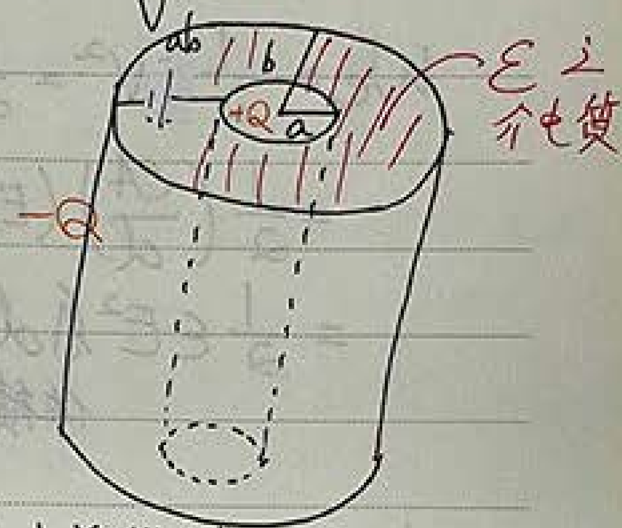
-sol- 令内外筒电位差為 V_{ab}

$$\vec{E} = \hat{a}_r \frac{Q}{2\pi\varepsilon L r}$$

$$\Rightarrow V_{ab} = -\int_b^a \vec{E} \cdot d\vec{l}$$

$$= -\int_b^a \frac{\hat{a}_r Q}{2\pi\varepsilon L r} \cdot \hat{a}_r dr$$

$$= \frac{Q}{2\pi\varepsilon L} \ln\frac{b}{a}$$ (得 V_{ab} 和 Q 之关係!)



$$\Rightarrow C = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

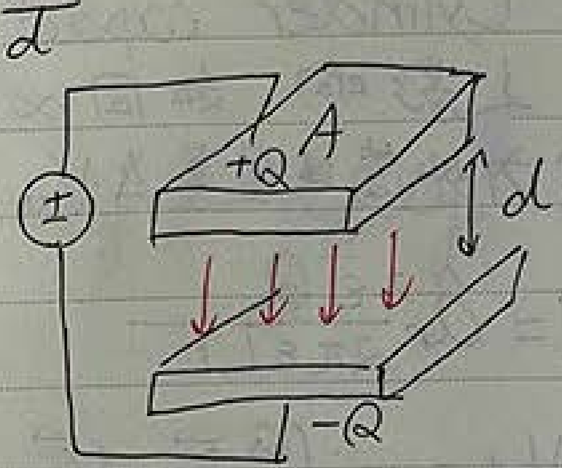
* per unit length 之电容 $C' \triangleq \frac{C}{L} = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}$

§3-11 电位能

1. 考虑一组平行电容板，$C=\frac{\varepsilon A}{d}$

$E=\frac{V}{d}=$ const. (uniform)

則 electrostatic energy



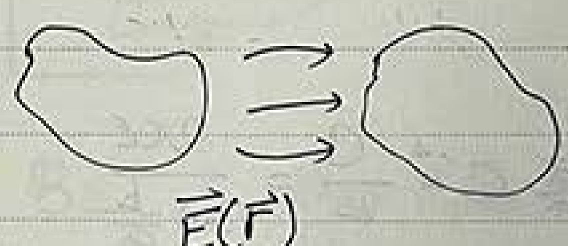
$$W_e=\frac{1}{2}CV^2=\frac{1}{2}\frac{Q^2}{C}$$

$$=\frac{1}{2}\left(\frac{\varepsilon A}{d}\right)(Ed)^2$$

$$=\frac{1}{2}\varepsilon E^2\underbrace{(Ad)}_{体積=V} \Rightarrow dW_e=\frac{1}{2}\varepsilon E^2 dv$$

electrostatic energy density $w_e=\frac{W_e}{V}=\frac{1}{2}\varepsilon E^2$

2.



静电能储存在"兩導体之間"or 电場之中.

假設$\vec{E}(\vec{r})$ (电場分佈已知)，則 (对体積積分)

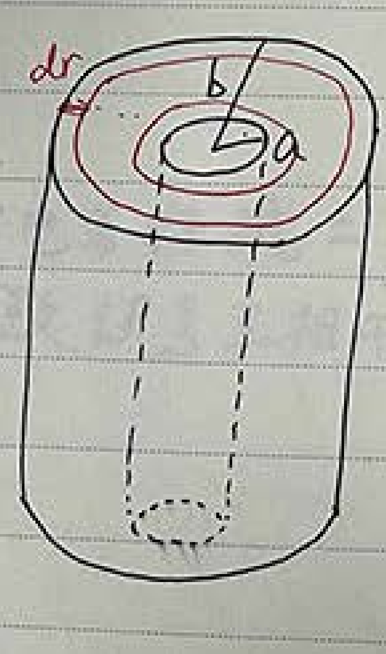
(1) $W_e=\int_V w_e\, dv=\frac{1}{2}\int_V \varepsilon E^2 dv$

(2) 得W_e後可用 $W_e=\frac{1}{2}\frac{Q^2}{C}$ 求C！

ex 3-25. 和前一例相同之电容器

(1) 求其儲存之靜电能？

(2) 求 C？



-sol-

(1) $\vec{E}=\hat{a}_r\frac{Q}{2\pi\varepsilon L r}$, for $a<r<b$

Total $W_e=\int w_e\, dv$

$$= \frac{1}{2}\int_a^b \varepsilon\left(\frac{Q}{2\pi\varepsilon L r}\right)^2 (2\pi r\,dr)\cdot L$$

$$= \cdots$$

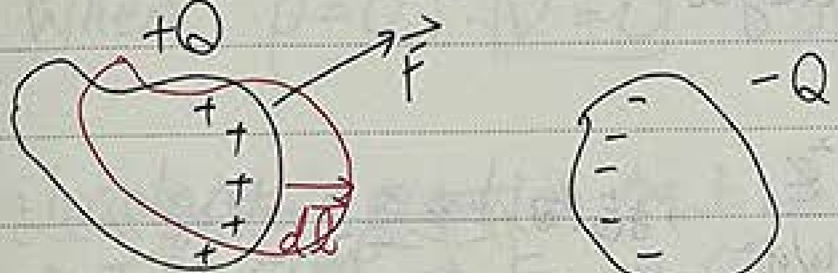
$$= \frac{Q^2}{4\pi\varepsilon L}\ln\frac{b}{a}$$

(2) $\frac{Q^2}{4\pi\varepsilon L}\ln\frac{b}{a} = \frac{1}{2}\frac{Q^2}{C}$

$$\Rightarrow C = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}} \quad ※$$

3. Electrostatic store

假設兩導体 A与B上分别有 $+Q$ 与 $-Q$ 的电量，A 受B之力為 $\vec{F}$，移动 $d\vec{l}$，則作功 (If Q is const)



$dW = \vec{F}\cdot d\vec{l}$

令兩者間位能(為位置函数)為 $W_e(\vec{r})$，則令

$W_e(\vec{r}+d\vec{l}) \triangleq W_e(\vec{r}) + dW_e$

$\Rightarrow dW_e = \nabla W_e \cdot d\vec{l}$

$\because dW + dW_e = 0$

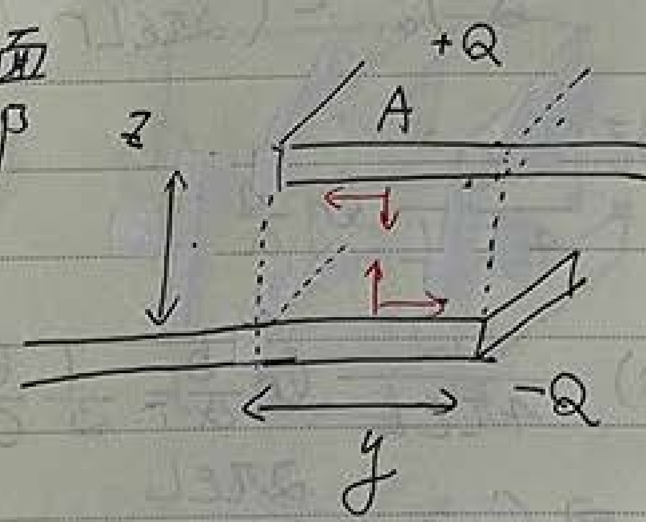
$\therefore (\vec{F} + \nabla W_e)\cdot d\vec{l} = 0$

$\Rightarrow \vec{F} = -\nabla W_e$

$\Rightarrow$ 力指向能量變化最劇烈的方向 (for const Q).

4. parallel plate capacitor

導体長條平板 重疊部份面積為A，相距z，重疊部份長為y，寬為w，求兩平板間之力？



-sol-

$\because C = \varepsilon \frac{A}{d}$

$\therefore W_e = \frac{1}{2} C V^2 = \frac{Q^2}{2C} = \frac{Q^2 z}{2\varepsilon w y}$

in z direction,

$F_z = -\frac{\partial W_e}{\partial z} = -\frac{Q^2}{2\varepsilon y w}$ → 指向距離縮小之方向

in y direction

$F_y = -\frac{\partial W_e}{\partial y} = \frac{Q^2 z}{2\varepsilon w y^2}$ → 指向使重疊區域增加之方向

⇒ 有使C值增加之趨勢

Ch4 静电問題的解.

§4.2. Poisson eq. & Laplace eq.

1. $\vec{\nabla}\cdot\vec{D}=\rho \Rightarrow \vec{\nabla}\cdot(\varepsilon\vec{E})=\rho$

   $\vec{\nabla}\times\vec{E}=0 \Rightarrow \vec{E}=-\vec{\nabla}V$

2. If $\varepsilon=$ const. (極化不均勻可能導致 ε 隨時間变)

   $\vec{\nabla}\cdot(\vec{\nabla}V)=-\dfrac{\rho}{\varepsilon}$

   $\Rightarrow \vec{\nabla}\cdot\vec{\nabla}V \triangleq \boxed{\nabla^2 V=-\dfrac{\rho}{\varepsilon}}$ poisson's eq.

3. $\nabla^2 V=\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=\dfrac{-\rho}{\varepsilon}$

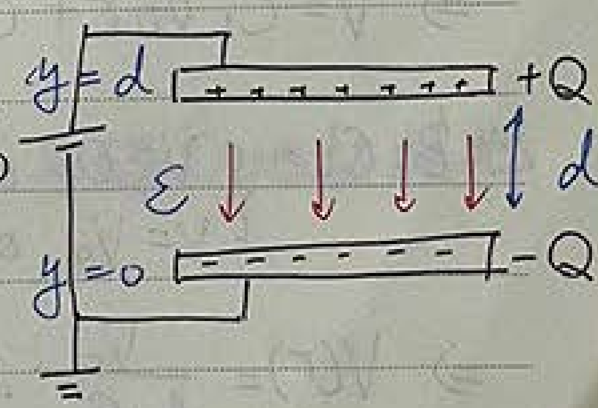
4. When $\rho=0$, $\boxed{\nabla^2=0}$ Laplace eq.

5. Laplace equation 加上 B.C. 可解出 $V(\vec{r})$

   $\Rightarrow$ 可解 $E(\vec{r}) \Rightarrow E_n \Rightarrow \rho_s \Rightarrow Q \Rightarrow C$

   $\vec{E}=-\vec{\nabla}V$; $E_n=\hat{a}_n\cdot\vec{E}$; see 3-6. $E_n=\dfrac{\rho_s}{\varepsilon_0}$; $Q=\rho_s A$; $C=\dfrac{Q}{V}$

ex 4-1. 解 ① $V(x,y,z)$? ② 电容値?



$\nabla^2 V=0=\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=\dfrac{d^2 V}{dy^2}$

B.C.: $\begin{cases} V=0 & \text{at } y=0 \\ V=V_0 & \text{at } y=d \end{cases}$

$\Rightarrow V(y)=C_1 y+C_2$, 代入 B.C. 得 $C_2=0$, $C_1=\dfrac{V_0}{d}$

$\Rightarrow \vec{E}=-\vec{\nabla}V=\dfrac{dV}{dy}=\dfrac{V_0}{d}$

surface charge:

ρ_s at lower plate: $\hat{a}_n = \hat{a}_y$

$$E_n = \hat{a}_n \cdot \vec{E} = -\frac{V_0}{d} = \frac{\rho_{sl}}{\varepsilon} \Rightarrow \rho_{sl} = -\frac{\varepsilon V_0}{d}$$

ρ_s at upper " : $\hat{a}_n = -\hat{a}_y$

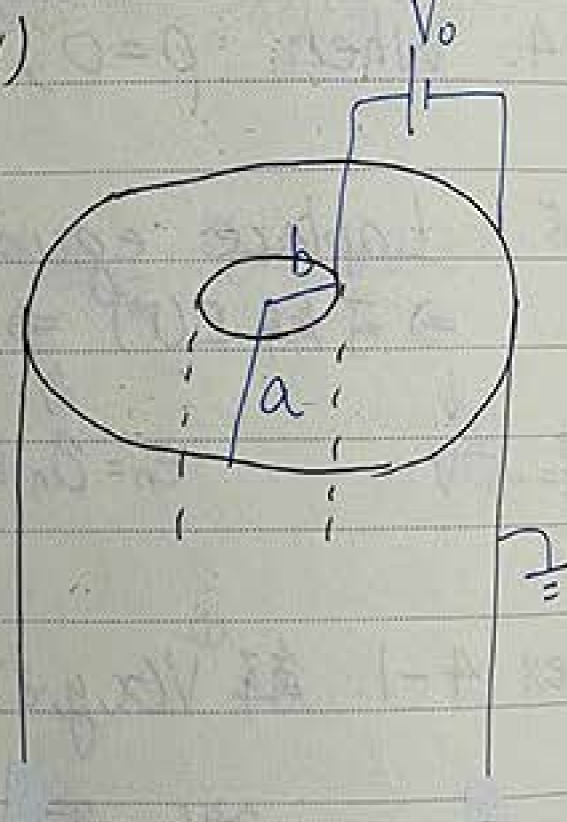
$$E_n = \hat{a}_n \cdot \vec{E} = \frac{V_0}{d} = \frac{\rho_{su}}{\varepsilon} \Rightarrow \rho_{su} = \frac{\varepsilon V_0}{d}$$

$\Rightarrow$ Total Charge $Q = \rho_{su} \cdot A = \dfrac{\varepsilon V_0 A}{d}$

$$\Rightarrow C = \frac{Q}{V} = \frac{\frac{\varepsilon V_0 A}{d}}{V_0} = \varepsilon \frac{A}{d} \quad ※$$

ex. long coax cable (同軸电缆)

求 $V(r)$?



-sol-

$$\nabla^2 V(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow r\frac{\partial V}{\partial r} = C_1$$

$$\Rightarrow V = C_1 \ln r + C_2$$

$$\text{B.C.}: \begin{cases} V = 0 \text{ at } r = a \Rightarrow C_1 \ln a + C_2 = 0 \\ V = V_0 \text{ at } r = b \Rightarrow C_1 \ln b + C_2 = 0 \end{cases} \Rightarrow \cdots$$

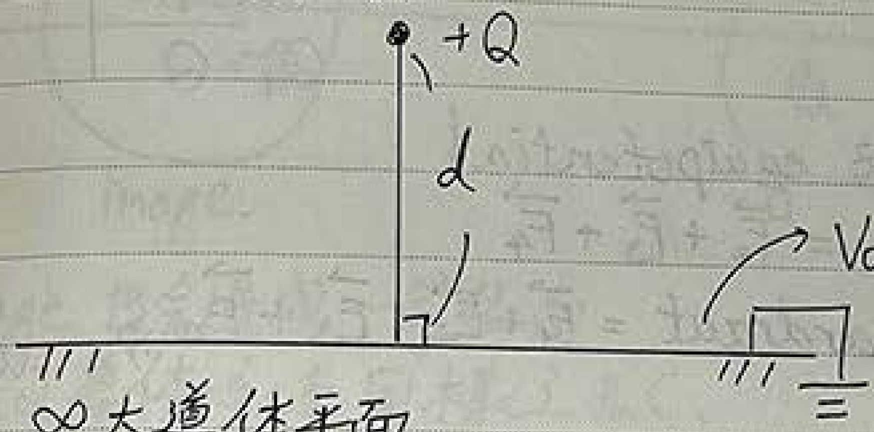
$$\Rightarrow V(r) = \frac{V_0}{\ln\frac{a}{b}} \ln\frac{r}{b} \quad ※$$

§4-4 Method of Image

$\nabla^2 O \langle {}^{0}_{-\frac{\rho}{\varepsilon}}$ (有極化电荷)

→若ODE太難解?

1.



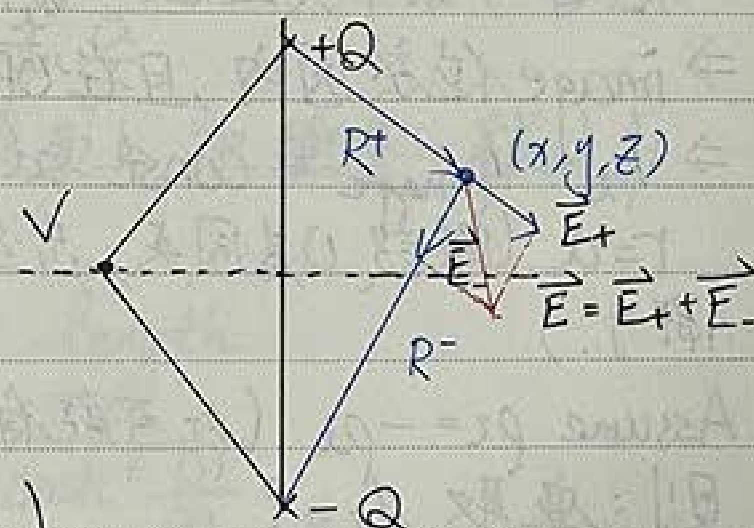
∞大導体平面.

求解平板上之V & $\vec{E}$?

-sol-. 抽掉板子

$V = V_+ + V_- = 0$

⇒加一虛擬电荷, B.C.仍成立.

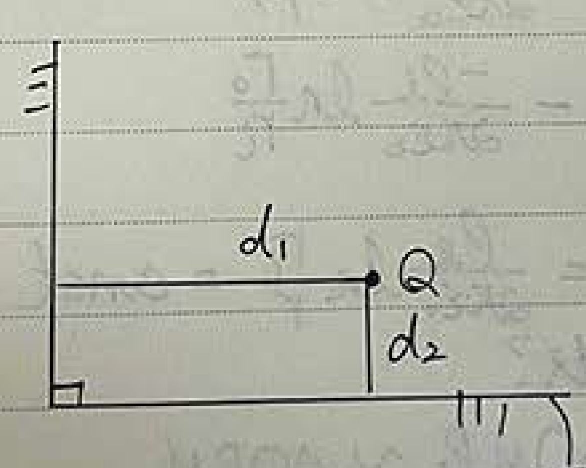


$V(x,y,z) = V_+ + V_-$

$= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R_+} + \frac{-Q}{R_-}\right)$

以x,y,z, d表示R_+与R_-, 即得 ☆

ex4-3. 金屬牆&地板接地



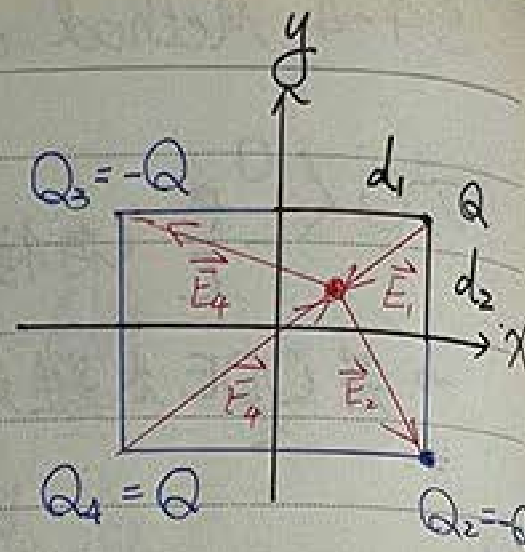
-sol-

For xz plane equipotential

→ Q_2;

" yz "

→ Q_3.



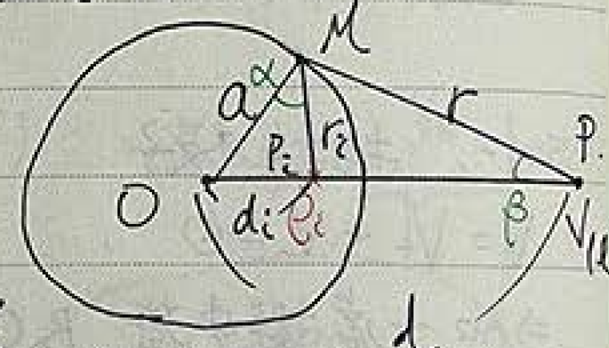
Add Q_4 → xz, yz equipotential

令 $\vec{F}_1$ = force on Q_1 = $\vec{F}_2 + \vec{F}_3 + \vec{F}_4$

$\vec{E}$ in first guardraut = $\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4$ ※

2. 园柱型 (俯視)

边界為园柱 ⇒ 园柱表面上之potential 相等，求P点之电位 V_{ρ_ℓ}?



⇒ image位於园内，且在 $\overline{OP}$ 上

⇒ $V_{\rho_\ell} + V_{\rho_{image}}$ = const at $r=a$.(假設O為园点，出纸為z方向.)

Assume $\rho_i = -\rho_\ell$ (ρ_i可能有多种 ∵量值不同→位置不同)，則：应取 d_i = ?

$E_r = \frac{\rho_\ell}{2\pi\varepsilon_0 r}$，任取一数 r_0 做為參考点

$V_{\rho_\ell} = -\int_{r_0}^{r} E_r \, dr = \frac{\rho_\ell}{2\pi\varepsilon_0} \ln \frac{r_0}{r}$

$V_{\rho_i} = -\int_{r_0}^{r_i} = -\frac{-\rho_\ell}{2\pi\varepsilon_0} \ln \frac{r_0}{r_i}$

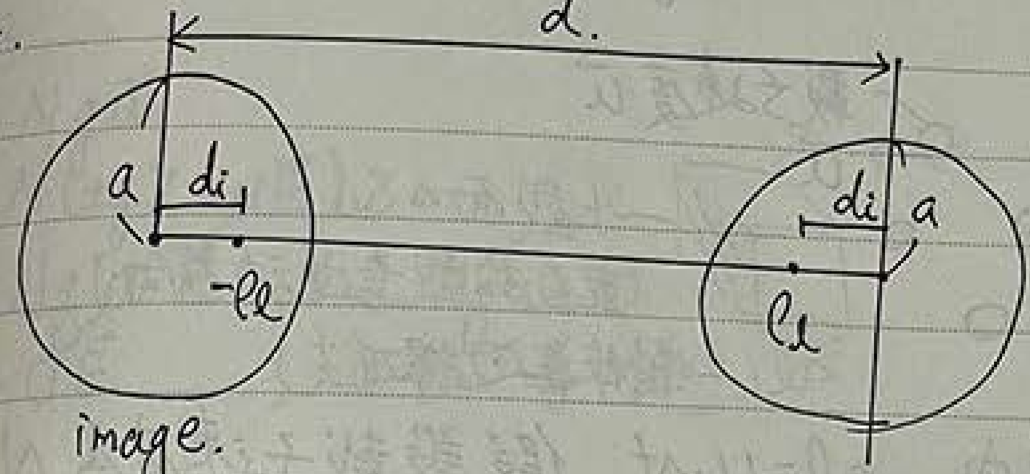
⇒ $V_M = V_{\rho_\ell} + V_{\rho_i} = \frac{\rho_\ell}{2\pi\varepsilon_0} \ln \frac{r_i}{r}$ = const ⟺ $\frac{r_i}{r}$ = const.

→ 如何找出此常数?

If $\alpha = \beta$, then $\triangle OMP_i \sim \triangle OPM$

⟺ $\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d}$ ⇒ $d_i = \frac{a^2}{d}$ ⇒ 將image放在此處※

ex4-4.



∵ image 也会造成一等位面

∴ 故2導体之有同樣之B.C.

⇒ E在园外相同!($V_2=V_\mu$)

$$\begin{cases} V_2 \text{ (due to } \rho_l) = V_\mu = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{r_i}{r} = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{a}{d} \\ V_1 \text{ (due to } -\rho_l) = -\dfrac{\rho_l}{2\pi\varepsilon_0}\ln\dfrac{a}{d} \end{cases}$$

(把集中一点的电荷打散到表面上,但總量一樣!)

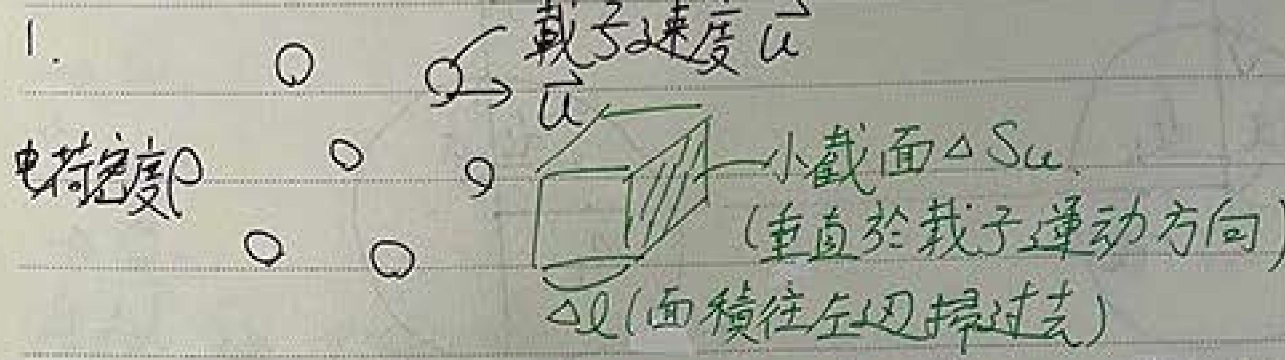
⇒ Capacitance per unit length

$$C' = \frac{\rho_0}{V_1 - V_2} = \frac{\pi\varepsilon_0}{\ln\frac{d}{a}} = \frac{\pi\varepsilon_0}{\cosh^{-1}\frac{D}{2a}}$$ ※

Ch5 穩定电流

§5.2 Current density

1.



在 Δt 時間内，$\Delta l = u \cdot \Delta t$. 假設載子密度為 N ($1/m^3$)

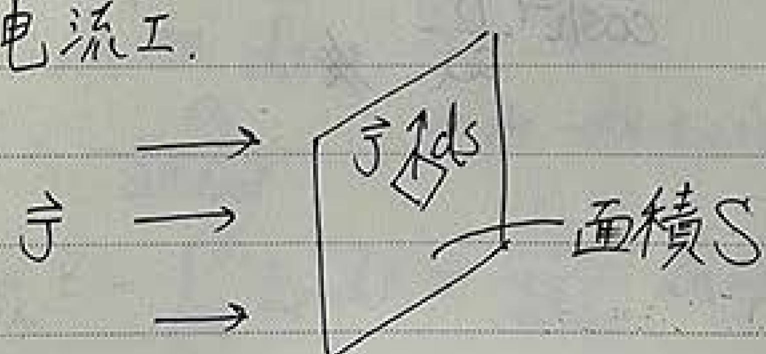
$$\Rightarrow \Delta Q = \Delta V \cdot N \cdot q$$
$$= \Delta l \cdot \Delta S_u \cdot N \cdot q$$
$$= u \cdot \Delta t \cdot \Delta S_u \cdot Nq$$
$$\Rightarrow \Delta I_u = \frac{\Delta Q}{\Delta t} = u \cdot \Delta S_u \cdot Nq.$$

定義 current density $\vec{J}$ (為 (x,y,z) 之函數) $= \hat{a}_u \dfrac{\Delta I}{\Delta S_u}$ (A/m^2)

$$= \underbrace{\hat{a}_u \cdot u}_{\vec{u}} \underbrace{Nq}_{\rho} = \rho \vec{u} \quad (\vec{u} \text{ 亦為 } (x,y,z) \text{ 之函數})$$

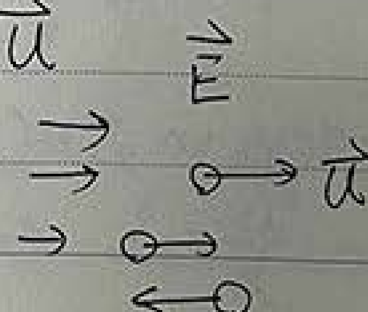
$\Rightarrow$ $\vec{J}$ 為位置的函數！(一种分佈)

2. 电流 I.



$$I = \int_S \vec{J} \cdot ds.$$

3. Drift velocity. $\vec{u}$



⇒ $\vec{u} = -\mu_e \vec{E} = \mu_p \vec{E}$，$\mu_e$, μ_p 称移动率 (mobility)，为一常数.

⇒ $\vec{J} = Nq\vec{u}$

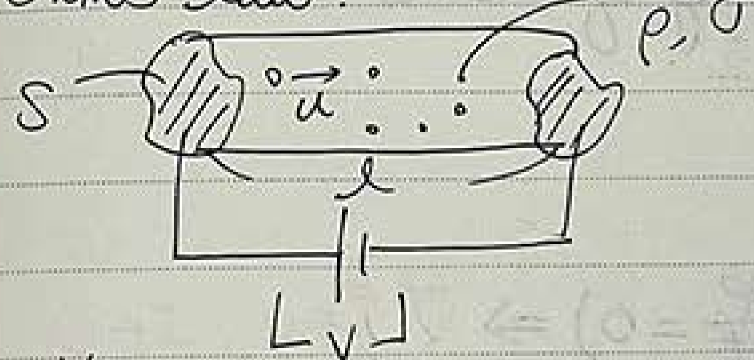
$= N(\pm q)(\pm \mu \vec{E})$

$= Nq\mu\vec{E}$，令 $Nq\mu = \sigma$

$= \sigma\vec{E}$，σ 称「导电率」

⇒ 很多种电流時 $\vec{J} = \sum_i \vec{J}_i = \sum_i (N_i q_i \mu_i)\vec{E}$

4. Ohm's law.



$E = \frac{V}{l}$，$I = \int_S \vec{J}\cdot d\vec{s} = JS$

∵ $\vec{J}$, $\vec{E}$ 方向相同且不变

∴ $J = \sigma E$

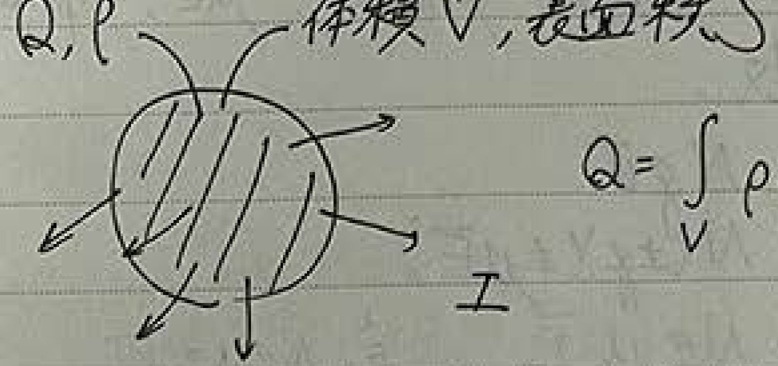
⇒ $\frac{I}{S} = \sigma\frac{V}{l}$

⇒ $V = \frac{1}{\sigma}\frac{l}{S} I = (\rho\frac{l}{S}) I$.

令 $R = \rho\frac{l}{S}$ 称「电阻」(resistance)

§5.4. Continuity equation

1. 流出表面S的總电流 = I



$$Q = \int_V \rho \, dV$$

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int \rho \, dV$$

$$= -\int \frac{\partial \rho}{\partial t} dV.$$

(I為 (x,y,z) 之函數!)

又因為 $I = \oint_S \vec{J}\cdot d\vec{S} = \int_V \nabla\cdot\vec{J} \, dv$

divergence Thm

$\Rightarrow \nabla\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$ (continuity equation)

物理意义: 电荷守恆.

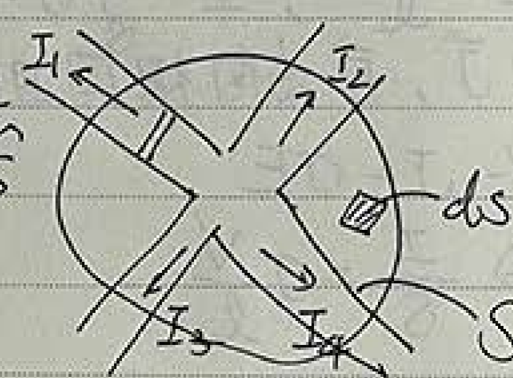
2. KCL.

假設电流穩定 $\left(\frac{\partial \rho}{\partial t} = 0\right) \Rightarrow \nabla\cdot\vec{J} = 0$, $I = 0$.

$\Rightarrow \vec{J}$ is divergenless

$\Rightarrow I = \oint_S \vec{J}\cdot d\vec{S} = 0$, $\forall$ any surface $\hat{S}$

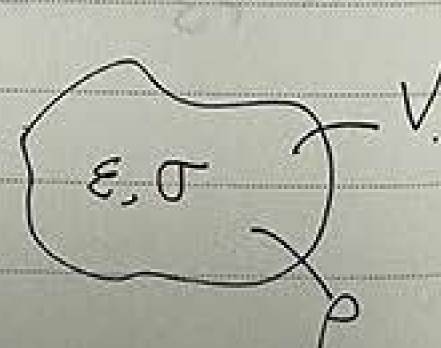
$\Rightarrow \sum_i I_i = 0$, 即KCL



3. relax time.

回憶: §3.6 → 放入導体内部的电荷会移至表面, why? 需多少時間?

$\Rightarrow \because \rho$ 產生 $\vec{E} \Rightarrow \vec{J} = \sigma\vec{E}$



$$\Rightarrow \nabla\cdot\vec{J} = \sigma\nabla\cdot\vec{E}$$

$$= \sigma\frac{\rho(t)}{\varepsilon}$$

$$= -\frac{\partial \rho(t)}{\partial t}$$

$$\Rightarrow \rho(t) = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}, \quad \rho_0 = \rho(0)$$

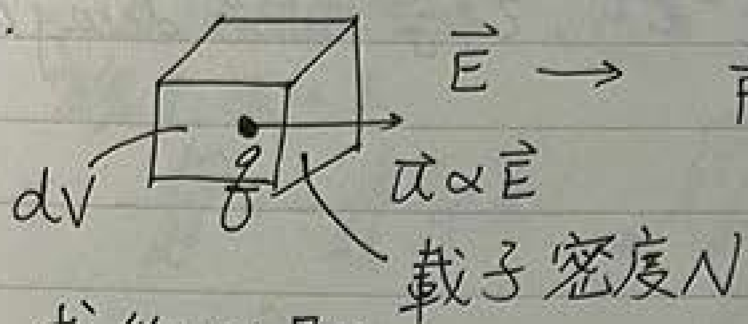
or $\rho(\vec{r},t)=\rho(\vec{r},t=0)e^{-\frac{\sigma}{\varepsilon}t}$, $\tau=\frac{\varepsilon}{\sigma}$ 稱為 decay/relaxation time.

*結論: $\vec{\nabla}\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$

MINIATURE LIFE EXHIBITION

§5.5 Power dissipation

1.



$\vec{E} \longrightarrow \qquad \vec{F}=q\vec{E}$

求總动量$\mathbb{P}$.

一粒子 $dp=\vec{F}_e\cdot\vec{u}=q\vec{E}\cdot\vec{u}$

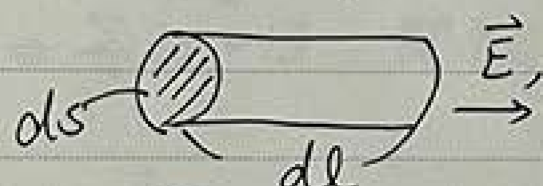
$$\Rightarrow d\mathbb{P}=(N\,dv)\cdot q\vec{E}\cdot\vec{u}$$
$$=\vec{E}\cdot(Nq\vec{u})\,dv$$
$$=\vec{E}\cdot\vec{J}\,dv$$
, $\vec{E}$, $\vec{J}$ 皆為位置的函數

$$\Rightarrow \frac{d\mathbb{P}}{dv}=\vec{E}\cdot\vec{J} \triangleq$$ power dissipation density (W/m³)

物理-\-\: 單位体積消耗多少功率.

$$\Rightarrow \mathbb{P}=\int_V \frac{d\mathbb{P}}{dv}\cdot dv=\int_V \vec{E}\cdot\vec{J}\,dv$$ (Joule's law)

ex. For a resistor



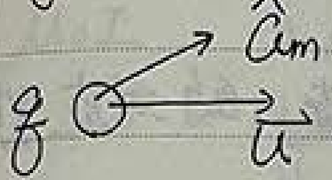
$\vec{E}$, $\vec{J}$ (const)

$$\mathbb{P}=\int \vec{E}\cdot\vec{J}\,dv$$
$$=E\cdot J\,ds\,dl$$
$$=I(E\,dl)$$
$$=IV \quad (=I^2R=\cdots)$$

Ch6 靜磁場

§6-1 Introduction

1. Magnetic field



由實驗可知：磁力 $\vec{F}_m$

(a) $\vec{F}_m \perp \vec{u}$，$\vec{F}_m \perp \hat{a}_m$，$\hat{a}_m$ 為一特定方向

(b) $|\vec{F}_m| \propto |u_{m\perp}| = \vec{u}$ 在 $\hat{a}_m$ 上的投影量

令 $\vec{B} \triangleq \hat{a}_m B$，$B$ 為磁通量密度，$\vec{B}$ 為磁場

$\Rightarrow \vec{F} = q\vec{u} \times \vec{B}$

2. $\vec{F} = \vec{F}_e + \vec{F}_m = q(\vec{E} + \vec{u} \times \vec{B})$ (Lorentz's force equation)

§6-2 自由空間靜磁學的基本假說

1. 2個基本假說

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 & -(1) \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} & -(2) \end{cases}$$

(1) 之物理意義：磁學中沒有「電荷」的對應觀念！

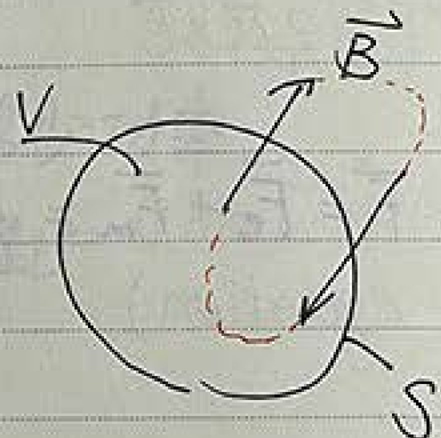
($cf.\ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon$ → 電荷)

⇒ 不存在 magnetic charge

(2) 之物理意義：$\vec{J}$ is the vortex source of $\vec{B}$

2. Integral form

(1) $\oint \vec{B}\cdot d\vec{s} = \int_V \vec{\nabla}\cdot\vec{B}\, dv = 0$



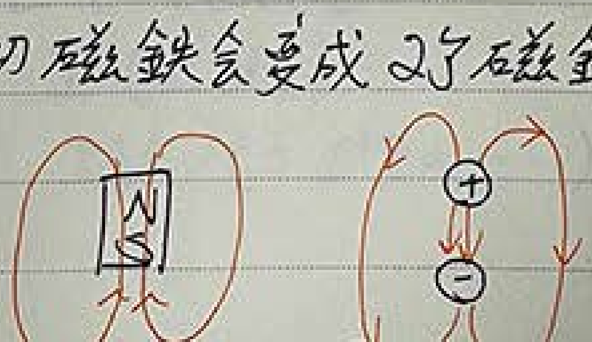
→ 物理意義：有往外之flux，就必有一往內的flux與之抵消

→ 磁力線必為封閉迴圈！

ex. permanent magnetic

類似 dipole ⊕ ⊖

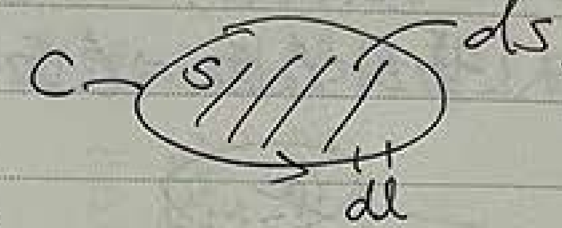
但切磁鐵會變成2個磁鐵



(2). $\oint_C \vec{B}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{B}\, d\vec{s}$

$= \int_S \mu_0 \vec{J} \cdot d\vec{s}$

$= \mu_0 \int_S \vec{J}\, d\vec{s}$



$= \mu_0 I$ → Ampere's law.

$d\vec{l}$, $d\vec{s}$ → right hand rule.

ex6-1. ∞長园柱狀導体，园截面半徑=b，载有穩定电流I_0，求：磁通密度B_ϕ隨距园心r之变化？

-sol-. 令 $\vec{B} = \hat{a}_\phi B_\phi(r)$

① $r<b$:

$$\oint_{C_1} \vec{B} \cdot d\vec{l} = \oint_{C_1} \hat{a}_\phi B_\phi(r) \cdot \hat{a}_\phi\, dl$$

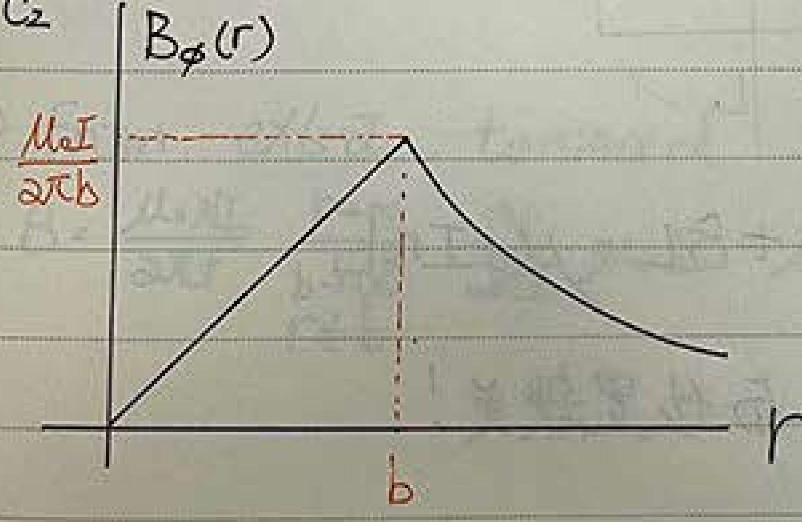
$$= B_\phi(r) \cdot 2\pi r$$

$$= \mu_0 \cdot I \left(\frac{r}{b}\right)^2$$

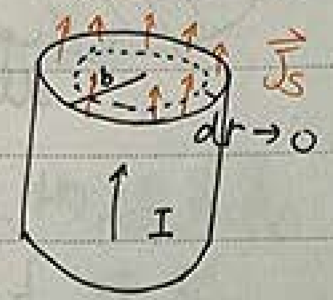
$$\Rightarrow \vec{B} = \hat{a}_\phi B_\phi = \hat{a}_\phi \frac{\mu_0 r I}{2\pi b^2}$$

② $r>b$:

$$\oint_{C_2} \vec{B} \cdot d\vec{l} = B_\phi(r) \cdot 2\pi r = \mu_0 I_0 \Rightarrow \vec{B} = \hat{a}_\phi \frac{\mu_0 I}{2\pi r}$$



3. 一般而言，I之频率越高（交流电），I越会跑到导线表面 ⇒ 表面电流 ⇒ 影响导线电感

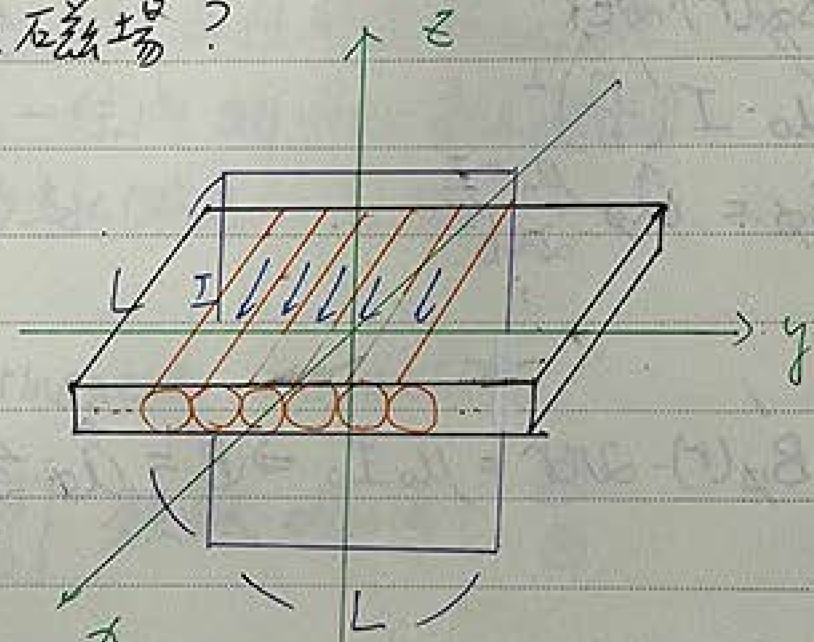


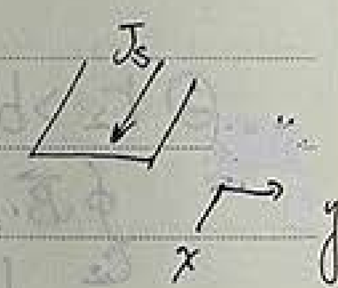
∞长导线，但只有表面电流 ⇒ $J_s = \frac{I}{2\pi b}$

⇒ From Ampere's law

$$\vec{B} = \begin{cases} 0, & r<b \\ \hat{a}_\phi \frac{\mu_0 I}{2\pi r}, & r>b \end{cases}$$

4. 一 ∞大导体平板，可視為多條載电流I之导线並排。設导线密度為 n wire/length，求距平板L處之磁場？



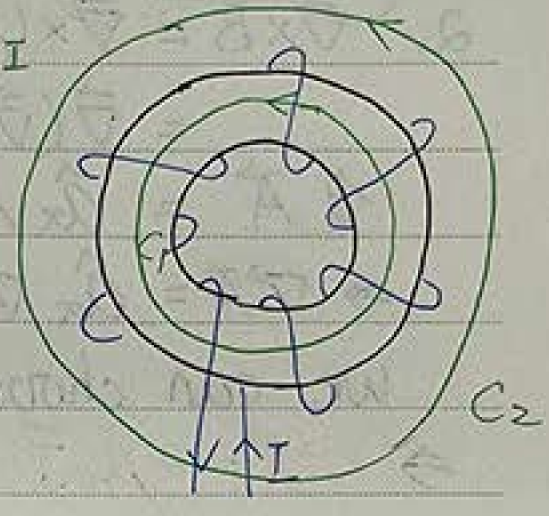


$$\oint_C \vec{B}\cdot d\vec{l} = B\cdot L + BL = \mu_0 I n L$$

⇒ $B = \frac{\mu_0}{2} n I$ → 与位置無关！

ex6-2. Toroidal.

N匝, 載有电流I, 半徑=b, 求 toroidal 内部之磁場?

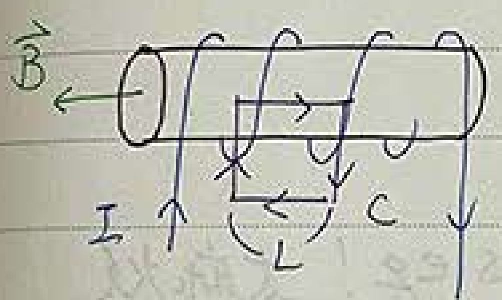


-sol-. $\oint_{C_1} \vec{B}\cdot d\vec{l} = B\cdot 2\pi r = \mu_0 NI$

$\Rightarrow \vec{B} = \hat{a}_\phi \frac{\mu_0 NI}{2\pi r}$ #

* 在外部 (C_2), ∵抵消 ∴ $\vec{B}=0$

ex6-3. 螺線管, 求内部磁場 $\vec{B}$=?



-sol- <法I>. From Ampere's law

$\oint_C \vec{B}\cdot d\vec{l} = BL = \mu_0 (nL) I$

$\Rightarrow B = \mu_0 n I$

<法II> From ex6-2, toronoid

$B = \frac{\mu_0 NI}{2\pi r} \xrightarrow[b\to\infty,\ r\doteq b]{r\to\infty} \frac{N}{2\pi b}\mu_0 I = \mu_0 n I$ #

§6-3 向量磁位

1. $\vec{E}=-\vec{\nabla}V \Leftrightarrow \vec{\nabla}\times\vec{E}=0$

   $\vec{B}=\vec{\nabla}\times\vec{A} \Leftrightarrow \vec{\nabla}\cdot\vec{B}=0$. A: Wb/m

2. $\vec{\nabla}\times\vec{B}=\vec{\nabla}\times(\vec{\nabla}\times A)$

   $=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$

   令 $\vec{A}=\hat{a}_x A_x+\hat{a}_y A_y+\hat{a}_z A_z$

   $\Rightarrow \nabla^2\vec{A}=\hat{a}_x\nabla^2 A_x+\hat{a}_y\nabla^2 A_y+\hat{a}_z\nabla^2 A_z$

   We can choose $\vec{\nabla}\cdot\vec{A}=0$ (Coulomb's condition)

   $\Rightarrow \nabla^2\vec{A}=-\mu_0\vec{J}$

3. $\because \nabla^2\vec{A}=-\mu_0\vec{J}$

   $\therefore \begin{cases}\nabla^2 A_x=-\mu_0 J_x\\ \nabla^2 A_y=-\mu_0 J_y\\ \nabla^2 A_z=-\mu_0 J_z\end{cases}$

   $\Rightarrow$ 要解 $\vec{A}$，需解3條 poisson's eq! 太麻煩.

   cf. $\nabla^2 V=-\dfrac{\rho}{\varepsilon_0} \Longleftrightarrow V=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{V'}\dfrac{\rho}{R}dv'$

   $\Rightarrow A_x=\dfrac{\mu_0}{4\pi}\displaystyle\int_{V'}\dfrac{J_x}{R}dv'$

   $\Rightarrow \vec{A}=\hat{a}_x A_x+\hat{a}_y A_y+\hat{a}_z A_z$

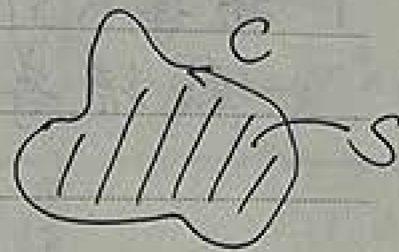
   $=\dfrac{\mu_0}{4\pi}\displaystyle\int_{V'}\dfrac{1}{R}(\hat{a}_x J_x+\hat{a}_y J_y+\hat{a}_z J_z)\,dv'$

   $=\dfrac{\mu_0}{4\pi}\displaystyle\int_{V'}\dfrac{\vec{J}}{R}dv'$

$\Rightarrow$ 定義向量磁位 $\vec{A}=\dfrac{\mu_0}{4\pi}\displaystyle\int_{V'}\dfrac{\vec{J}}{R}dv'$

4. 磁通量 $\Phi \doteq \int_C \vec{B}\cdot d\vec{S}$

$$= \int_S \nabla \times \vec{A}\, d\vec{S} = \oint_C \vec{A}\cdot d\vec{l}$$

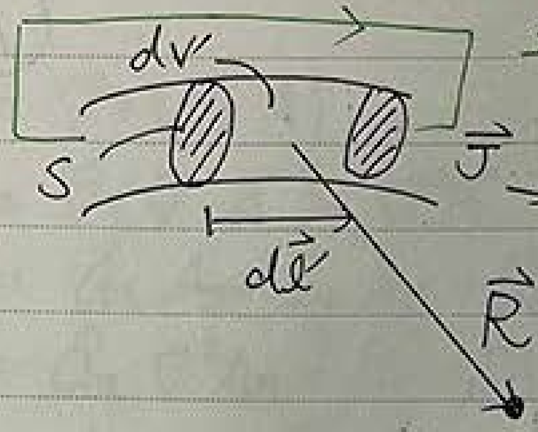


§6.4 Biot-Savart law.

1. 推導BSL:

一載流導線其中一小段

導線所形成之迴路為C'



$(\vec J /\!/ d\vec l')$

$$\vec J\cdot dv' = \vec J\cdot S\,d\vec l' = JS\,dl' = I\,d\vec l'$$

$$\Rightarrow \vec A = \frac{\mu_0}{4\pi}\oint_{C'}\frac{I\,d\vec l'}{R}$$

在沒有source的坐標微分

source

$$\Rightarrow \vec B = \vec\nabla\times\vec A = \frac{\mu_0 I}{4\pi}\left(\vec\nabla\times\oint_{C'}\frac{d\vec l'}{R}\right)$$

$$= \frac{\mu_0 I}{4\pi}\oint_{C'}\vec\nabla\times\frac{d\vec l'}{R}$$

$$\because\ \vec\nabla\times\frac{d\vec l'}{R} = \frac{1}{R}\cancelto{0}{(\vec\nabla\times d\vec l')} + \nabla\left(\frac{1}{R}\right)d\vec l' = -\frac{\hat a_x}{R^2}\,d\vec l'$$

$$\therefore\ \vec B = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\vec l'\times\hat a_R}{R^2}\ \left(=\oint_{C'} d\vec B\right)$$ — Biot Savart law

(or $d\vec B = \dfrac{\mu_0 I}{4\pi}\dfrac{d\vec l'\times\hat a_R}{R^2}$)

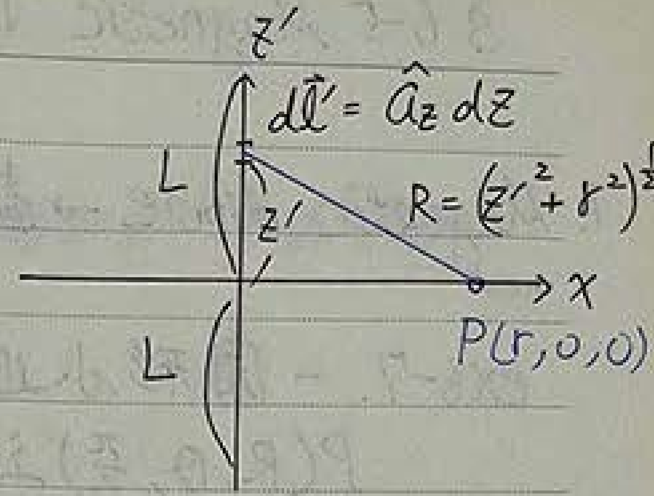
ex6-4. 長度為2L之導線上有直流电I，求距導線 (水平面上) r處之磁通密度$\vec B$?

-sol-. 〈法I〉先求$\vec A$

$$\vec A = \frac{\mu_0}{4\pi}\oint_C\frac{\vec J}{R}dv' = \frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{\hat a_z\,dz'}{(z'^2+r^2)^{\frac12}}$$

$d\vec l'$, R

$$= \cdots = \hat{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}$$



$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & \hat{a}_\phi r & \hat{a}_z \\ \frac{\partial}{\partial r} & \cancel{\frac{\partial}{\partial \phi}}^{\,0} & \frac{\partial}{\partial z} \\ \cancel{A_r}^{\,0} & \cancel{rA_\phi}^{\,0} & A_z \end{vmatrix}$$

$$= -\hat{a}_\phi \frac{\partial A_z}{\partial r}$$

$$\Rightarrow \vec{B}\Big|_{z=0} = \cdots = -\hat{a}_\phi \frac{\mu_0 I}{4\pi} \frac{\partial}{\partial r} \ln \frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}$$

$$= \hat{a}_\phi \frac{\mu_0 I L}{2\pi r \sqrt{L^2+r^2}} \quad ※$$

〈法II〉. use BSL

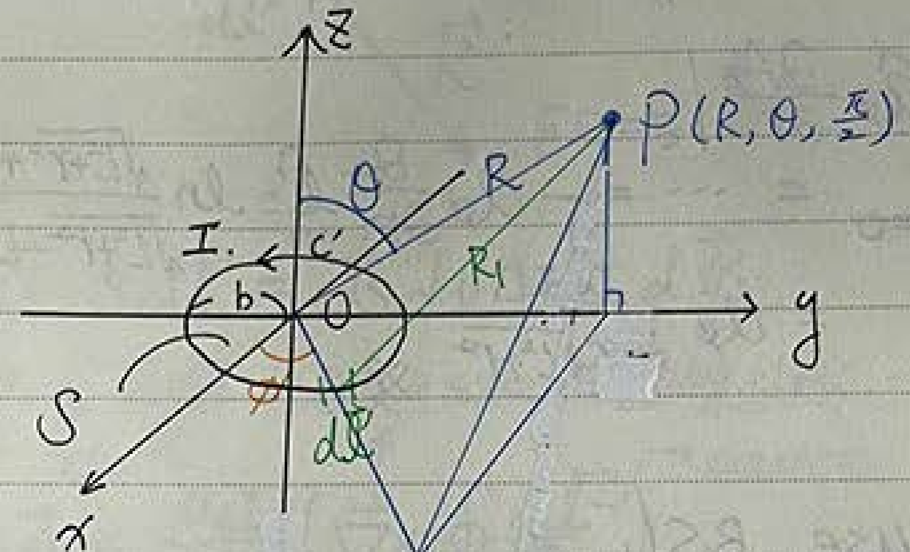
$$\left\langle \begin{array}{l} \vec{R} = \hat{a}_r r - \hat{a}_z z', \quad d\vec{l}' = \hat{a}_z dz' \\ d\vec{l}' \times \vec{R} = \hat{a}_\phi r\, dz' \end{array} \right.$$

$$\vec{B} = \int d\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{\hat{a}_\phi r\, dz}{(r^2+z'^2)^{\frac{3}{2}}} = \hat{a}_\phi \frac{\mu_0 I L}{2\pi r \sqrt{L^2+r^2}} \quad ※$$

§ 6-5 Magnetic Dipole

1. Magnetic dipole → 產生磁場的最基本單元

ex6-7. 一园形小迴路，半徑 b，电流 I，求（球坐標下）
$P(R, \theta, \frac{\pi}{2})$ 之磁場 $\vec{B} = ?$ $(R \gg b)$.



-sol-. 先求 $\vec{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{\ell}}{R_1}$，$d\vec{\ell} = \hat{a}_\phi b\, d\phi'$

$$= (-\hat{a}_x \sin\phi' + \hat{a}_y \cos\phi')\, b\, d\phi'$$

$$\Rightarrow \vec{A} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{-\hat{a}_x \sin\phi' + \hat{a}_y \cos\phi'}{R_1}\, b\, d\phi'$$

$$= \frac{\mu_0 I b}{4\pi} \int_0^{2\pi} \frac{-\hat{a}_x \sin\phi'}{R_1}\, d\phi'$$

∵ R_1 為 R, ϕ, θ 之函数

∴ 由餘弦定理，

$$R_1^2 = R^2 + b^2 - 2bR\cos\psi'$$

$$= R^2 + b^2 - 2bR\sin\theta\cos\left(\frac{\pi}{2} - \phi'\right)$$

$$= R^2 + b^2 - 2bR\sin\theta\sin\phi'$$

$$\Rightarrow \frac{1}{R_1} = \frac{1}{R}\left(1 + \left(\frac{b}{R}\right)^2 - 2\frac{b}{R}\sin\theta\sin\phi'\right)^{-\frac{1}{2}}$$

$$\approx \frac{1}{R}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right)$$

$$\Rightarrow \vec{A} \approx -\hat{a}_x \frac{\mu_0 I b}{4\pi R}\int_0^{2\pi}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right)\sin\phi'\,d\phi'$$

$$= -\hat{a}_x \frac{\mu_0 I b}{4\pi R}\frac{b}{R}\sin\theta\,\pi$$

$$= \hat{a}_\phi \frac{\mu_0 I b^2}{4R^2}\sin\theta$$

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} = \cdots = \frac{\mu_0 I b^2}{4R^3}\left(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta\right) \#$$

* cf. electrical dipole $\vec{E} \doteq \frac{P}{4\pi\varepsilon_0 R^3}\left(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta\right)$

① 皆反比於 R^3

② 帶电流線圈為產生磁場最基本的source

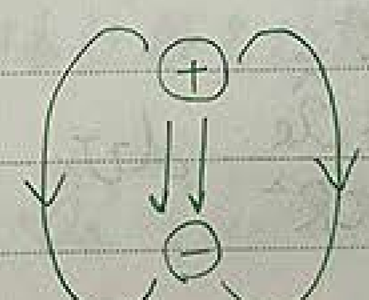
③ $P(=qd)$ 之角色与 Ib^2(与線圈面積)有关.

2. define $\vec{m} = \hat{a}_z IS = \hat{a}_z I\pi b^2$

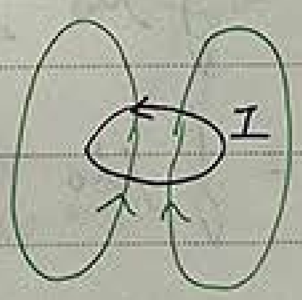
$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi R^2}\left(\hat{a}_R I\pi b^2 \sin\theta\right), \quad \hat{a}_\phi \sin\theta = \hat{a}_z \times \hat{a}_R$$

$$= \frac{\mu_0}{4\pi R^2}\vec{m}\times\hat{a}_R \quad (R \gg b)$$

$$* \begin{cases} \vec{p} \Leftrightarrow \vec{m} \\ \varepsilon_0 \Leftrightarrow \frac{1}{\mu_0} \end{cases}$$



未封閉



封閉

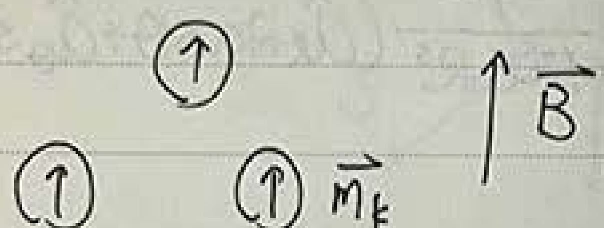
§6.6 Magnetization

1. 原子

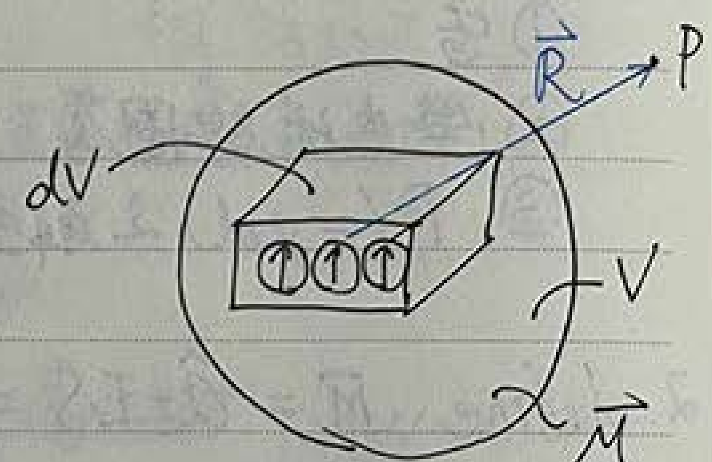
公転 → 都有等效磁場

2. 無外加磁場時，物質中原子的 magnetic dipole 散乱排列 $\Rightarrow \sum \vec{m}_k = 0$.

3. 給外加磁場 ⇒ 物質中原子的磁偶極整齊排列



4. Magnetization vector



$$\vec{M} \triangleq \lim_{\Delta v \to 0} \frac{\sum_k \vec{m}_k}{\Delta v}$$

$$\begin{cases} d\vec{m} = \vec{M}\, dv \\ d\vec{A} = \mu_0 \dfrac{d\vec{m} \times \hat{a}_R}{4\pi R^2} \end{cases}$$

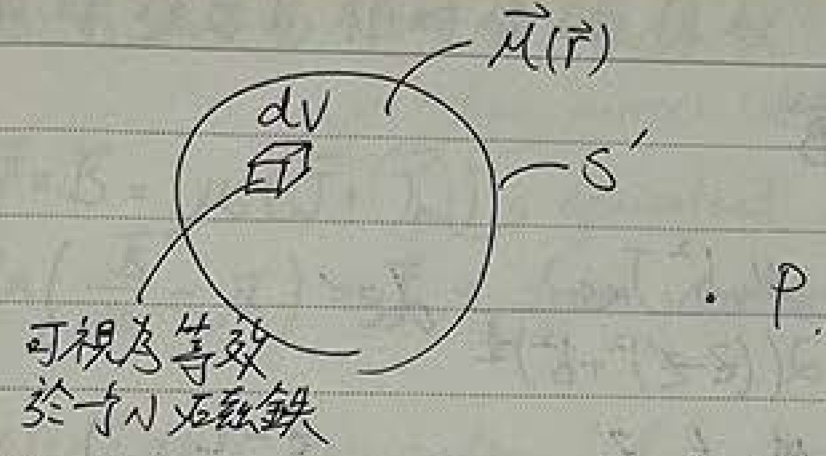
$$\Rightarrow \vec{A} = \int_V d\vec{A}$$

$$= \int_V \mu_0 \frac{d\vec{m} \times \hat{a}_R}{4\pi R^2}$$

$$= \int_V \mu_0 \frac{\vec{M} \times \hat{a}_R}{4\pi R^2}\, dv$$

5. $M(\vec{r})$ = magnetic moment per unit volume

$$M(\vec{r}) = \frac{\sum_k \vec{m}_k}{\Delta v}$$



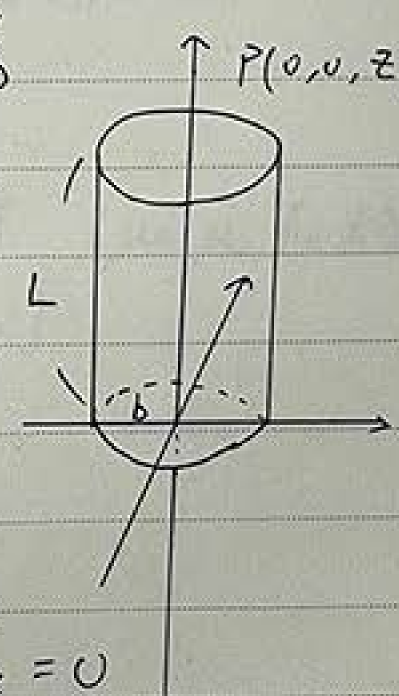
$$d\vec{m} = \vec{M}\, dv \Rightarrow d\vec{A} = \frac{\mu_0 \vec{M} \times \hat{a}_R}{4\pi R^2} dv'$$

$$\Rightarrow \vec{A} = \int d\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{\nabla}' \times \vec{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{M} \times \hat{a}_n'}{R} ds'$$

$$\left(\text{cf: } \vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}}{R} dv' \right)$$

$$\Rightarrow \begin{cases} \vec{\nabla}' \times \vec{M} \triangleq \vec{J}_m = \text{equivalent volume magnetization current density} \\ \vec{M} \times \hat{a}_n' \triangleq \vec{J}_{ms} \quad \text{" surface "} \end{cases}$$

ex6-8 圓柱均勻磁化，z軸上磁化向量為 $\vec{M} = \hat{a}_z M_0$，求 $P(0,0,z)$ 上之磁通密度？



-sol- (1) side:

$$\vec{J}_{ms側} = \vec{M} \times \hat{a}_n' = (\hat{a}_z M_0) \times \hat{a}_r = \hat{a}_\phi M_0$$

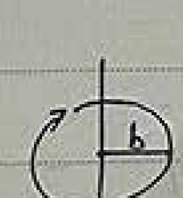
(环繞側面 ⟺ 相當於很多線圈)

(2) top, bottom:

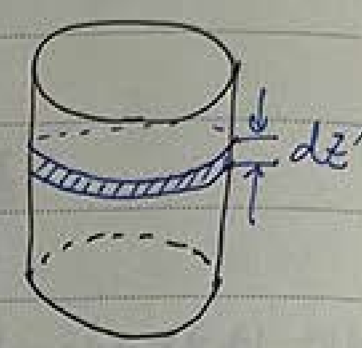
$$\vec{J}_{ms上} = \vec{J}_{ms下} = \vec{M} \times \hat{a}_n' = (\hat{a}_z M_0) \times \hat{a}_z = 0$$

⇒ from ex6-6, eq 6-38

$$d\vec{B} = \hat{a}_z \frac{\mu_0\, dI \cdot b^2}{2\left((z-z')^2 + b^2\right)^{\frac{3}{2}}}$$



$$\Rightarrow \vec{B} = \int d\vec{B}$$



$$= \hat{a}_z \int \frac{\mu_0 b^2 J_{ms}}{2\left((z-z')^2+b^2\right)^{\frac{3}{2}}}\, dz'$$

$$= \hat{a}_z \frac{\mu_0 M_0}{2}\left(\frac{z}{\sqrt{z^2+b^2}} - \frac{z-L}{\sqrt{(z-L)^2+b^2}}\right) \#$$

6-7 磁場強度与相对導磁係数

1. $\because \nabla\times\vec{B} = \mu_0(\vec{J}+\vec{J}_m)$

free current (通常较能控制) → $\vec{J}$; equivalent current → $\vec{J}_m$

$\therefore \nabla\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right) = \vec{J}$ = free current only

定義 磁場強度 (magnetic field intensity)

$$H \triangleq \frac{\vec{B}}{\mu_0}-\vec{M} \quad (A/m)$$

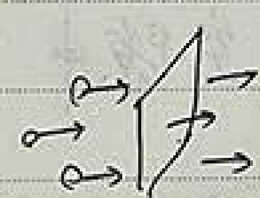
$\Rightarrow \nabla\times\vec{H} = \vec{J}$ (適用於有物質存在時)

$\left\{ \oint_C \vec{H}\, d\vec{l} = I \right.$ (integral form: Ampere's law)

* 体積电流 → e^- 在 v 中運动

表面 〃 → 载子被限制在表面.

断句: $\boxed{体積电流}\boxed{密度}$ A/M^2.



2. $\vec{M} = \chi_m \vec{H}$

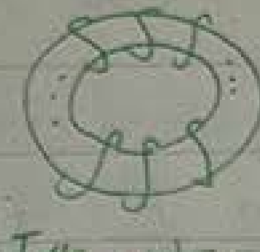
$\Rightarrow \vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu_0\mu_r\vec{H} = \mu\vec{H}$ ($\mu \doteq \mu_0$ for 大部份物質)

χ_m: 磁化率 (magnetic susceptibility)

§6-8 磁路

1. $\begin{cases} \vec{\nabla}\cdot\vec{B}=0 \Rightarrow \text{conservation of mag. flux} \\ \vec{\nabla}\times\vec{H}=\vec{J} \end{cases}$

$\Leftrightarrow \begin{cases} \oint_S \vec{B}\cdot d\vec{s}=0 \\ \oint_C \vec{H}\cdot d\vec{l}=I \end{cases}$ $\longrightarrow$



$I_{總}=NI=V_m$
(mmf, 磁動勢)

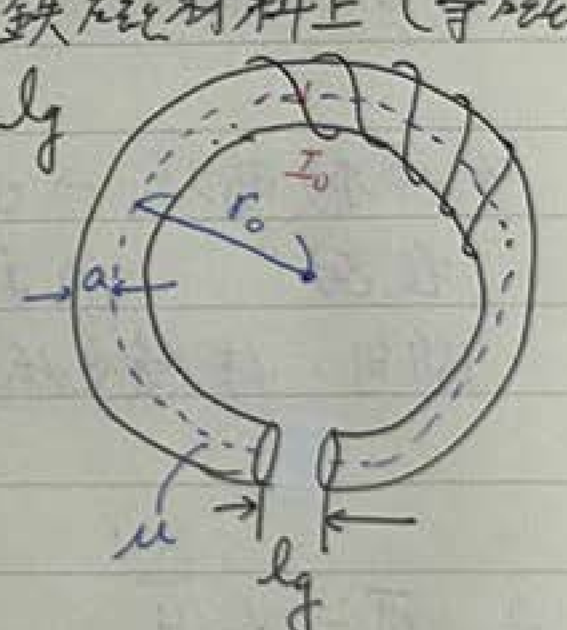
$\Leftrightarrow \begin{cases} \text{磁動勢可對外作功} \\ \text{電〃 不可〃} \end{cases}$

ex 6-10. N匝導線繞在環形鐵磁材料上(導磁係數$=\mu$), $a \ll r_0$, 缺口長 l_g

問:(1) 鐵芯內磁通密度 $\vec{B}_f$

(2) 〃 磁場強度 $\vec{H}_f$

(3) 空隙中 〃 $\vec{H}_g$



-sol-. 令座線逆時針為C, 則

$$\oint_C \vec{H}\cdot d\vec{l}=NI_0 \quad (*),$$

$$\because \begin{cases} \vec{B}_f=\vec{B}_g=\hat{a}_\phi B_f \\ H_f=\hat{a}_\phi \dfrac{B_f}{\mu_0} \\ H_g=\hat{a}_\phi \dfrac{B_f}{\mu_0} \end{cases}$$

$\therefore$ 代入$(*)$,

$$\oint_C \vec{H}\cdot d\vec{l}=\frac{B_f}{\mu}(2\pi r_0-l_g)+\frac{B_f}{\mu_0}l_g=NI_0$$

$$\Rightarrow \vec{B}_f=\hat{a}_\phi \frac{\mu_0\mu N I_0}{\mu_0(2\pi r_0-l_g)+\mu l_g}$$

$$\vec{H}_f = \hat{a}_\phi \frac{\mu_0 N I_0}{\mu_0(2\pi r_0 - l_g) + \mu l_g}$$

$$\vec{H}_g = \hat{a}_\phi \frac{\mu N I_0}{\mu_0(2\pi r_0 - l_g) + \mu l_g}$$ #

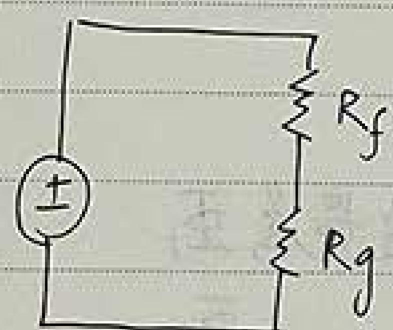
2. 承上例，可知

$$\Phi = \int_S \vec{B} \cdot d\vec{s} = \Phi_{core} = \Phi_{gap} = BS = B_f S$$

S → 截面

$$= \frac{N I_0}{\underbrace{\frac{2\pi r_0 - l_g}{\mu S}}_{R_f} + \underbrace{\frac{l_g}{\mu_0 S}}_{R_g}} = \frac{V_m}{R_f + R_g}$$

(reluctance) ⇒ 对应到电阻



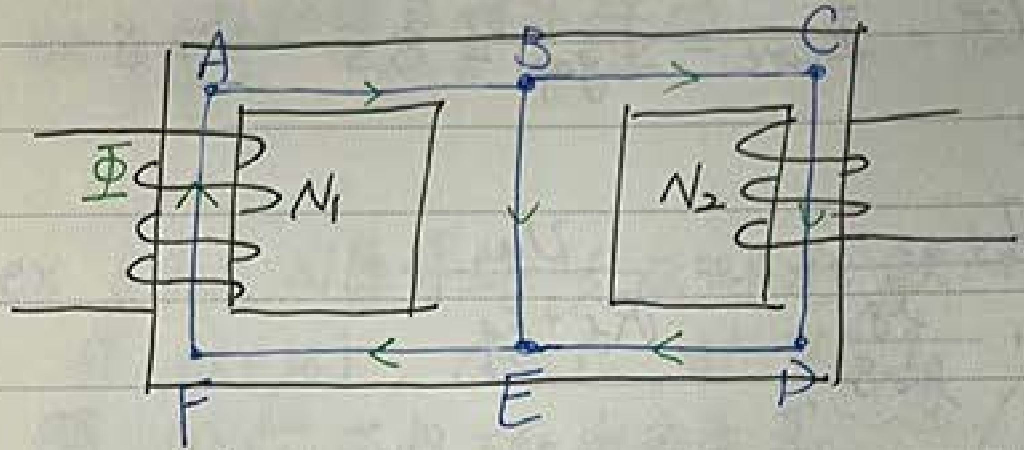
3. KVL: $\sum_i V_{mi} = \sum_i N_i I_i = \sum_k R_k \Phi_k$

KCL: $\sum_j \Phi_j = 0$.

电路	磁路
emf V	mmf $V_m = NI$
电流 I	磁通量 Φ
电阻 R	磁阻 $\mathcal{R}$
導电率 σ	導磁係数 σ.

$$\Rightarrow \Phi = \frac{NI}{R} = \frac{\mathcal{V}_m}{R} \begin{matrix} \leftarrow 磁动势 \\ \leftarrow 磁阻 \end{matrix}, \quad R = \frac{l}{\mu A}.$$

ex6-11 以下磁路中，左右各有 3 N_1, N_2 匝線圈，分别載电流 I_1, I_2，截面積 $= S_c$. 導磁係数為 μ，求 $\overline{BE}$ 上的磁通量？



其中，設 $\begin{cases} \overline{AB} + \overline{AF} + \overline{FE} = l_1 \\ \overline{BC} + \overline{CD} + \overline{DE} = l_2 \end{cases}$

-sol- 令迴圈 ABEF 上的磁通量為 Φ_1

" ACDF " Φ_2

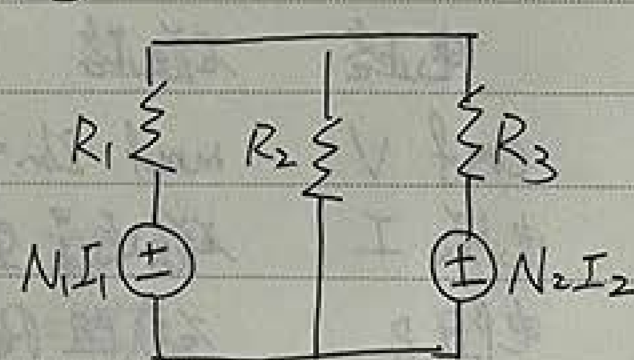
$\Rightarrow$ $\overline{AF}$ 上之磁通量為 $\Phi_1 + \Phi_2$.

再令 $\overline{AF}$ 上之磁阻為 R_1

$\overline{CD}$ " R_2

$\overline{BE}$ " R_3

$$\Rightarrow \begin{cases} R_1 = \dfrac{l_1}{\mu S_c} \\ R_2 = \dfrac{l_2}{\mu S_c} \\ R_3 = \dfrac{l_3}{\mu S_c} \end{cases}$$

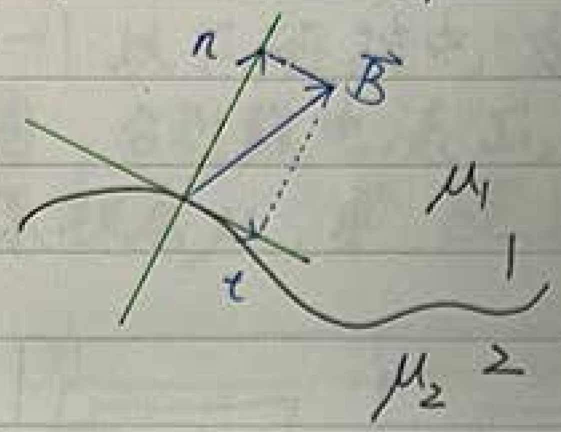


$$\Rightarrow \begin{cases} N_1 I_1 = R_1(\Phi_1+\Phi_2) + R_3\Phi_1 \\ N_1 I_1 - N_2 I_2 = R_1(\Phi_1+\Phi_2) + R_2\Phi_2 \end{cases}$$

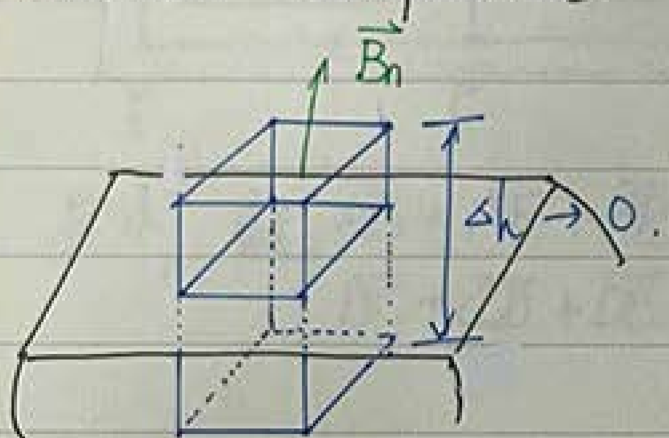
$$\Rightarrow \Phi_1 = \frac{R_2 N_1 I_1 + R_1 N_2 I_2}{R_1R_2 + R_1R_3 + R_2R_3} \quad ※$$

§6-10 靜磁場邊界條件

Q：考慮兩材質（導磁係數分別為 μ_1, μ_2）之邊界，則磁場在邊界上如何分佈？



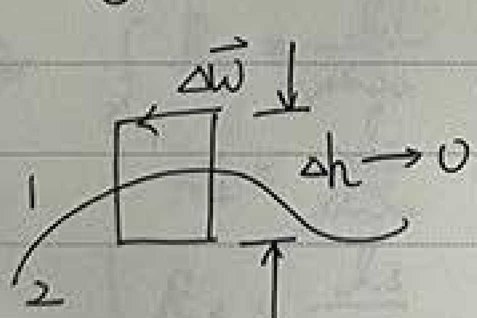
1. Normal Component.



取一極小之盒子

$$\nabla \cdot \vec{B} = 0 \Rightarrow \begin{cases} B_{1n} = B_{2n} \\ \mu_1 H_{1n} = \mu_2 H_{2n} \end{cases}$$

2. Tangentical component



$\because \vec{\nabla} \times \vec{H} = \vec{J} =$ free current

$\therefore \oint \vec{H} \cdot d\vec{l} = I = \vec{H_1} \cdot \Delta\vec{w} - \vec{H_2} \cdot \Delta\vec{w} = \underbrace{J_{sn}}_{\downarrow} \Delta\vec{w}$

$\Rightarrow (H_{1t} - H_{2t}) = J_{sn}$

$\Rightarrow \hat{a}_{n2} \times (\vec{H_1} - \vec{H_2}) = \vec{J_s}$

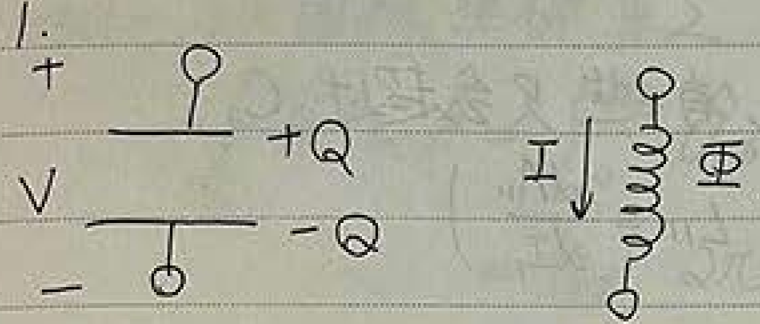
界面上垂直於边線
C的面电流密度.

＊結論：在兩介質之边界上

1. normal: $B_{1n} = B_{2n}$, $\mu_1 H_{1n} = \mu_2 H_{2n}$
2. Tangent: $\hat{a}_{n2} \times (\vec{H_1} - \vec{H_2}) = \vec{J_s}$

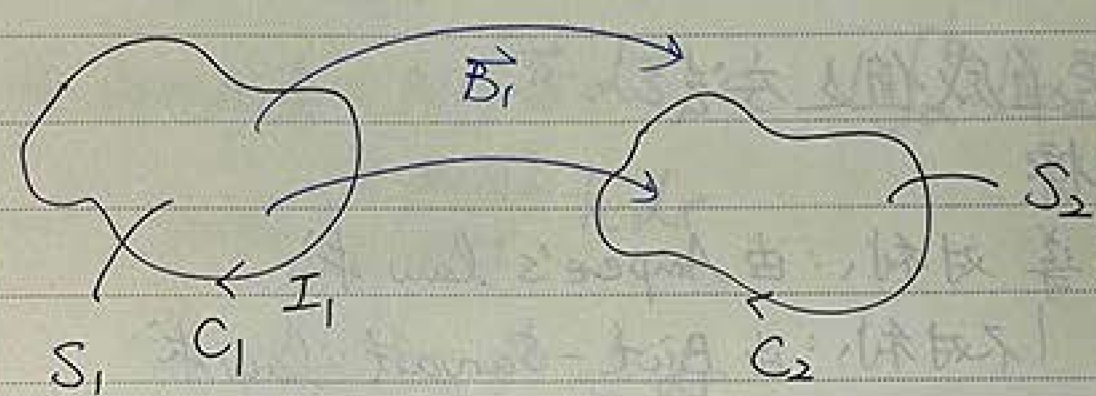
§6-11 Inductor

1.



$Q=CV$ $\qquad$ $\Phi = LI$

2.



(1) ∵ I_1 in $C_1 \xrightarrow{\text{產生}} B_1$

$\xRightarrow{\text{產生}}$ flux Φ_{12} in C_2.

∴ $\Phi_{12} = \oint_{S_2} \vec{B}_1 \cdot d\vec{s} \propto I_1$

↳ by Biot-Sarvat rule

⇒ $\Phi_{12} = L_{12} I_1$. 稱比例常數 L_{12} 為 C_1, C_2 之間的互感 (mutual inductance), 單位: H.

(2). 若 C_2 有 N_2 匝, 則 Φ_{12} 之磁通匝連數 (flux linkage)

$\Lambda_{12} = N_2 \Phi_{12}$, 單位: Wb

⇒ general form: $L_{12} = \dfrac{\Lambda_{12}}{I_1}$ → 線性介質, or

$L_{12} = \dfrac{\Delta\Lambda_{12}}{\Delta I_1} = \dfrac{d\Lambda_{12}}{dI_1}$ (for 非線性介質)

3. Self inductance

I_1 所產生的磁通量中，有一些只會通過 C_1

$$L_{11} = \frac{N_1 \Phi_1}{I_1} = \frac{\Lambda_{11}}{I_1} \quad \left(\text{or } L_{11} = \frac{d\Lambda_{11}}{dI_1}\right)$$

其中 $\begin{cases} \Phi_{11} = L_{11} I_1 \\ \Lambda_{11} = N_1 \Phi_1 \end{cases}$

4. 求电感器自感值之方法

(1) 選坐標

(2) 求 $\vec{B}$ $\begin{cases} \text{对称：由 Ampere's law 求} \\ \text{不对称：由 Biot-Savart law 求} \end{cases}$

(3) 求 $\Phi = \int_{S_1} \vec{B} \cdot d\vec{S}$

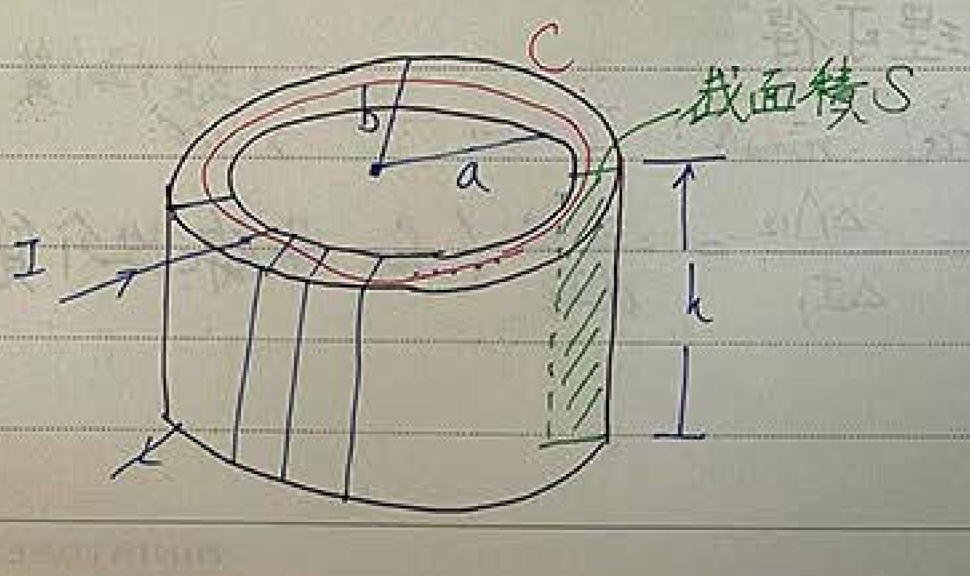
(4) $\Lambda = \Phi \cdot N$ (N=匝數)

(5) 由 $L = \Lambda / I$ 求 L.

若求互感 $L_{12} \Rightarrow$ 对另一線圈之 S_2 積分以求 $\Phi_{12} \Rightarrow L_{12} \Rightarrow \Lambda_{12}$

(3)

ex6-14. 有一环形框架，導磁係數為 μ_0. 設其內徑為 a，外徑為 b，若 高 h，繞 N_1 匝線圈於其上，並通以电流 I，求其自感？



-sol-. 以圓柱坐標算之

$$\oint_C \vec{B}\cdot d\vec{l} = \int_0^{2\pi} \hat{a}_\phi B_\phi \cdot \hat{a}_\phi r d\phi$$

$$= B_\phi \cdot 2\pi r$$

$$= \mu_0 N I$$

$$\Rightarrow B_\phi = \frac{\mu_0 N I}{2\pi r} \Rightarrow \vec{B} = B_\phi \hat{a}_\phi$$

$$\Rightarrow \Lambda = N\Phi = N\int \vec{B}\cdot d\vec{s} = N\int_a^b B_\phi \underbrace{h\,dr}_{ds}$$

$$= N\int_a^b \frac{\mu_0 N I}{2\pi r} h\,dr$$

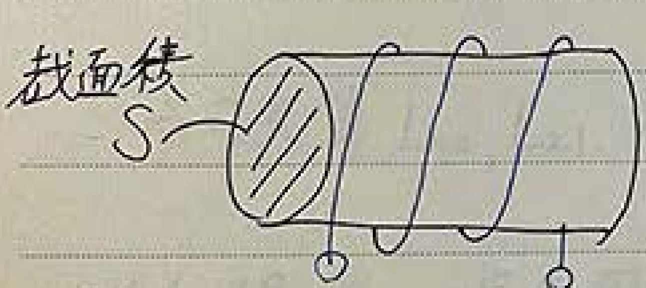
$$= \frac{\mu_0 N^2 h}{2\pi}\int_a^b \frac{1}{r} dr$$

$$= \frac{\mu_0 N^2 h}{2\pi} \ln\frac{b}{a} \quad \#$$

→ 可知多繞10倍 ⇒ L值差100倍!

ex 6-15. solenoid

一長空心螺線管上每單位長了 n匝線圈,試求每單位長度之电感 (空氣導磁係數 $= \mu_0$)



-sol-. ∵ $B = \mu_0 n I$ (by ex 6-3) 為一定值

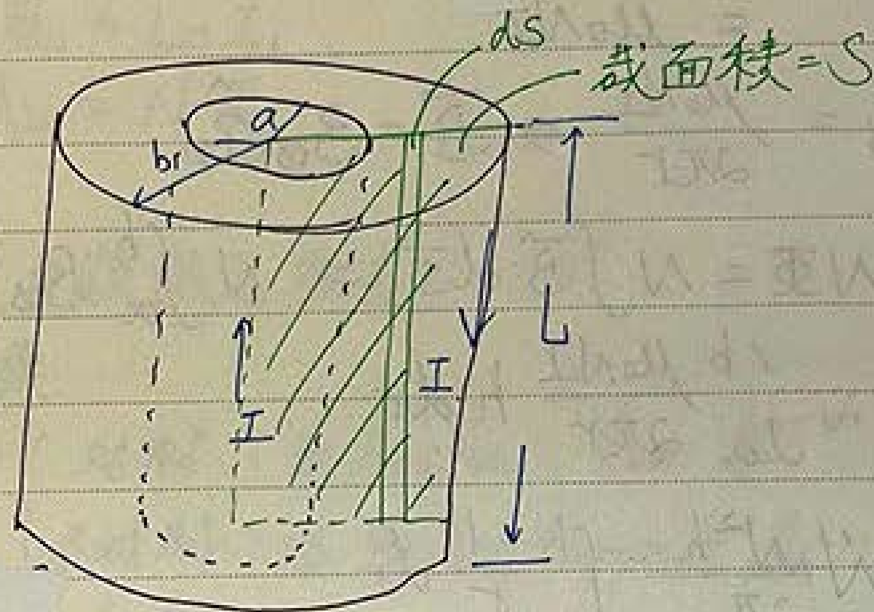
∴ $\Phi = \mu_0 n S I$

⇒ self inductance per unit length:

$$\Lambda' = n\Phi = \mu_0 n^2 S I$$

$$\Rightarrow L' = \mu_0 n^2 S \ (H/m) \quad \#$$

ex6-16. 有一同軸电缆，內導体半徑 a，外層薄導体殼之半徑為 b. 內外層間為空気，且內外兩層載反向电流 I. 求此傳輸線空気部份之电感？



-sol- $\because \oint_C \vec{B}\cdot d\vec{l} = B_\phi \cdot 2\pi r = \mu_0 I$，其中 C 為半徑 $b>r>a$ 之圓圈

$\therefore B_\phi = \dfrac{\mu_0 I}{2\pi r}$

$\Rightarrow \Phi = \int \vec{B}\cdot d\vec{S}$

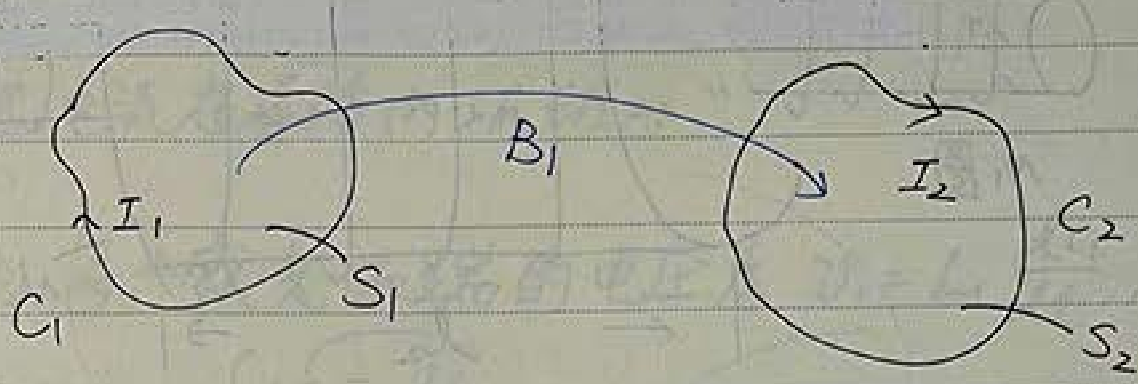
$= \int_a^b \dfrac{\mu_0 I}{2\pi r}\, dr \cdot L$

$= \dfrac{\mu_0 I L}{2\pi} \ln \dfrac{b}{a}$

$\Rightarrow L = \dfrac{\Phi}{I} = \dfrac{\mu_0 L}{2\pi} \ln \dfrac{b}{a}$ 爻.

* 包含內導体之情形有美難懂，請見課本 p277 - p288.

5. 兩迴路之互感



線圈1在線圈2上的flux：$\Phi_{12}=L_{12}I_1$

"　2"　1"　：$\Phi_{21}=L_{21}I_2$

Q: Does $L_{12}=L_{21}$?

A: Yes!

∵ $L_{12}=\frac{N_2}{I_1}\int_{S_2}\vec{B}_1\cdot d\vec{S}_2=\frac{N_2}{I_1}\oint_{C_2}\vec{A}_1\cdot d\vec{l}_2$，且

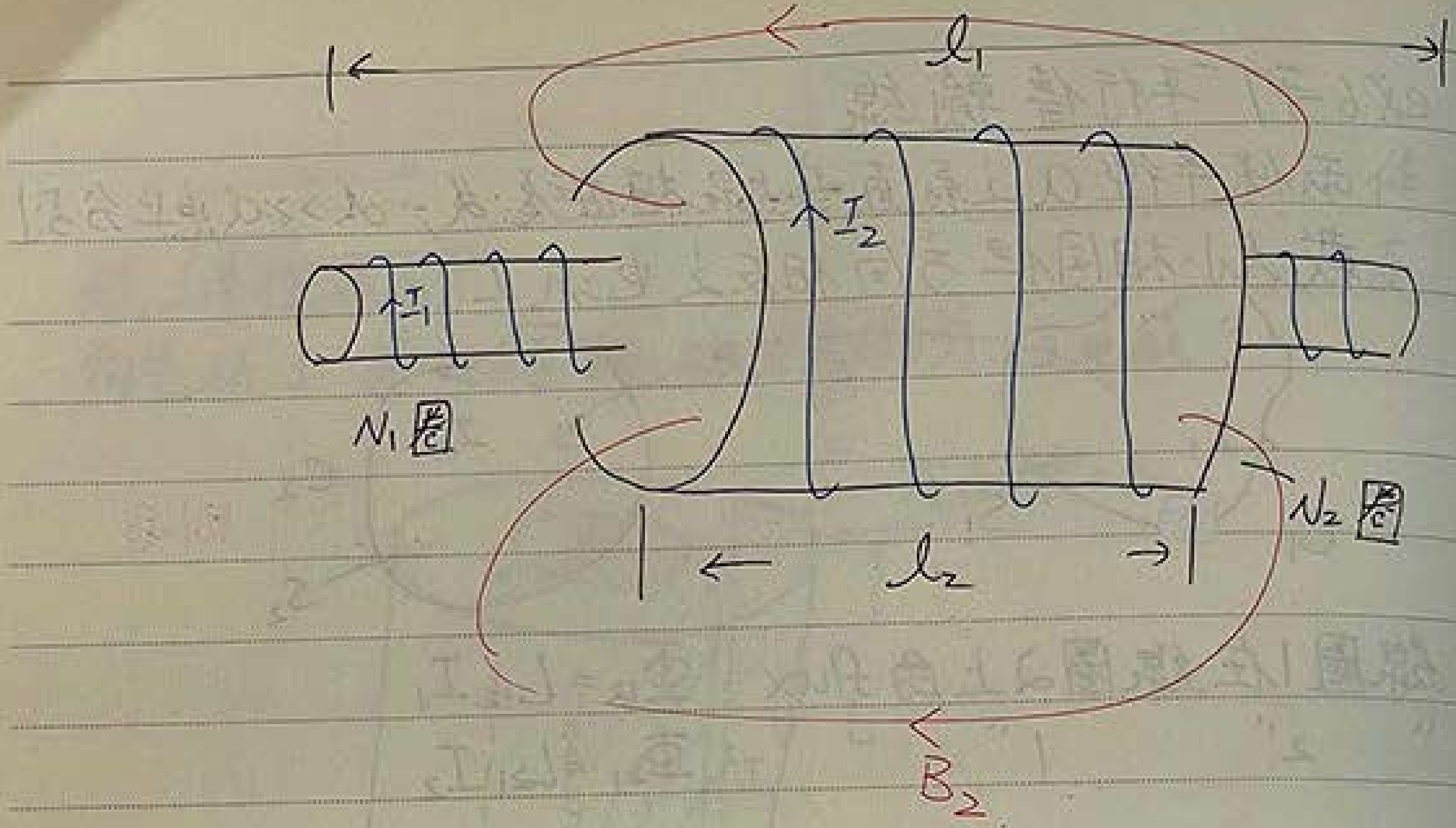
$$\vec{A}_1=\frac{\mu_0 N_1 I_1}{4\pi}\oint_{C_1}\frac{d\vec{l}_1}{R}$$

∴ $L_{12}=\frac{\mu_0 N_1 N_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{l}_1\cdot d\vec{l}_2}{R}=L_{21}$

symmetric!

⇒可只算 L_{12}, L_{21} 中 較好算者以求互感

ex 6-18. 有一長直圓柱形鐵芯（導磁係數$=\mu$）半徑為a，長l_1，上繞N_1匝線圈．有另一空心螺線管半徑比a大，長$l_2(<l_1)$，上繞N_2匝線圈．今把鐵芯穿入螺線管中心，求兩者之互感值？



-sol- ∵ The flux in coil2 due to coil1

$\triangleq \Phi_{12} = \mu n_1 I_1 = \mu \frac{N_1}{l_1} \pi a^2 \cdot I_1$

∴ $\Lambda_{12} = N_2 \Phi_{12} = \frac{\mu}{l} N_1 N_2 \pi a^2 I_1$

$\Rightarrow L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu}{l_1} N_1 N_2 \pi a^2$ #

§6-12 磁能

1. 把一群电荷組合在一起→要做功→功以电能的形式儲存

把电流在導体內流动→〃 →〃 磁〃

⇒ 令电感气兩端的电压為 $v_1 = L_1\frac{di_1}{dt}$，則需

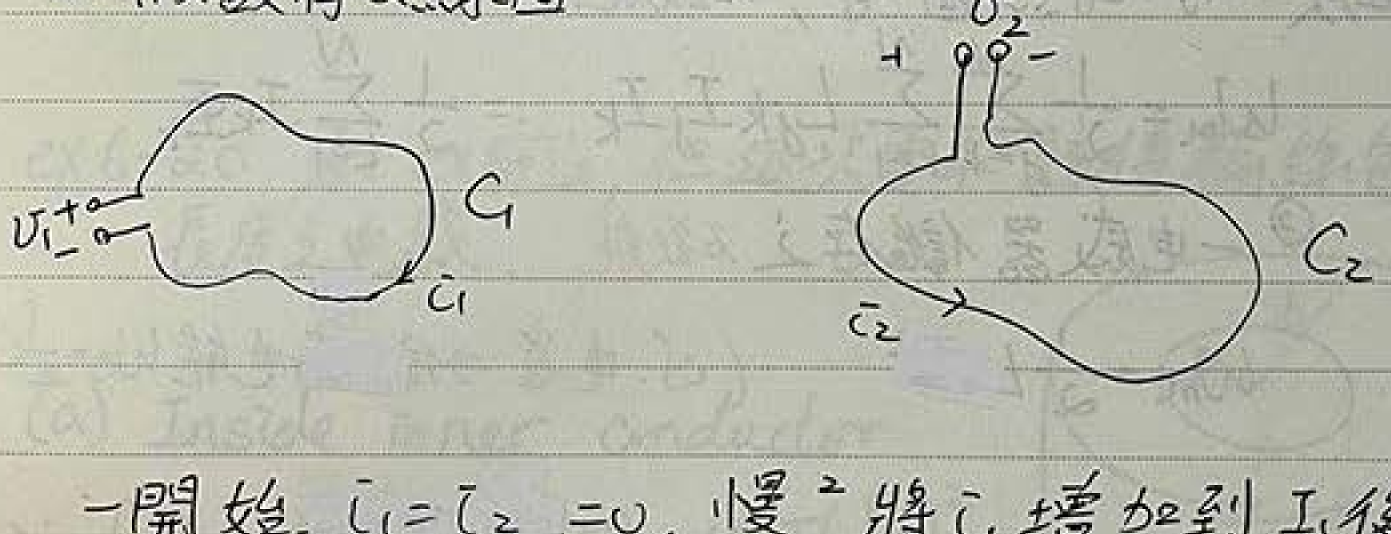
作功 $W_1 = \int v_1 i_1 \, dt$

$$= L_1\int_0^{I_1} i_1 \, di_1$$

$$= \frac{1}{2}L_1 I_1^2$$

2. 对線性介質而言，$W_1 = \frac{1}{2}L_1 I_1^2 \overset{L_1=\frac{\Phi_1}{I_1}}{=} \frac{1}{2}I_1\Phi_1^2$

3. 假設有2線圈



一開始 $i_1=i_2=0$，慢慢將 i_1 增加到 I_1 後，再慢慢將 i_2 增加到 I_2（維持 $i_1=I_1$），求所需之功？

(a) $i_2=0$, $i_1=0\to I_1$：需作功 $W_{11}=\frac{1}{2}L_{11}I_1^2$

(b) $i_2=0\to I_2$, $i_1=I_1$ in C_1.

∵ C_1, C_2 之間互感

∴ 產生感应电动勢，但要維持 $i_1 = I_1$

⇒ 需被电压 $V_{21} = L_{21}\frac{di_2}{dt}$ 抵消.

$$\Rightarrow W_{21} = \int V_{21} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$$

(C_2 对 C_1 作功)

(c) In C_2，為抵消感应电动勢 & $i_2 = 0 \to I_2$

⇒ 需作功 $W_{22} = \frac{1}{2} L_2 I_2^2$.

$$\Rightarrow W_2 = W_{11} + W_{12} + W_{22}$$

需作的功 $= \frac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_{22} I_2^2$ $\quad = \frac{L_{12}+L_{21}}{2} I_1 I_2$

‖

(以)磁能(的形式保存) $= \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} L_{jk} I_j I_k$

4. 推廣：① 有 n 个線圈時，磁能

$$\boxed{W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k}$$

② 一电感器儲存之磁能

$$\boxed{W_m = \frac{1}{2} L I^2}$$

(cf. 电容 ⇒ 儲存电能 $W_e = \frac{1}{2}CV^2$)

5. 以場量表示磁能

recall: $W_e = \frac{1}{2}\int_{V'} \vec{D}\cdot\vec{E}\, dv' = \int_{V'} w_e\, dv'$

⇒ by a similar process in Ch3 (see p296 in textbook), we may also write

$$W_m=\frac{1}{2}\int_{V'} \vec{A}\cdot\vec{J}\,dv' = \frac{1}{2}\int_{V'}\vec{B}\cdot\vec{H}\,dv' = \int_{V'} w_m\,dv'$$

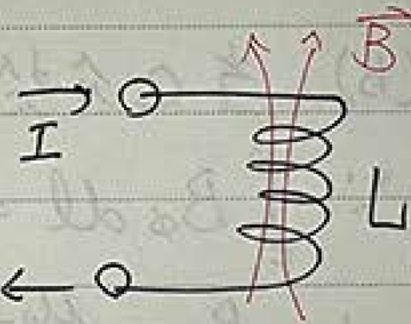
→ 迴路之体積 or $\vec{J}$所在之線性介質之体積.

recall 电能密度 $w_e=\frac{1}{2}\vec{D}\cdot\vec{E}$，故也可定義一磁能密度 $w_m=\frac{1}{2}\vec{B}\cdot\vec{H}=\frac{1}{2}\mu H^2=\frac{1}{2}\frac{B^2}{\mu}$

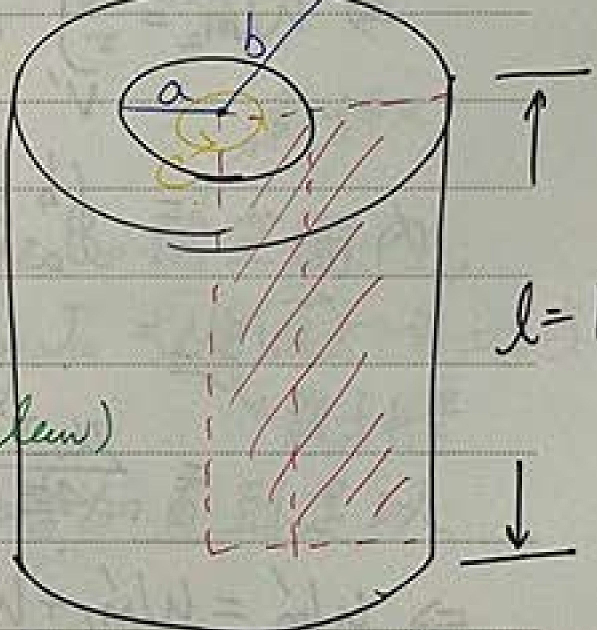
⇒ 只要能求出$\vec{B}$這个向量場，就可以求出 W_m（而不必先求$\vec{A}$）

6. 用磁能算自感

⇒ $L=\frac{2W_m}{I^2}$ → 先求出 W_m 可免去由磁通匝數算L之苦!!



ex6-20. 承 ex6-16，但這次要求該傳輸線每單位長度之电感.



(a) Inside inner conductor

$$\because \oint_C B_\phi\,dl = \mu_0 I(r) = \mu_0 I\left(\frac{r}{a}\right)^2$$

$$= B_\phi(r)\cdot 2\pi r \quad \text{(by Ampere's law)}$$

$$\therefore B_\phi = \frac{\mu_0 r I}{2\pi a^2}$$

內導體所儲存之能量

$$\Rightarrow W_{m_1} = \int_{V'} w_m \, dv'$$

$$= \int_{V'} \frac{B^2}{2\mu_0} dv'$$

$$= \frac{1}{2\mu_0} \int_0^a B_{\phi_1}^2 \cdot \underbrace{\boxed{2\pi r dr}}_{= dv'}$$

dv' ($\because$ unit length $\therefore l=1$)

$$= \frac{\pi r}{\mu_0} \int_0^a \frac{\mu_0 r I}{2\pi a^2} \cdot dr$$

$$= \frac{\mu_0 I^2}{16\pi}$$

(b). 当 C 介於 $r=a$ 与 $r=b$ 之間，

$$\because \int_C B_\phi \, dl = \mu_0 I = B_\phi \cdot 2\pi r \quad \text{(by Ampere's law)}$$

$$\therefore B_\phi = \frac{\mu_0 I}{2\pi r}$$

$$\Rightarrow w_{m_2} = \frac{B_\phi^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2}$$

$$\Rightarrow W_{m_2} = \int_{V'} w_{m_2} \, dv'$$

$$= \int_a^b \frac{\mu_0 I^2}{8\pi^2 r^2} \cdot \underbrace{\boxed{2\pi r dr}}_{= dv'}$$

dv' ($\because$ unit length)

$$= \cdots$$

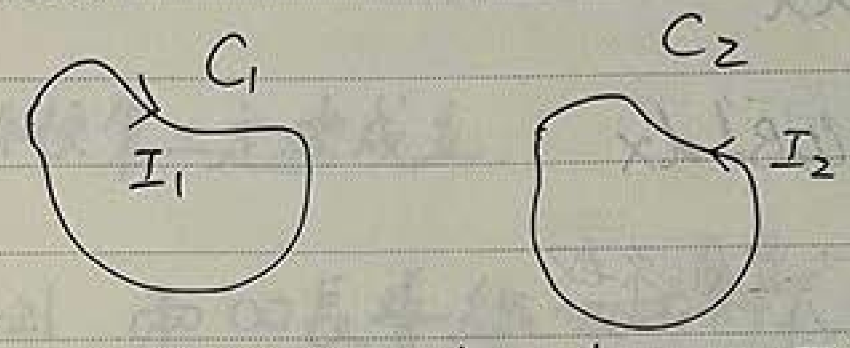
$$= \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}$$

$$\Rightarrow W_m' = W_{m_1}' + W_{m_2}'$$

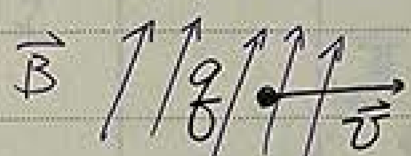
$$\Rightarrow L' = \frac{2W_m'}{I^2} = \underbrace{\frac{\mu_0}{8\pi}}_{\text{internal}} + \underbrace{\frac{\mu_0}{2\pi} \ln \frac{b}{a}}_{\text{external}} \quad \#$$

§6-13 磁力与磁矩

1. recall

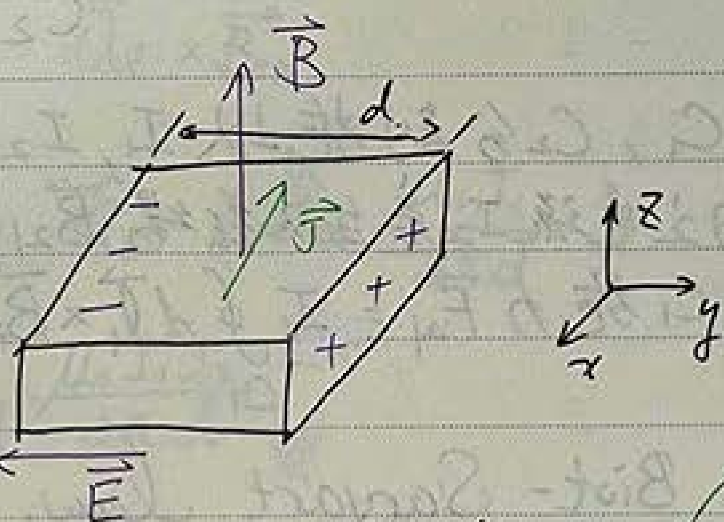


For N loop, $W_m = \frac{1}{2}\sum_k I_k \Phi_k$



$F_m = q\vec{v}\times\vec{B}$

2. 霍爾效應



載子數

假設一導体放在磁場為 $\vec{B}=\hat{a}_z B_0$ 的空間內,

令其上有一沿 y 方向之电流 $\vec{J}=\hat{a}_y J_0 = Nq\vec{u}$ → 載子速度

載子电量

⇒ 磁場想讓 e^- 往 +x 方向移动

⇒ 造成橫电場 $\vec{E}_h$, $\vec{E}_h$ 会逐漸增加直到足以阻止 e^- 飄移

⇒ 穩態時, $\vec{E}_h + \vec{u}\times\vec{B} = 0$

⇒ 稱 $\vec{E}_h = -\vec{u}\times\vec{B}$ 為霍爾电場 (Hall field)

⇒ 此 $\vec{E}_h$ 造成之电压為

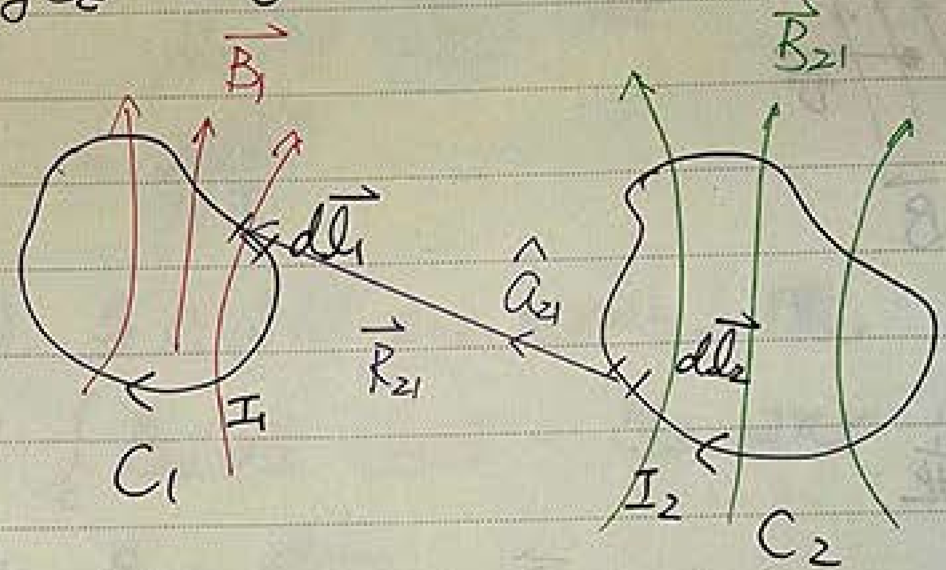
$$V_h = -\int_0^d E_h \, dx$$

$$= -\int_0^d u_0 B_0 \, dx$$

$$= u_0 B_0 d$$ → 霍爾电压

$$\frac{E_x}{J_y B_z} = \frac{1}{Nq}$$ 称霍爾系数.

3.



兩線圈 C_1，C_2 分别施以 I_1，I_2 之电流，

則 C_2 内之电流 I_2 產生磁場 $\vec{B_{21}}$

⇒ C_2 对 C_1 施力 $\vec{F_{21}} = I_1 \oint_{C_1} d\vec{l_1} \times \vec{B_{21}}$

∵ From Biot-Sarvart law,

$$\vec{B_{21}} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\vec{l_2} \times \hat{a}_{R_{21}}}{R_{21}^2}$$

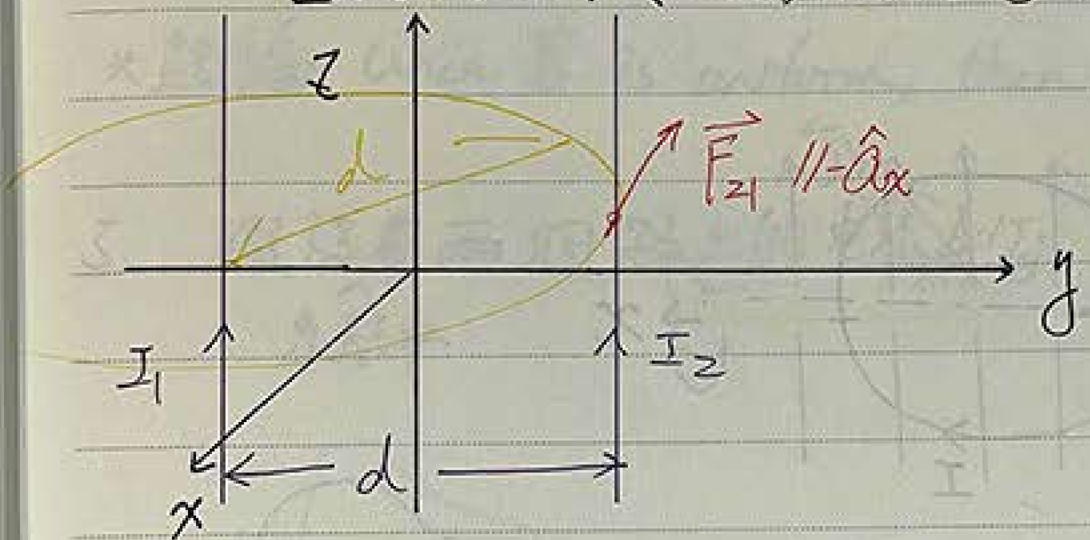
$$\therefore \ \vec{F_{21}} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l_1} \times (d\vec{l_2} \times \hat{a}_{R_{21}})}{R_{21}^2}$$

⇒ 同理可推得

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1}\oint_{C_2} \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \hat{a}_{R_{21}})}{R_{21}^2} = -\vec{F}_{21}$$

⇒ 牛頓第三定律成立.

ex6-21. 兩∞長導線彼此平行, 分別通以同向電流 I_1, I_2. 令2線間距為 d, 求兩線間每單位長度作用力?



-sol-. $F_{21} = I_2 \cdot \hat{a}_z \times \vec{B}_{12}$

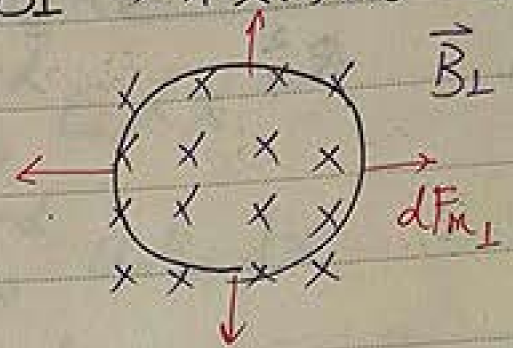
$$= I_2 \cdot \hat{a}_z \times \left(-\hat{a}_x \frac{\mu_0 I_1}{2\pi d}\right)$$

$$= -\hat{a}_y \frac{\mu_0 I_1 I_2}{2\pi d}$$ ※

4. 一圓形迴路, 半徑 b, 通以电流 I 置於磁場 $\vec{B}$ 之中. 現假設 $\vec{B}$ 可被分解為水平分量與垂直分量 $\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$

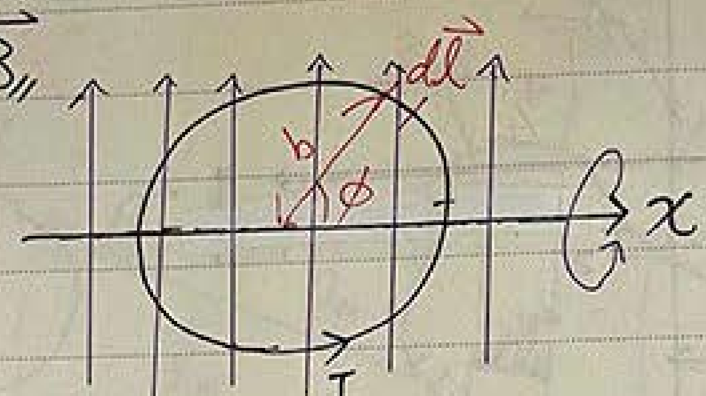
$\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$

(1) For $\vec{B}_\perp$ ⇒ 傾向將迴路撐大 or 縮小.



$$\Rightarrow d\vec{F}_{m\perp} = I\,d\vec{l} \times \vec{B}_\perp$$
$$= \hat{a}_R \cdot I|d\vec{l}|B_\perp$$

(2) For $\vec{B}_{/\!/}$



⇒ 会造成上半面入纸面之力
　　　下〃　出〃

⇒ 迴路会繞 x 軸転动，此時靜力=0 但靜力矩≠0，故不会移动

$$\Rightarrow d\vec{F}_{m/\!/} = I\,d\vec{l} \times \vec{B}_0$$
$$= \begin{cases} I\hat{a}_z, & \text{for upper half} \\ -I\hat{a}_z, & \text{〃 lower〃} \end{cases}$$

令上半部份線圈每一小段 $d\vec{l}$ 所受之向上之力為 $d\vec{F}_1$
〃下〃　　　　　　　　　　　　　　　下〃　$d\vec{F}_2$

$$\Rightarrow \begin{cases} d\vec{F}_1 = I\,d\vec{l}_1 \times \vec{B}_{/\!/} = \hat{a}_z I\,dl\,B_{/\!/}\sin\phi \\ d\vec{F}_2 = I\,d\vec{l}_2 \times \vec{B}_{/\!/} = -\hat{a}_z I\,dl\,B_{/\!/}\sin\phi \end{cases}$$

⇒ $d\vec{F}_1$, $d\vec{F}_2$ 產生之力矩為

$$d\vec{T} = \hat{a}_x\, 2\,dF\, b\sin\phi$$
$$= \hat{a}_x\, 2Ib^2 B_{/\!/} \sin\phi\, d\phi$$

⇒ 總力矩 $T=\int_0^{\pi} d\vec{T}$

$$= \hat{a}_x \overset{=m(\text{磁偶極矩})}{\boxed{I(\pi b^2)}} B_{//}$$

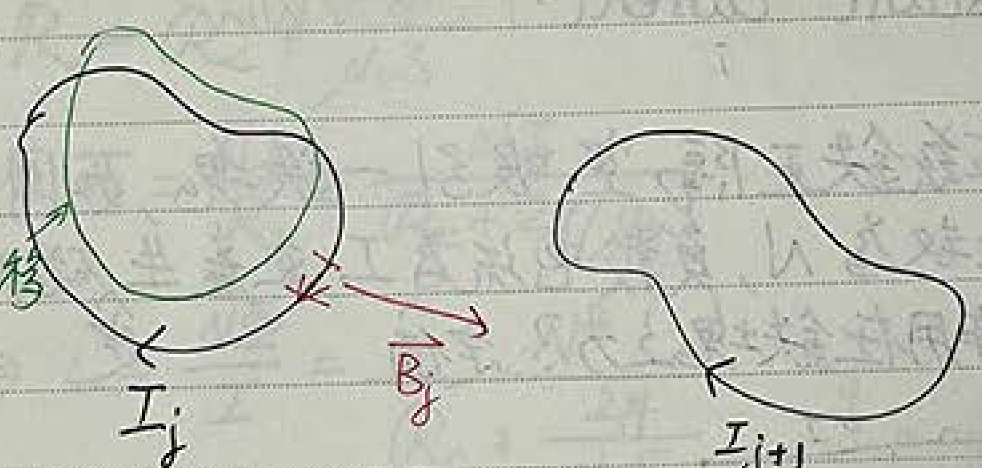
$$= \vec{m}\times\vec{B}_{//}$$

$\vec{m}\times(\vec{B}_{\perp}\times\vec{B}_{//})=\vec{m}\times\vec{B}_{//}$

$$= \vec{m}\times\vec{B} \quad \text{(assume } \vec{B} \text{ is uniform)}$$

*結論: When $\vec{B}$ is uniform, then $\boxed{\vec{T}=\vec{m}\times\vec{B}}$

5. 假設有兩迴路上的电流各 I_j, I_{j+1}
系統中



⇒ $W_m=\frac{1}{2}\sum_{j=1}^{n} I_j \Phi_j$

(a) 若迴路各 constant flux ⇒ 不会產生感应电动勢.
令通量之作用力各 $\vec{F}_\Phi$、位移各 $d\vec{l}$，則系統所作之机械功: $\vec{F}_\Phi d\vec{l}+dW_m=0$

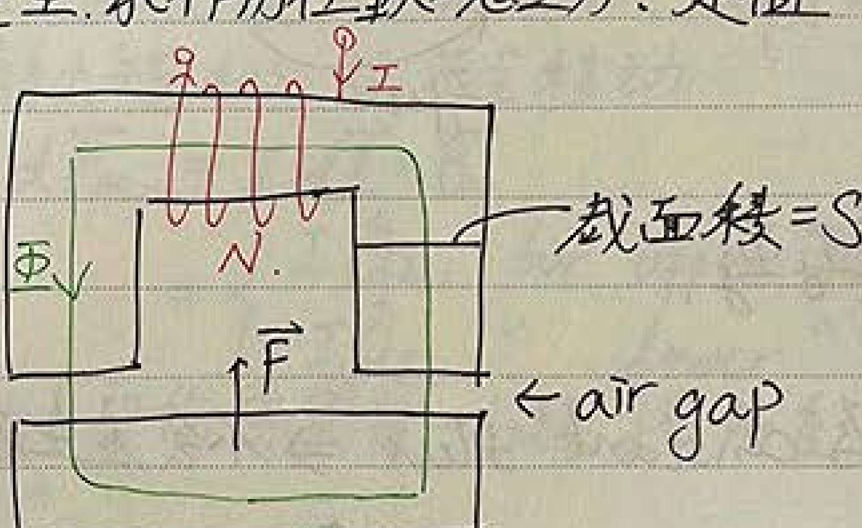
$$\Rightarrow \vec{F}_\Phi\cdot d\vec{l}=-dW_m$$
$$=-\left(W_m(\vec{r}_j+d\vec{l})-W_m(\vec{r}_j)\right)$$
$$=-\vec{\nabla}_j W_m\cdot d\vec{l}$$

⇒ $\vec{F}_{\Phi} = -\nabla W_m$.
其中 $\vec{r}_j$ 表迴圈 j 之位置向量.

(b) 若电流在系統中為常数
⇒ 电流源所作之功 $dW_s = \sum_{k=1}^{n} I_k d\Phi_k = dW + dW_m$
∵ $dW_m = \frac{1}{2}\sum_{k=1}^{n} I_k d\Phi_k = \frac{1}{2} dW_s$
∴ $dW = \vec{F}_I \cdot d\vec{l} = dW_m = \nabla W_m \cdot d\vec{l}$
⇒ $\vec{F}_I = \nabla W_m$

*結論: $\begin{cases} \text{constant flux}: & \vec{F}_{\Phi} = -\nabla W_m \\ \text{constant current}: & \vec{F}_I = \nabla W_m \end{cases}$

ex6-23 有一电磁鐵正隔空吸引一鐵塊,如圖,
設線圈匝数為 N, 負載电流為 I, 並產生定磁通量 Φ. 求作用在鐵塊之力? 定值



-sol-. 〈法I〉∵ constant flux, 令 y 為兩者間距.
∴ $dW_m = dW_{air\,gap}$

$$= 2\left(\frac{\Phi^2}{2\mu_0} dy\right)$$

(6-172)

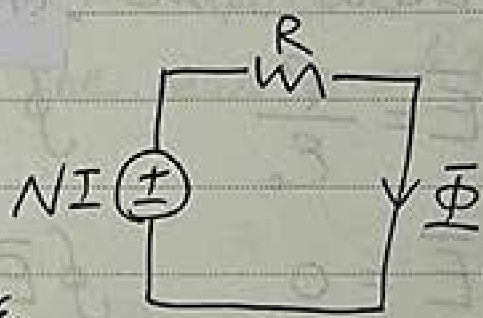
$*W_m = \frac{1}{2}\int_{V'} \frac{B^2}{\mu} dv'$

$$= \frac{\Phi^2}{\mu_0 S}\, dy$$

$$\Rightarrow \vec{F}_\Phi = -\nabla W_m = -\hat{a}_y \frac{dW_m}{dy} = -\hat{a}_y \frac{\Phi^2}{\mu_0 S} \text{ (吸引力)}$$

〈法II〉 ∵ constant current

∴ $\vec{F}_I = \nabla W_m$



令鐵塊之磁阻為 R_c，air gap之磁阻為 $\frac{y}{\mu_0 S}$，則總磁阻為

$$R = R_c + \frac{2y}{\mu_0 S}$$

$$\Rightarrow \Phi = \frac{NI}{R} = \frac{NI}{R_c + 2y/\mu_0 S}$$

$$\Rightarrow L = \frac{N\Phi}{I} = \frac{N^2}{R_c + \frac{2y}{\mu_0 S}}$$

$$\Rightarrow W_m = \frac{1}{2} L I^2 = \frac{N^2 I}{2R_c + \frac{4y}{\mu_0 S}}$$

$$\Rightarrow \vec{F}_I = \nabla W_m$$

$$= \frac{I^2}{2} \nabla L$$

$$= \frac{I^2}{2} \hat{a}_y \frac{dL}{dy}$$

$$= \cdots\cdots = -\hat{a}_y \frac{\Phi^2}{\mu_0 S} \quad \#$$

△總結

电磁学(一)討論的都是静电場与静磁場，也就是$\vec{B}$, $\vec{E}$不隨時間而变动. Ch1－Ch6之理論，都可由以下几條式子推出来：

① Differential form | Integral form | Name

Differential form	Integral form	Name
$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$	$\oint\vec{E}\cdot d\vec{s}=\frac{Q}{\varepsilon_0}$	(Gauss law)
$\vec{\nabla}\times\vec{E}=0$	$\oint\vec{E}\cdot d\vec{l}=0$	
$\vec{\nabla}\cdot\vec{B}=0$	$\oint\vec{B}\cdot d\vec{s}=0$	
$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$	$\oint\vec{B}\cdot d\vec{l}=\mu_0 I$	(Ampere's law)

Ch7. Time Varying Fields

§7.1 Introduction

1. 静电、磁場的基本假設

$$\begin{cases} \vec{\nabla}\times\vec{E}=0 & \text{(conservative field)} \\ \vec{\nabla}\cdot\vec{D}=\rho & \text{(Charge is the source of } \vec{E}\text{)} \end{cases}$$

$$\begin{cases} \vec{\nabla}\cdot\vec{B}=0 & \text{(no magnetic monopole)} \\ \vec{\nabla}\times\vec{H}=\vec{J} & \text{(Current is the source of } \vec{H}\text{)} \end{cases}$$

2. $$\begin{cases} \vec{D}=\varepsilon\vec{E} \\ \vec{H}=\dfrac{\vec{B}}{\mu} \end{cases}$$

电流帶走电荷

3. $\vec{\nabla}\cdot\vec{J}+\dfrac{\partial\rho}{\partial t}=0$ (continuity eq. / charge neutrality)

⇒ 某地方消失的电荷一定会出現在其它地方

4. Lorentz force : $\vec{F}=q(\vec{E}+\vec{u}\times\vec{B})$

5. $$\begin{cases} \vec{J}=\sigma\vec{E} \\ \vec{J}=\rho\vec{u} \end{cases}$$ $\vec{J}=\rho\vec{u}$ → 知道电荷的速度，就知道电流

电場 ($\vec{E}$) → 电荷運动 ($\vec{u}$) → 电流 ($\vec{E}$)

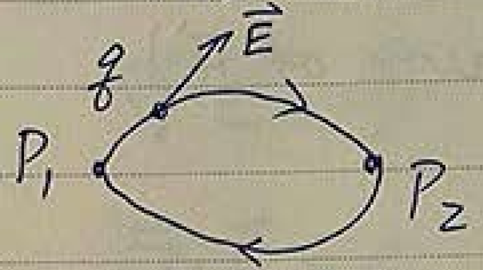
6. $$\begin{cases} \vec{\nabla}\times\vec{E}=0 \longleftrightarrow \vec{E}=-\vec{\nabla}V \\ \vec{\nabla}\cdot\vec{B}=0 \longleftrightarrow \vec{B}=\vec{\nabla}\times\vec{A} \end{cases}$$

以上実驗整理出之公理 & 定義可推 Ch2~6 .

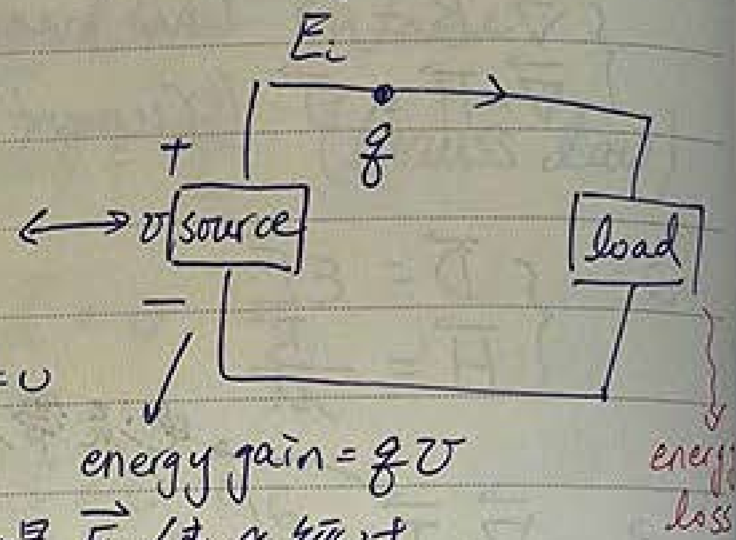
§ 7-2 Faraday's Law

1. Recall:

物理量	單位
$\vec{E}$	N/C ; V/m
V	V
emf (v)	V



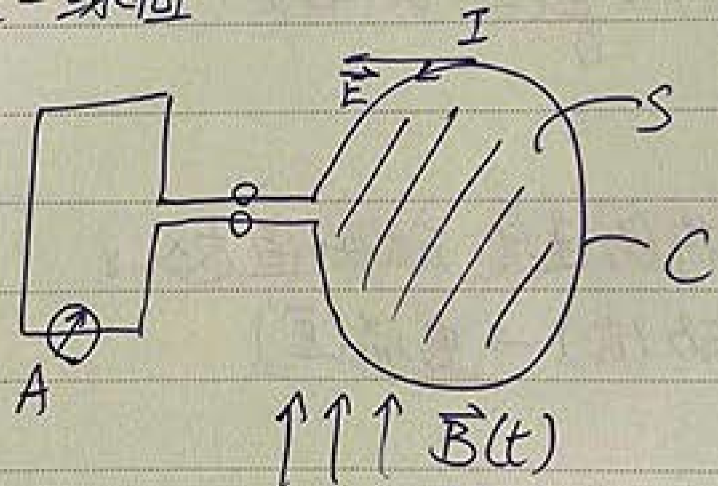
conservative, 繞一圈做功=0



→ 右圖中可假設有一等效电場 $\vec{E}_i$ 使 q 經过 source 之後得到 qv 之能量

$\Rightarrow$ emf $v \triangleq \int_{-}^{+} \vec{E}_i \cdot d\vec{l}$

2. 考慮一線圈



$$\Rightarrow \begin{cases} \text{emf}: v = \oint_C \vec{E} \cdot d\vec{l} \\ \Phi(t) = \int_S \vec{B}(t) \cdot d\vec{S} \end{cases}$$

(1) 由实驗可知 $v = -\dfrac{d\Phi}{dt}$ (Faraday's law of electromagnetic induction)

⇒ 电場与磁場在時变下是有關的!

⇒ 時变电場 & 時变磁場所感应出之电場
非保守場!

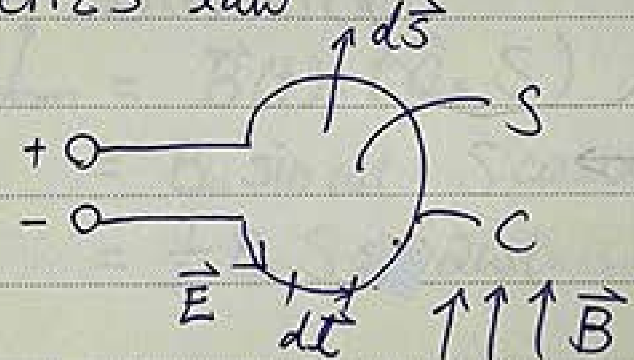
⇒ fundamental postulate 需修正

∵ Integral form: $v = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$

又 $\oint_C \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\, d\vec{s}$

∴ $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (electromagnetic induction)

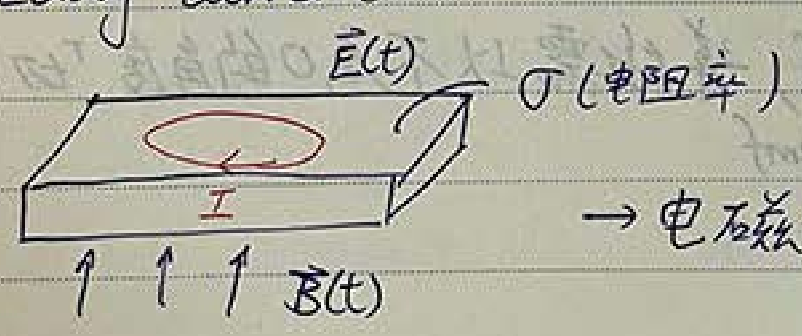
(2) Lenz's law



If $\vec{B}\uparrow \Rightarrow \Phi\uparrow \Rightarrow v<0$

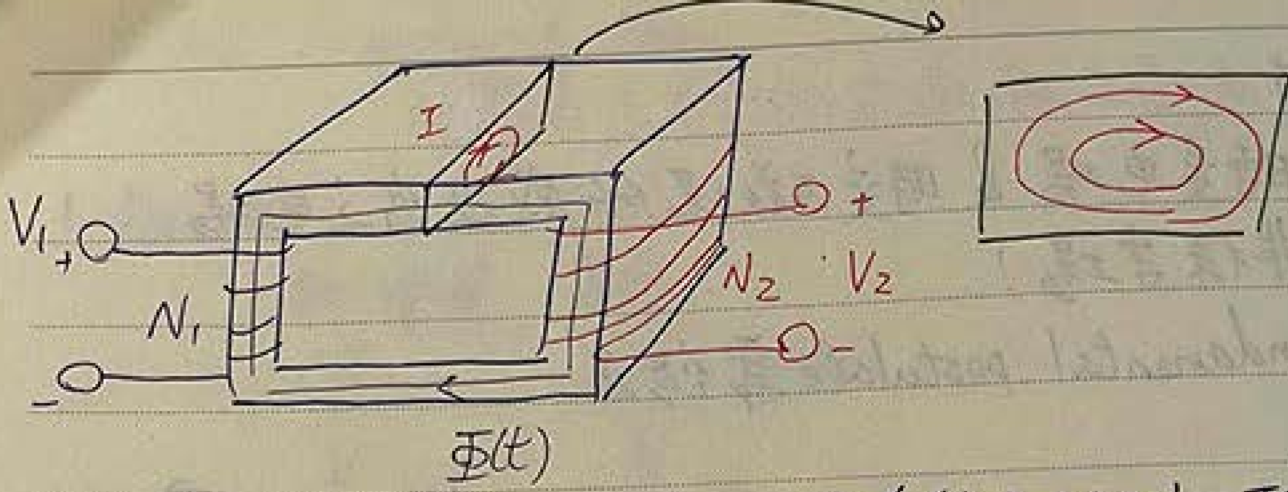
⇒ induced $\vec{E}$, I is negative

(3) Eddy current



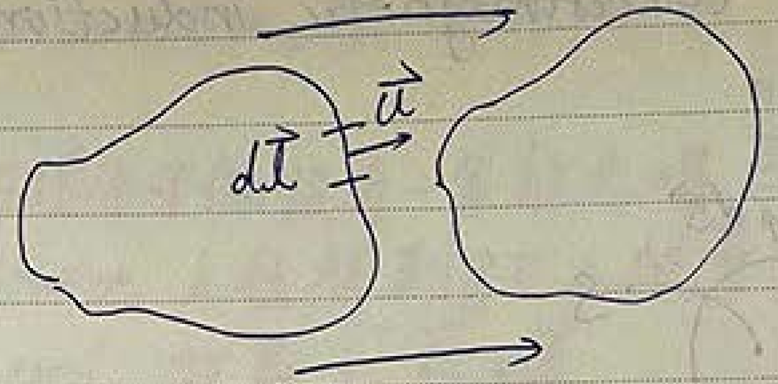
→ 电磁炉

transformer (变压器)



⇒ 使可能絕緣但又是良好磁導体之材料，可降低 transformer 之發熱

3. 若線圈与 $\vec{B}$ 都是時变



路徑之移动速度為 $\vec{u}$

$$v = \oint \vec{E} \cdot d\vec{l}$$

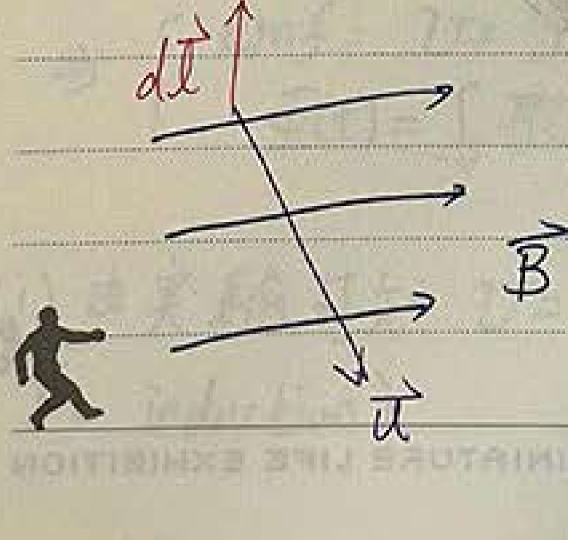
$$= \cdots$$

$$= -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C \vec{u} \times \vec{B}\, d\vec{l}$$

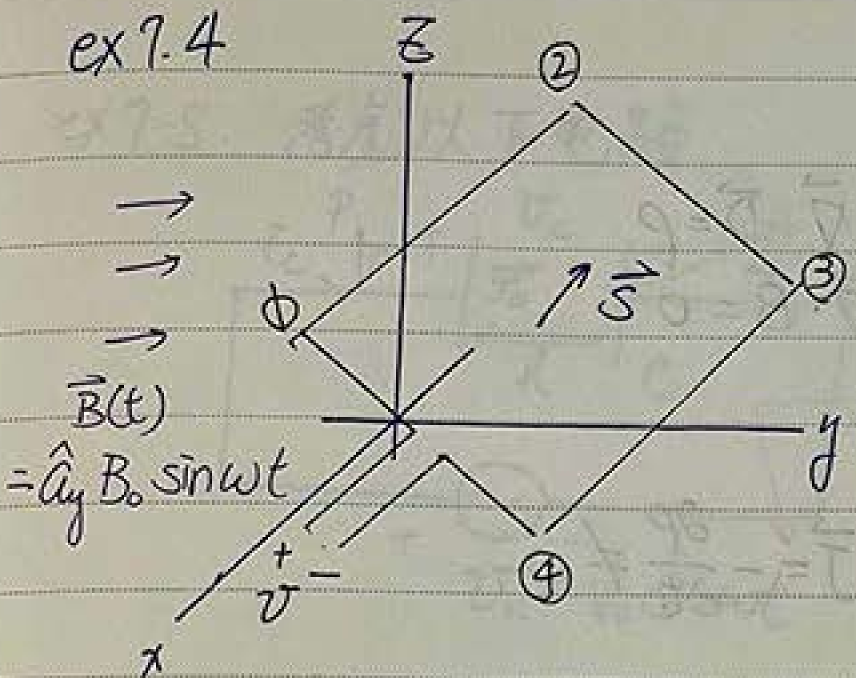
$$\triangleq v_a + v_a'$$

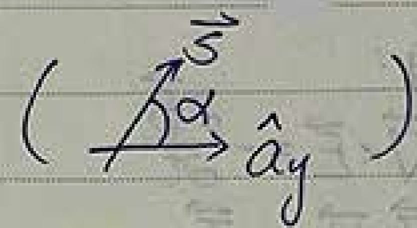
(transformer emf) (motional emf)

要產生 motional emf，導線需以不為0的角度「切開」$\vec{B}$ 才能產生 motional emf



ex 7.4





求 $\Phi(t) = ?$ $v = ?$

-sol-. no motional emf in $\overline{②③}$, $\overline{①④}$!

$$\Phi(t) = \vec{B}(t) \cdot \vec{S}(t)$$

uniform $= \vec{B}(t) \cdot (\hat{a}_n S)$

$$= B_0 \sin\omega t \cdot S\cos\alpha$$

$$= \frac{1}{2} B_0 S \sin 2\omega t$$

$$\Rightarrow v = -\frac{d\Phi}{dt} = -B_0 S \omega \cos 2\omega t \quad \#$$

§7-3 Maxwell equation

1. 目前:

$$\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{H} = \vec{J} \end{cases} \qquad \begin{cases} \vec{\nabla}\cdot\vec{D} = \rho \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

and $\vec{\nabla}\cdot\vec{J} + \dfrac{\partial\rho}{\partial t} = 0$

可得 $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})}_{\equiv 0} = \vec{\nabla}\cdot\vec{J} = -\dfrac{\partial\rho}{\partial t} \neq 0$

⇒ 矛盾! 必有缺項

$$\because \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0 = \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}$$
$$= \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})$$
$$= \vec{\nabla}\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$\therefore \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \triangleq \vec{J} + \vec{J}_D$$

$\vec{J}_D \triangleq \dfrac{\partial\vec{D}}{\partial t}$ = displacement current (單位: A/m^2)

物理-\-\-: ① 時变磁場 → 時变电場 → 电流效应 → 產生磁場

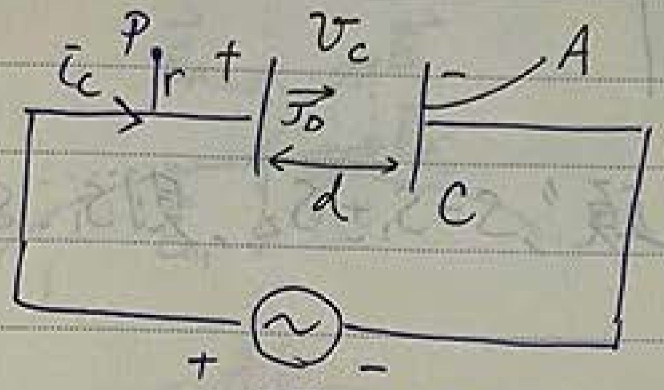
② 時变电場与电流等效

③ 电場、磁場只要其中一个是時变的，就会感应出另一者，然後又感应出另一者

2. Maxwell equation

$$\begin{cases} \vec{\nabla}\cdot\vec{D} = \rho \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \end{cases}$$

ex7-5. 考虑以下电路



$v_c = V_0 \sin\omega t$

(a) 求 i_c = ? (b) 找距導線 r 的位置 P 上之磁場強度?

-sol-. (a) $i_c = C_1 \cdot \frac{dv_c}{dt} = C_1 V_0 \omega \cos\omega t$

(b) 令金层板面積 = A, 距離 = d

∵ C_1 中間的电場会隨時間而变

∴ 有 $\frac{\partial \vec{D}}{\partial t}$ ⇒ 有 $\vec{J_D}$

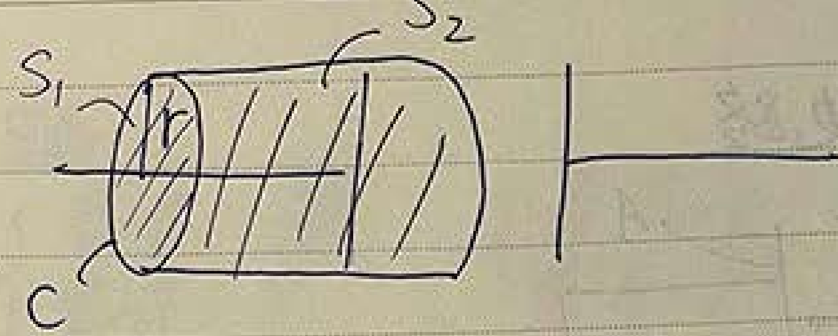
$$\Rightarrow i_d = \int_A \frac{\partial \vec{D}}{\partial t} d\vec{s}$$

$$= \int_A \frac{\partial (\epsilon \vec{E})}{\partial t} d\vec{s}$$

$$= \frac{\epsilon}{d} \cdot A \cdot \frac{\partial v_c}{\partial t}$$

$$= C_1 V_0 \omega \cos\omega t = i_c$$

⇒ total current $i = i_c + i_d$ is conti. along the circuit



考虑一"向左打開之鍋頭" $S=S_1 \cup S_2$，則 S_1、S_2 的边界都是C

(i) 对 S_1 積分：

$$\oint_C \vec{H}\cdot d\vec{l} = I_C\Big|_{S_1} + \cancel{I_D}\Big|_{S_1}$$ 0

$$\Rightarrow H\cdot 2\pi r = i_c \Rightarrow H = \frac{i_c}{2\pi r}$$

(b) 对 S_2 積分：

$$\oint_C \vec{H}\cdot d\vec{l} = \cancel{I_C}\Big|_{S_2} + I_D\Big|_{S_2}$$ 0

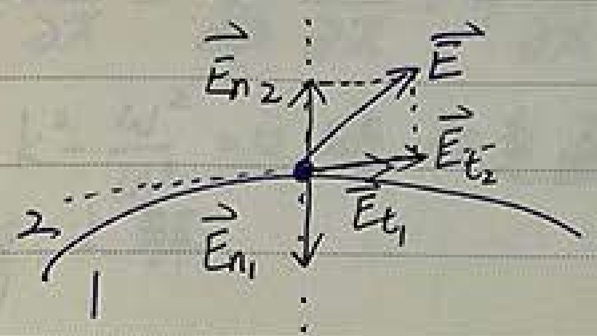
$$\Rightarrow H\cdot 2\pi r = i_d = i_c$$

$$\Rightarrow H = \frac{i_c}{2\pi r}$$

⇒無論用 S_1 还是 S_2 算，H都是 $\frac{C_1 V_0}{2\pi r}\omega\cos\omega t$，符合預期

§7.5 電磁边界條件

1. 考虑两介質 1, 2



表面

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$ 体積内的source

$$\begin{cases} E_{1t}=E_{2t}\rightarrow\dfrac{D_{1t}}{D_{2t}}=\dfrac{\varepsilon_1}{\varepsilon_2} \\ H_{1t}=H_{2t}\rightarrow\dfrac{B_{1t}}{B_{2t}}=\dfrac{\mu_1}{\mu_2} \\ D_{1n}=D_{2n}\rightarrow\varepsilon_1E_{1n}=\varepsilon_2E_{2n} \\ B_{1n}=B_{2n}\rightarrow\mu_1H_{1n}=\mu_2H_{2n} \end{cases}$$

§7.6 Wave equation

1. 設 $\rho = 0$

$$\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} = -\mu\dfrac{\partial\vec{H}}{\partial t} \\ \vec{\nabla}\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t} = \varepsilon\dfrac{\partial\vec{H}}{\partial t} \\ \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{H} = 0 \end{cases}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$= -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\because \vec{\nabla}\times\vec{\nabla}\times\vec{E} = \nabla(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$$

$$\therefore \nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

let $u = \dfrac{1}{\sqrt{\mu\varepsilon}}$, then $\boxed{\nabla^2\vec{E} - \dfrac{1}{u^2}\dfrac{\partial^2\vec{E}}{\partial t^2} = 0}$

wave equation

Similarly,

$$\nabla^2\vec{H} - \frac{1}{u^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

2. $\nabla^2 f - \dfrac{1}{u^2}\dfrac{\partial^2 f}{\partial t} = 0$ 稱為波方程式

$(f = f(x, y, z, t))$

3. Let $f = f(x, t)$

$\rightarrow f'' k^2 - \dfrac{1}{u^2} f'' \omega^2 = 0$

Let $f(x,t) = f(\phi) \triangleq f(kx \pm \omega t)$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{df}{d\phi}\frac{\partial \phi}{\partial x} = f\frac{\partial \phi}{\partial x}$$

$\Rightarrow k^2 - \frac{\omega^2}{u^2} = 0$ 時，任意函数 $f(kx \pm \omega x)$ 都是 wave eq. 之解

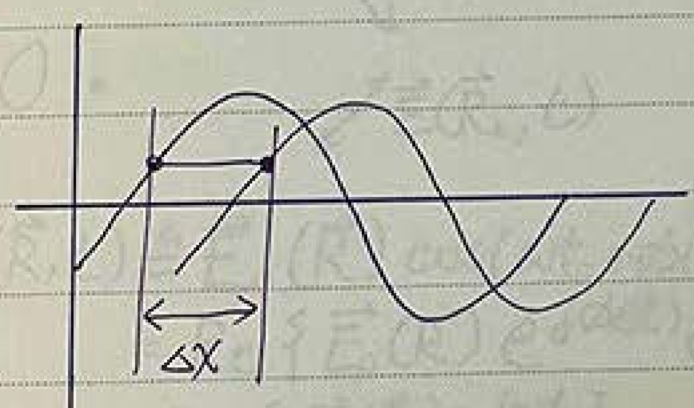
ex. Let $f(x,t) = f_0 \sin(kx - \omega t)$

$\omega = 2\pi f = \frac{2\pi}{T}$ = freq. in space

λ = wavelength = period in space

4. Phase velocity

$\sin(kx - \omega t) \triangleq \sin \underset{相位}{\underline{\phi}}$



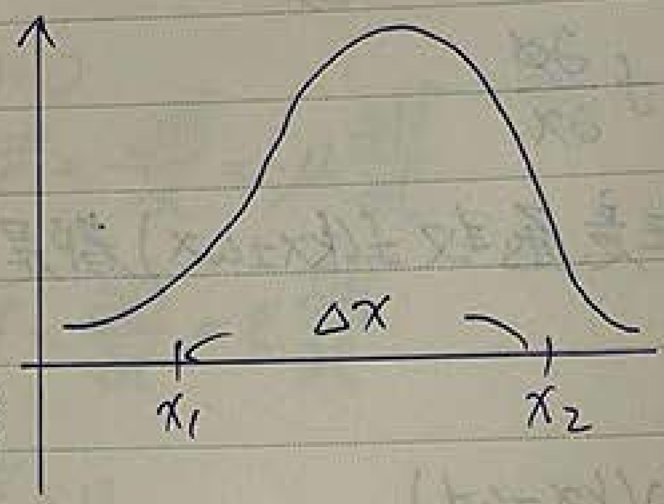
phase velocity $v_p = \left.\frac{\Delta x}{\Delta t}\right|_{\phi = kx - \omega t = const}$

$$= \left.\frac{dx}{dt}\right|_{\phi = const}$$

$$= \frac{\omega}{k}$$

$= \pm u \begin{cases} +: 往「+」向傳播 \\ -: 〃 \; - \; 〃 \end{cases}$

5.

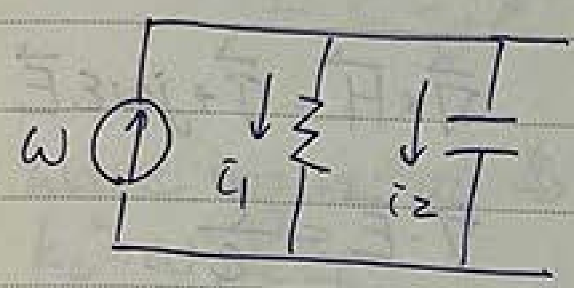


x_2看到之波形

$$
\begin{aligned}
f(kx_2-\omega t) &= f(kx_1-\omega t+kx_2-kx_1)\\
&= f\left(kx_1-\omega t+\frac{\omega}{u}\Delta x\right)\\
&= f\Big(kx_1-\omega\big(t-\underbrace{\frac{\Delta x}{u}}_{\triangleq\Delta t}\big)\Big)\\
&= f(kx_1-\omega(t-t'))
\end{aligned}
$$

§7-7 時間諧和場

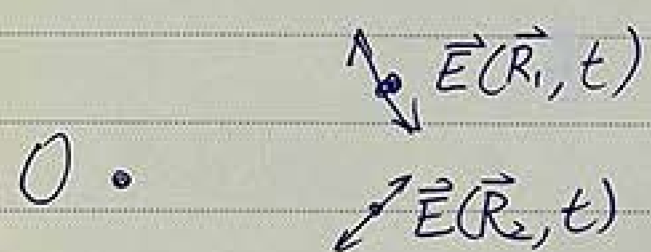
1. 考慮电路



$$i_1(t) = I_{01}\cos(\omega t+\phi_1)$$
$$= \mathrm{Re}\{I_{01}e^{j\phi_1}e^{j\omega t}\}$$

$$\Rightarrow \left.\begin{aligned} I_1 &\triangleq I_{01}e^{j\phi_1} \\ I_2 &\triangleq \underbrace{I_{02}}_{\text{振幅}}\, e^{j\underbrace{\phi_2}_{\text{相位}}} \end{aligned}\right\}\ \text{phasor}$$

2. 可將空間中每一个点对应到一phasor



$$\vec{E}(\vec{R},t) \triangleq \vec{E}_0(\vec{R})\cos(\omega t+\phi(\vec{R}))$$
$$= \mathrm{Re}\{\vec{E}_0(\vec{R})e^{j\phi(\vec{R})}e^{j\omega t}\}$$
$$= \mathrm{Re}\{\vec{E}(\vec{R})e^{j\omega t}\}$$

$$\vec{E}(\vec{R}) \triangleq \vec{E}_0(\vec{R})e^{j\phi(\vec{R})}$$
= electric field phasor
$$= \hat{a}_x E_x(\vec{R}) + \hat{a}_y E_y(\vec{R}) + \hat{a}_z E_z(\vec{R})$$

$$E_x(\vec{R},t) = \mathrm{Re}\{E_x(\vec{R})e^{j\omega t}\}$$

instantaneous (time domain)

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

Phasor form

$$\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

(1) $\boxed{\frac{\partial}{\partial t} \to j\omega}$ $\left(\because \frac{\partial}{\partial t}e^{j\omega t} = j\omega e^{j\omega t}\right)$

phasor 跟原本的field在不搞混的前提下用同一符号

4. Source free wave eq.

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\xrightarrow{\text{phasor}} \begin{cases} \nabla^2\vec{E} + k^2\vec{E} = 0 \\ k^2 = \omega^2\mu\varepsilon = \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda}\right)^2 \\ \nabla^2\vec{H} + k^2\vec{H} = 0 \end{cases}$$

5. $\vec{J} = \sigma\vec{E}$

$$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$= (\sigma + j\omega\varepsilon)\vec{E}$$

$$= j\omega\underbrace{\left(\varepsilon + \frac{\sigma}{j\omega}\right)}_{\triangleq\, \varepsilon_c}\vec{E},$$

where $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega}$

$= \text{complex permittivity}$

$\triangleq \varepsilon' - j\varepsilon''$

$\Rightarrow$ wave equation 為 $\nabla^2\vec{E} + k_c^2\vec{E} = 0$

$k_c^2 = \omega^2\mu_c\varepsilon_c$

6. loss tangent

$\tan\delta_c = \dfrac{\varepsilon''}{\varepsilon'}$

δ_c = loss angle $\begin{cases} \text{实部} \gg \text{虚部}: \text{良绝缘体} \\ \text{虚}\ \text{"}\ \ \text{实}\ \text{"}\ :\ \text{"}\ \text{導体} \end{cases}$

Ch8 平面電磁波

8-2 在非損耗性介質中的平面波

1. 原本的 form: $\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$

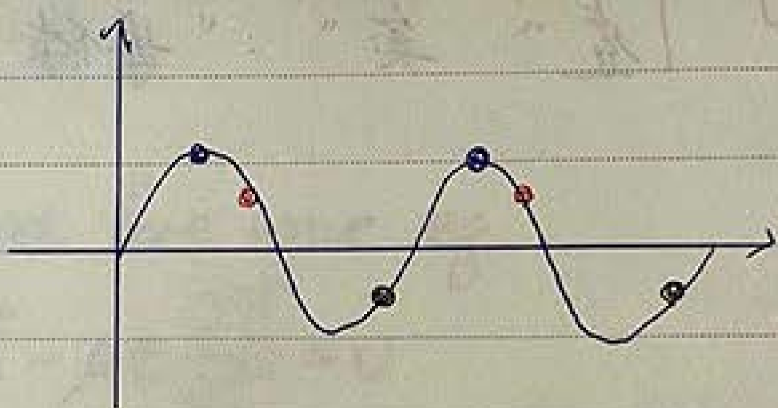
phasor

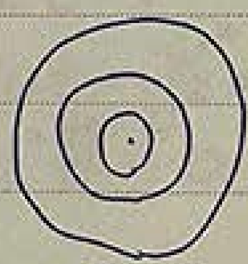
$\Rightarrow \nabla^2\vec{E} + k_0^2\vec{E} = 0$

$$\Rightarrow \begin{cases} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k_0^2\right)E_x(x,y,z)=0 \\ \text{〃} \qquad\qquad\qquad\qquad E_y \text{〃} \\ \text{〃} \qquad\qquad\qquad\qquad E_z \text{〃} \end{cases}$$

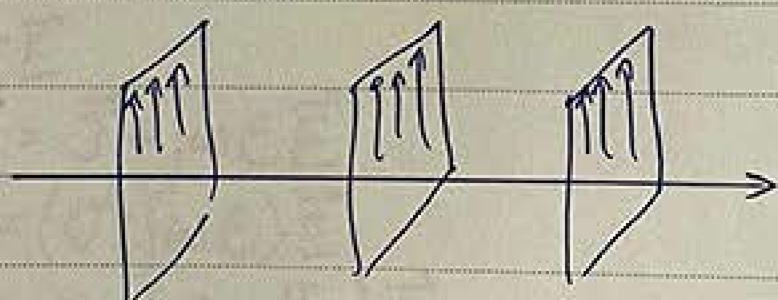
2. Wavefront：所有跟此點相同相位的點集合

ex.





水波 → wavefront 為圓形



波前為平面
→ 平面波

3. Uniform plane wave

考慮一組均勻平面波，在 xy 平面上以均勻之 E_x 描述。

即：設

$$\frac{\partial E_x}{\partial x}=0,\quad \frac{\partial E_y}{\partial y}=0$$ (方便起見，可設 $E_y=E_z=0$)

$$\Rightarrow \frac{d^2E_x(z)}{dz^2}+k_0^2E_x(z)=0$$

先假設是0。

$$\Rightarrow E_x(z)=E_0^+e^{-jk_0z}+E_0^-e^{-jk_0z}$$

$$\triangleq E_x^+(z)+E_x^-(z)$$

$$\Rightarrow E_x^+(z,t)=Re\{E_0^+e^{j(\omega t-k_0z)}\}$$ → 原本的 $E_x^+(z,t)$，非phasor

$$=E_0^+\cos(\omega t-k_0z)$$

定義 phase velocity 為

$$u_p\triangleq\left.\frac{dz}{dt}\right|_{\phi=\omega t-kz=\text{const}}$$

$$\underset{\substack{\uparrow\\ \text{在3中}}}{=}\frac{\omega}{k_0}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=c$$ (光速)

4. 求此時之 $\vec{H}$ 為？

由maxwell eq. 求 $\vec{H}$ 之phasor

$$\vec{\nabla}\times\vec{E}=-j\omega\mu_0\vec{H}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x(z) & 0 & 0\end{vmatrix}$$

$$\Rightarrow\begin{cases}H_x=0\\ H_y=-\dfrac{1}{j\omega\mu_0}\dfrac{\partial E_x(z)}{\partial z}\\ H_z=0\end{cases}$$

For $+z$ wave

$$\Rightarrow H_y^+(z) = -\frac{1}{j\omega\mu_0}(-jk_0)E_0^+ e^{-jk_0 z}$$

$$= \frac{k_0}{\omega\mu_0} E_x^+(z)$$

Let $\eta_0 \triangleq \sqrt{\frac{\mu_0}{\varepsilon_0}} \doteq 120\pi \doteq 377(\Omega)$

then $H_y^+(z) = \frac{E_x^+(z)}{\eta_0}$

η_0 稱為自由空間之本質阻抗 (intrinsic impedance of the free space)，為一 real const.

① 物理意義：电場与磁場振幅比例

② η_0 為电磁波行進之阻抗，但物理義意与电阻不同！

亦即：$\eta_0 = \frac{E_x^+(z)}{H_y^+(z)} = \frac{E_0^+}{H_0^+}$

5. 磁場 $\vec{H}(z,t) = ?$

$$\vec{H}(z,t) = \hat{a}_y H_y^+(z,t)$$
$$= \hat{a}_y \, Re\{H_y(z)e^{j\omega t}\}$$
$$= \cdots$$
$$= \hat{a}_y \underbrace{\frac{E_0^+}{\eta_0}}_{\triangleq H_0^+} \cos(\omega t - k_0 z)$$

结論：
$$\begin{cases} \vec{E}(z,t) = \hat{a}_z E_0^+ \cos(\omega t - k_0 z) \\ \vec{H}(z,t) = \hat{a}_y H_0^+ \cos(\omega t - k_0 z) \end{cases}$$

ex.8-1. 設 $\varepsilon_r=4,\ \mu_r=1,\ \sigma=0,\ f=100\,MHz$

(a) 求 $\vec{E}(z,t)=?$

(b) 求 $\vec{H}=?$

(c) 當 $t=10^{-8}$, E_x有最大正值之位置

-sol-. $\because k=\omega\sqrt{\mu\varepsilon}$

$$=\omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}$$

$$=\frac{2\pi\times10^8}{3\times10^8}\sqrt{4}$$

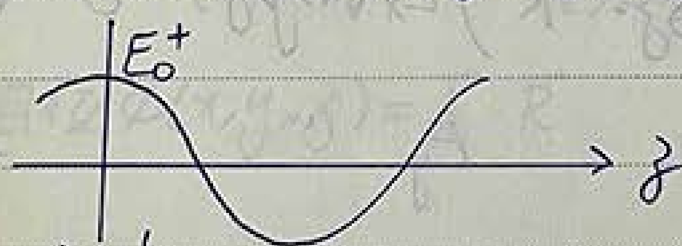
$$=4\pi/3 \ \text{rad/m}$$

$\therefore \lambda = 3/2\ m.$

(a). $\vec{E}(z,t)=\hat{a}_x E_0^+\cos(\omega t-kz)$

(Given at $t=0$ at $z=\frac{1}{8}m$, $E_x=E_{max}=10^{-4}V/m=E_0$)

忘記寫在題目裡了…



$\because \omega t-kz+\psi\big|_{z=\frac{1}{8},\,t=0}=0$

$\therefore \psi=\pi/6$

$\Rightarrow \vec{E}(z,t)=\hat{a}_x 10^{-4}\cos\left(2\pi\times10^8 t-\frac{4\pi}{3}\left(z-\frac{\pi}{8}\right)\right)$

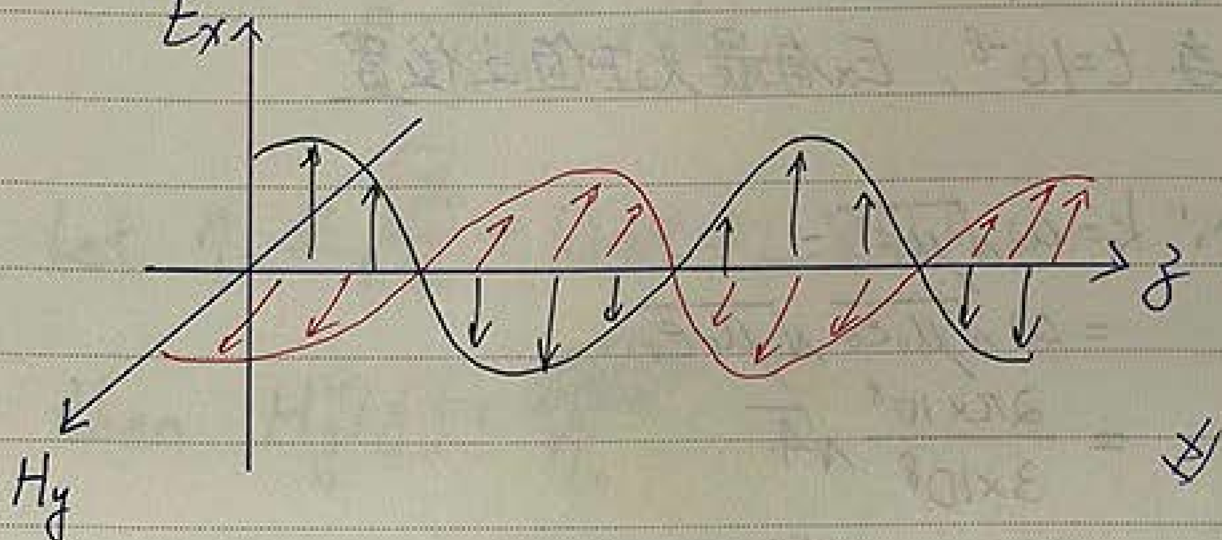
(b) $\because \eta=\sqrt{\frac{\mu}{\varepsilon}}=\sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}}=60\pi\,(\Omega)$

$\therefore \vec{H}(z,t)=\hat{a}_y\frac{10^{-4}}{60\pi}\cos\left(2\pi\times10^8 t-\frac{4}{3}\pi\left(z-\frac{\pi}{8}\right)\right)$

(c) Positive peaks at $t=10^{-8}s$ =正的波峰

$\omega t - k(z-z_0) = \pm 2n\pi$

$\Rightarrow z_n - z_0 = \frac{3}{2} \pm \frac{3}{2}n = \frac{3}{2} \pm n\lambda,\ n \in \mathbb{N}$ (m)



6. Wave in 3D

直接解 PDE of phasor.

$\nabla^2 \vec{E} + k^2 \vec{E} = 0$

$$\Rightarrow \begin{cases} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_x(x,y,z) = 0 \quad -① \\ \quad " \qquad E_y \quad " \\ \quad " \qquad E_z \quad " \end{cases}$$

三个PDE

Let $E_x(x,y,z) = X(x)\,Y(y)\,Z(z)$ 代回①

$$\Rightarrow \underbrace{\frac{\frac{d^2X}{dx^2}\cancel{YZ}}{X\cancel{YZ}}}_{-k_x^2} + \underbrace{\frac{\frac{d^2Y}{dy^2}\cancel{XZ}}{\cancel{X}Y\cancel{Z}}}_{-k_y^2} + \underbrace{\frac{\frac{d^2Z}{dz^2}\cancel{XY}}{\cancel{XY}Z}}_{-k_z^2} + k^2 = 0$$

$$\Rightarrow \begin{cases} X'' + k_x^2 X = 0 \\ Y'' + k_y^2 Y = 0 \\ Z'' + k_z^2 Z = 0 \end{cases}, \quad k^2 = k_x^2 + k_y^2 + k_z^2$$

$$\Rightarrow \begin{cases} X = X_0 e^{\overset{\to \text{取“}-\text{”}}{\pm} j k_x x} \\ Y = Y_0 e^{\pm j k_y y} \\ Z = Z_0 e^{\pm j k_z z} \end{cases}, \quad \text{Let } E_{x0} \triangleq X_0 Y_0 Z_0, \text{ then}$$

$$\Rightarrow E_x(x,y,z) = E_{x0}\, e^{-j(k_x x + k_y y + k_z z)}.$$

Let $\vec{k} = (k_x, k_y, k_z)$, $\vec{R} = (x, y, z)$, $E_0 = (E_{x0}, E_{y0}, E_{z0})$

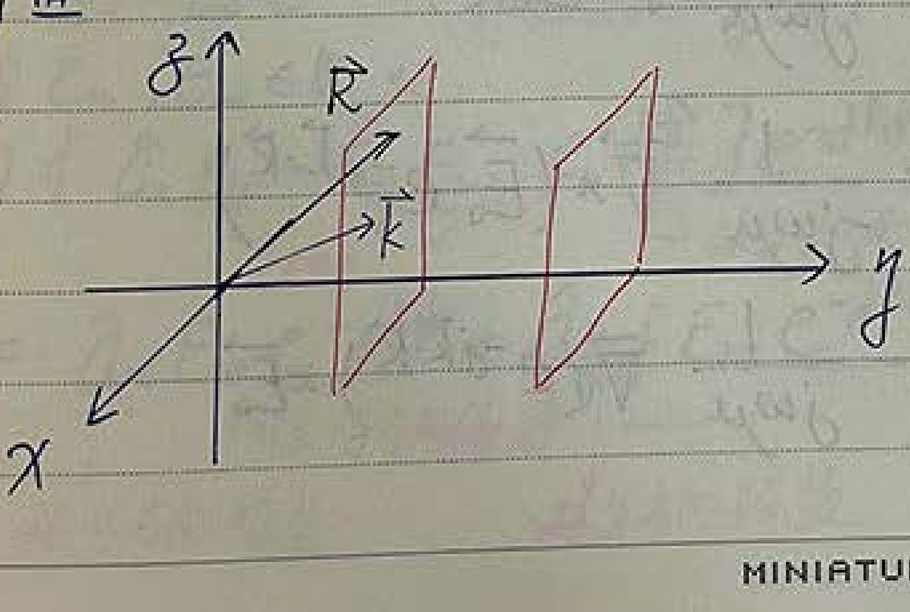
$\Rightarrow$ 同理可得 $E_y(x,y,z)$, $E_z(x,y,z)$，故

$$\left.\begin{aligned} \vec{E} &= \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z \\ &= \cdots \\ &= \vec{E}_0\, e^{-j\vec{k}\cdot\vec{R}} \end{aligned}\right\} \text{phasor!}$$

其中 $\begin{cases} |\vec{k}|^2 = k^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{2\pi}{\lambda}\right)^2 \\ \text{相位 } \phi(x,y,z) = \vec{k}\cdot\vec{R} \end{cases}$

7. 求此時之 wavefront 為？

$\Rightarrow \phi = \vec{k}\cdot\vec{R} = \text{const} = k_x x + k_y y + k_z z$，為一3維空間之平面



$\Rightarrow \vec{R}$(wavefront) = plane perpendicular to $\vec{k}$

$\because \vec{k} = \hat{a}_n k$, $\hat{a}_n$ = propagation direction, and

$|\vec{k}| = k = \omega\sqrt{\mu\varepsilon} = \cdots = \frac{2\pi}{\lambda}$

$\therefore \vec{E}$ is an uniform plane wave

8. 考慮一 source free space，代入則得出之 $\vec{E}(\vec{R})$ (phasor)

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 0 \\
&= \vec{\nabla}\cdot(\vec{E}_0 e^{-j\vec{k}\cdot\vec{R}}) \\
&= \vec{E}_0\cdot\vec{\nabla}(e^{-j\vec{k}\cdot\vec{R}}) \\
&= \vec{E}_0\cdot(-j\vec{k}e^{-j\vec{k}\cdot\vec{R}}) \\
&= -(j\vec{k})\cdot\vec{E}
\end{aligned}
$$

$\Rightarrow \boxed{\vec{k}\cdot\vec{E} = 0}$ for 平面EM wave

9. 求此時之 $\vec{H}(\vec{R}) = ?$

$$
\begin{aligned}
\vec{H}(\vec{R}) &= -\frac{1}{j\omega\mu}\vec{\nabla}\times\vec{E}(\vec{R}) \\
&= \frac{1}{-j\omega\mu}\vec{\nabla}\times(\vec{E}_0 e^{-j\vec{k}\cdot\vec{R}}) \\
&= -\frac{1}{j\omega\mu}\vec{\nabla}(e^{-j\vec{k}\cdot\vec{R}})\times\vec{E}_0
\end{aligned}
$$

$$= \frac{-j\vec{k}}{-j\omega\mu}\underbrace{e^{-j\vec{k}\cdot\vec{R}}\times\vec{E}_0}_{\vec{E}}$$

$$= \frac{k}{\omega\mu}\hat{a}_n\times\vec{E}$$

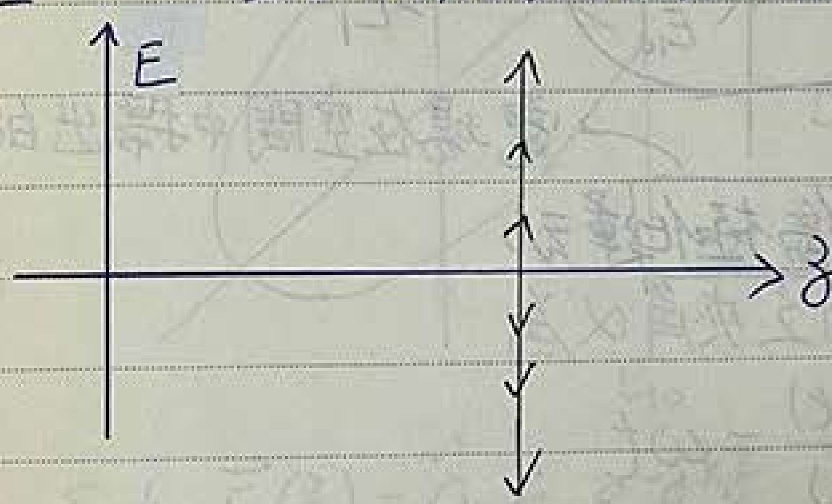
$$= \boxed{\frac{1}{\eta}\hat{a}_n\times\vec{E}} \qquad * \ \frac{|E|}{|H|}=\eta$$

10. Polarization

設 phasor $\vec{E}(z)=\hat{a}_x E_0^+ e^{-jkz}$

⇒ 原本的电場 $\vec{E}(z,t)=\hat{a}_x E_0^+\cos(\omega t-kz)$

物理意義: $\vec{E}$在一條線上來回振盪



ex. 有偏光效果的太陽眼鏡

11. 偏極化運动

設 $\vec{E}(z)=\vec{E}_0 e^{-jkz}=(\hat{a}_x \underline{E_1}+\hat{a}_y \underline{E_2})e^{-jkz}$ complex

則令 $E_{10}, E_{20}\in\mathbb{R}$ s.t.

$\vec{E}(z)=[\hat{a}_x \underline{E_{10}}+\hat{a}_y \underline{E_{20}}\, e^{-j\frac{\pi}{2}}]e^{-jkz}$ real

設y方向與x方向之相位差$\frac{\pi}{2}$

$= \hat{a}_x E_{10}\underline{e^{-jkz}}+\hat{a}_y\, j E_{20}\underline{e^{-jkz}}$

往+z方向傳播

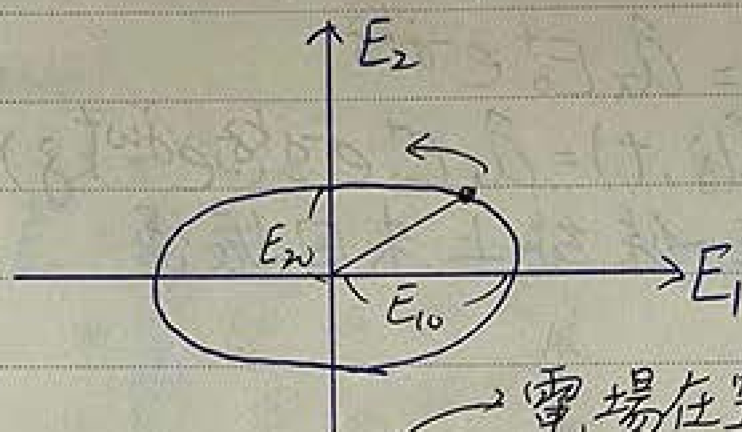
沿x方向振盪　　沿y方向振盪

$$\Rightarrow \vec{E}(z,t) = Re\{(\hat{a}_x E_1(z) + \hat{a}_y E_2(z))e^{j\omega t}\}$$
$$= \hat{a}_x E_{10}\cos(\omega t - kz) + \hat{a}_y E_{20}\cos(\omega t - kz - \tfrac{\pi}{2})$$

at $z=0$,

$$\vec{E}(0,t) = \underbrace{\hat{a}_x E_{10}\cos\omega t}_{E_1(0,t)} + \underbrace{\hat{a}_y E_{20}\sin\omega t}_{E_2(0,t)}$$

$$\Rightarrow \left(\frac{E_1(0,t)}{E_{10}}\right)^2 + \left(\frac{E_2(0,t)}{E_{20}}\right)^2 = 1$$



電場在空間中掃出的軌跡

此稱為橢圓$\boxed{\text{偏極化}}$運动

此時 $\vec{E}$ 與 x 軸之夾角 α 為

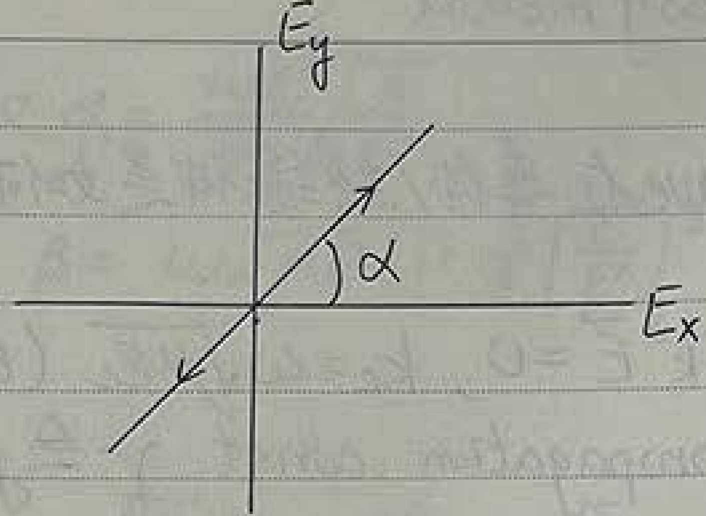
$$\alpha = \tan^{-1}\frac{E_2(0,t)}{E_1(0,t)} = \omega t$$

$\Rightarrow \vec{E}$ 以均勻角速度 ω 逆時針旋轉

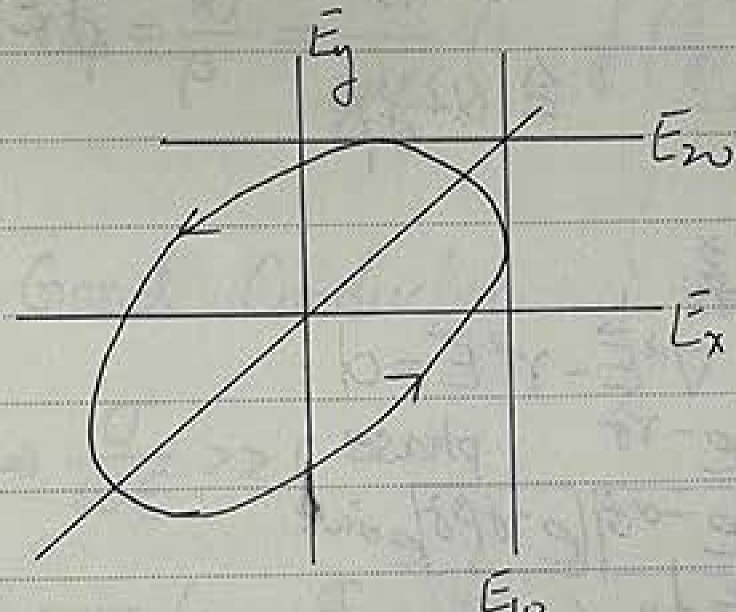
12. 線性偏極化

設 $\begin{cases} \vec{E}(z) = a_x E_{10} e^{-jkz} + \hat{a}_y j E_{20} e^{-jkz} \\ \vec{E}(0,t) = \hat{a}_x E_{10}\cos\omega t + \hat{a}_y E_{20}\cos\omega t \end{cases}$

$$\Rightarrow \alpha = \tan^{-1}\frac{E_{20}\cancel{\cos\omega t}}{E_{10}\cancel{\cos\omega t}} = \text{const}$$



13. For $E_{10} \neq E_{20}$, $\phi \neq 0, \pm\frac{\pi}{2}$, 統為 general elliptical polarization



ex8-3. Set $\vec{E}(z) = \hat{a}_x E_0 e^{-jkz}$

$$= \left(\frac{1}{2}(\hat{a}_x E_0 - \hat{a}_y j E_0) + \frac{1}{2}(\hat{a}_x E_0 + \hat{a}_y j E_0) \right) e^{-jkz}$$

$$\triangleq \underbrace{\vec{E}_{rc}(z)}_{\text{右手圓偏極化}} + \underbrace{\vec{E}_{lc}(z)}_{\text{左"}} \quad \#$$

§8.3 Lossy media

Q: EM wave 在導体/絕緣体是如何傳播的?

1. $\nabla^2\vec{E}+k^2\vec{E}=0$, $k_c=\omega\sqrt{\mu\varepsilon_c}$ (ε_c可以是複數)

定義 propagation const. $\gamma \triangleq jk_c$

$$\gamma_c=jk_c=j\omega\sqrt{\mu\varepsilon_c}=j\omega\sqrt{\mu\varepsilon}\left(1+\frac{\sigma}{j\omega\varepsilon}\right)^{\frac{1}{2}}$$

$$=j\omega\sqrt{\mu\varepsilon'}\left(1+\frac{\varepsilon''}{j\varepsilon'}\right)^{\frac{1}{2}}$$

$$\triangleq \alpha+j\beta$$

α之物理意義:

by wave eq. $\nabla^2\vec{E}-\gamma^2\vec{E}=0$

$$\Rightarrow \vec{E}=\hat{a}_x E_0 e^{-\gamma z}$$

$$=\hat{a}_x E_0 \underbrace{\boxed{e^{-\alpha z}}}_{\text{振幅}}\ \overbrace{\boxed{e^{-j\beta z}}}^{\text{phase}} e^{j\omega t}$$

α 稱衰減常數 (attenuation const.)

$\Rightarrow$ 走了 $1/\alpha$ m 後, 振幅減少 e 倍

§8.3.1. 低損耗性介電質

1. $\varepsilon_c=\varepsilon'-j\varepsilon''=\varepsilon-j\frac{\sigma}{\omega}$, 且 $\varepsilon''\ll\varepsilon'$

$$\Rightarrow \gamma=j\omega\sqrt{\mu\varepsilon'}\left(1-j\underbrace{\frac{\varepsilon''}{\varepsilon'}}_{\ll 1}\right)^{\frac{1}{2}}$$

$$=j\omega\sqrt{\mu\varepsilon'}\left(1-j\frac{\varepsilon''}{2\varepsilon'}+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2+\cdots\right)$$

此時 $\begin{cases} \text{衰減常数 } \alpha = \dfrac{\omega\varepsilon''}{2}\sqrt{\dfrac{\mu}{\varepsilon'}} & (1/m) \\ \text{相位〃 } \beta = \omega\sqrt{\mu\varepsilon'}\left(1+\dfrac{1}{8}\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2\right) & (rad/m) \end{cases}$

2. 本質阻抗

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \cdots = \sqrt{\frac{\mu}{\varepsilon'}}\left(1+j\frac{\varepsilon''}{2\varepsilon'}\right) = \frac{E_x}{H_y}$$

3. 相速度 $u_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{\mu\varepsilon'}}\left(1-\dfrac{1}{8}\underbrace{\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2}_{\ll 1}\right)$

§ 8.3.2. Good Conductor

1. 良導体 $\Rightarrow \dfrac{\sigma}{\omega\varepsilon} >> 1$

$$\Rightarrow \gamma = j\omega\sqrt{\mu\varepsilon}\left(1+\frac{\sigma}{j\omega\varepsilon}\right)^{\frac{1}{2}}$$
$$= j\omega\sqrt{\mu\varepsilon}\left(\frac{\sigma}{j\omega\varepsilon}\right)^{\frac{1}{2}}$$
$$= \sqrt{j}\sqrt{\omega\mu\sigma}$$
$$= (1+j)\sqrt{\pi f\mu\sigma}$$
$$= \alpha + j\beta$$

$\Rightarrow \alpha = \beta = \sqrt{\pi f\mu\sigma} \propto \sqrt{f}, \sqrt{\sigma}$

物理意義: f高 → 衰減快!

⇒ 防電磁干擾,用導体包住此招对低頻 EM wave 效果極差

2. $\eta_c = \sqrt{\dfrac{\mu}{\varepsilon_c}} = \sqrt{\dfrac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\dfrac{\pi f \mu}{\sigma}}$

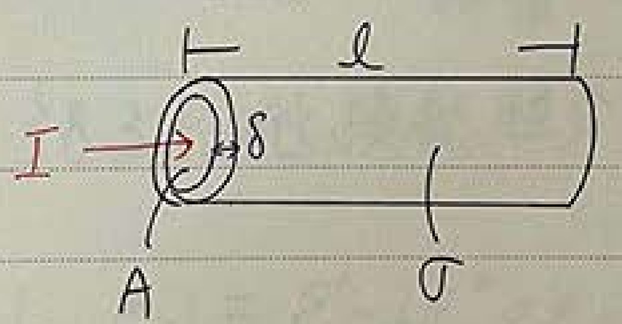
3. $u_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\sqrt{\pi f \mu \sigma}} = \sqrt{\dfrac{2\omega}{\mu\sigma}}$

4. 銅之$\sigma = 5.8\times10^{7}$ S/m，$\mu = \mu_0 = 4\pi\times10^{-7}$ H/m，
在3MHz時，$u_p \approx 720$ m/s
⇒ attenuation const $\alpha = \sqrt{\pi f \mu \sigma} = 2.6\times10^{4}$ Np/m

(1) 定義 skin depth (集膚深度) $\delta \triangleq \dfrac{1}{\alpha} \approx 0.038$ mm
⇒ 振幅在導体内0.038mm衰減為原本之0.368($\approx e^{-1}$)倍
⇒ EM wave 難傳導到導体内!

(2) Copper内之波長 $\lambda = \dfrac{2\pi}{\beta} = \dfrac{u_p}{f} = 0.24$ mm
(air内 → $\lambda = 100$ m)

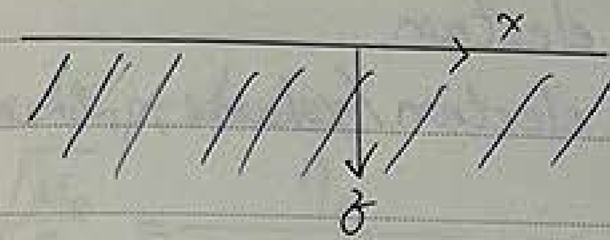
5. 在導線上傳一訊号



$R = \dfrac{1}{\sigma}\dfrac{l}{A}$

⇒ 過導線之高頻电流多在導線表面流过去(约δ)，故等效A会↓ ⇒ R↑，此称集膚效应

ex8-4. Seawater. 求λ与δ?



at $z=0$，設 $\vec{E}=\hat{a}_x 100\cos(10^7\pi t)$

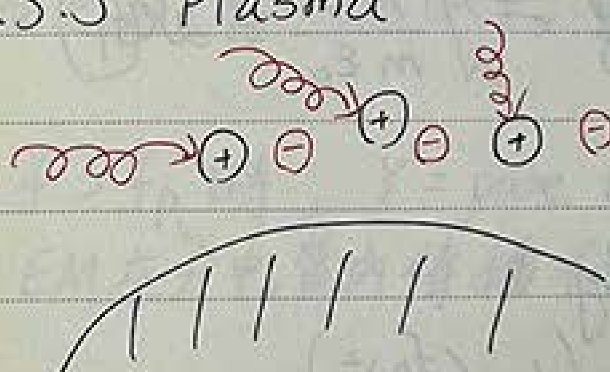
∵ at $f=5\,MHz$, $\frac{\sigma}{\omega\varepsilon}=200>>1$ → 海水是 good conductor

∴ $\alpha=\beta=8.8\ (1/m)$

⇒ $\lambda=\frac{2\pi}{\beta}=0.7m$

⇒ $\delta=\frac{1}{\alpha}=0.112\,m$ #

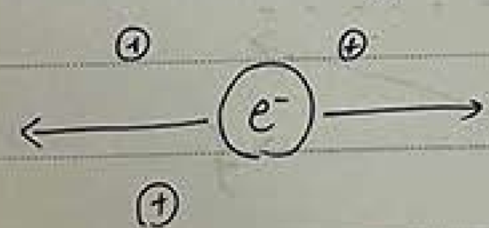
§8.3.3 Plasma



1. 考虑一个电子 e^- → 带电量 $-e$

⇒ $\vec{F}=-e\vec{E}=m\vec{a}=m\frac{d^2\vec{x}}{dt^2}$

in phasor form $-e\vec{E}=m(j\omega)^2\vec{x} \Rightarrow \vec{x}=\frac{e}{m\omega^2}\vec{E}$



⇒ 在⊕之background下来回運动 (也叫 polarization)

$\because$ $\vec{p} = (-e)\vec{x}$ for an electron

$\therefore$ If there are N electron/volume, then the polarization vector

$\vec{\mathbb{P}}$ = volume density of dipole moment

$$= N\vec{p}$$

$$= -\frac{Ne^2}{m\omega^2}\vec{E}$$

$$\Rightarrow \vec{D} = \varepsilon_0\vec{E} + \vec{\mathbb{P}}$$

$$= \varepsilon_0\left(1 - \frac{Ne^2}{m\omega^2\varepsilon_0}\right)\vec{E}$$

$$= \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\vec{E}$$

where $\omega_p \triangleq$ plasma freq. $\triangleq \sqrt{\frac{Ne^2}{m\varepsilon_0}} \triangleq 2\pi f_p$ —(*)

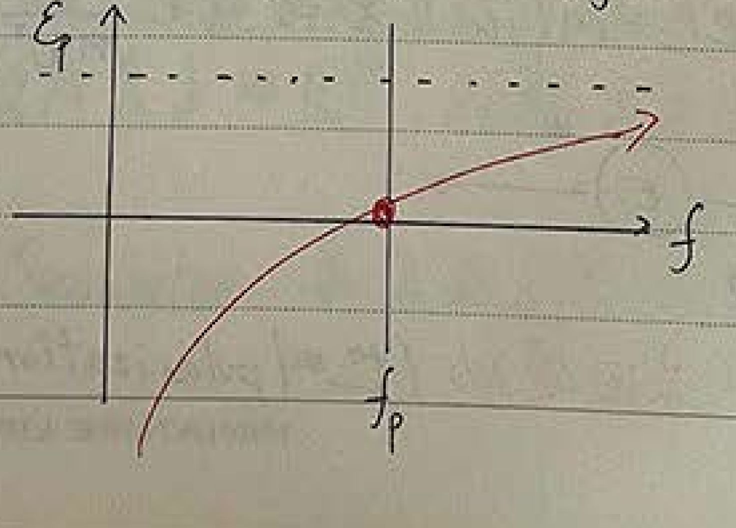
plasma freq.

2. Equivalent permittivity

$$\varepsilon_p = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \varepsilon_0\underbrace{\left(1 - \left(\frac{f_p}{f}\right)^2\right)}_{\varepsilon_r}$$

propagation const.

$$\gamma = j\omega\sqrt{\mu\varepsilon_p} = j\omega\sqrt{\mu\varepsilon_0\left(1-\left(\frac{f_p}{f}\right)^2\right)}$$



intrinsic impedance

$$\eta_p = \sqrt{\frac{\mu_0}{\varepsilon_p}} = \frac{\eta_0}{\sqrt{1-\left(\frac{f_p}{f}\right)^2}}$$

(a) 当 $f < f_p$，此時 γ 為 pure real number $= \alpha$

$\Rightarrow E \propto \underbrace{e^{-\alpha z}}_{\text{只衰減，不傳播}} \cancel{e^{-j\beta z}}$，

$\eta_p = \frac{E}{H}$ = pure imaginary

→ 不消耗能量，no power transmission

（cf. 純実数阻抗 → 消耗能量）

→ 將 e, m, ε_0 代入 (*)，可得 $f_p \approx 9\sqrt{N}$

(b) $f > f_p$ 時，γ = pure imaginary $= j\beta$

→ EM在介电質内傳播不会衰減！

ex 8-5. $N = 2\times10^4\ m^{-3}$ 時，$f_p = 9\sqrt{N} = 127$ MHz

$\Rightarrow$ 需 $> f_p$ 才能通訊

§8.4 Group velocity

1. 相速度 $u_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\omega\sqrt{\mu\varepsilon}}$

非損耗性介質

2. 當相位常數 β 非 ω 的線性函數 (ex. 損耗性介質)

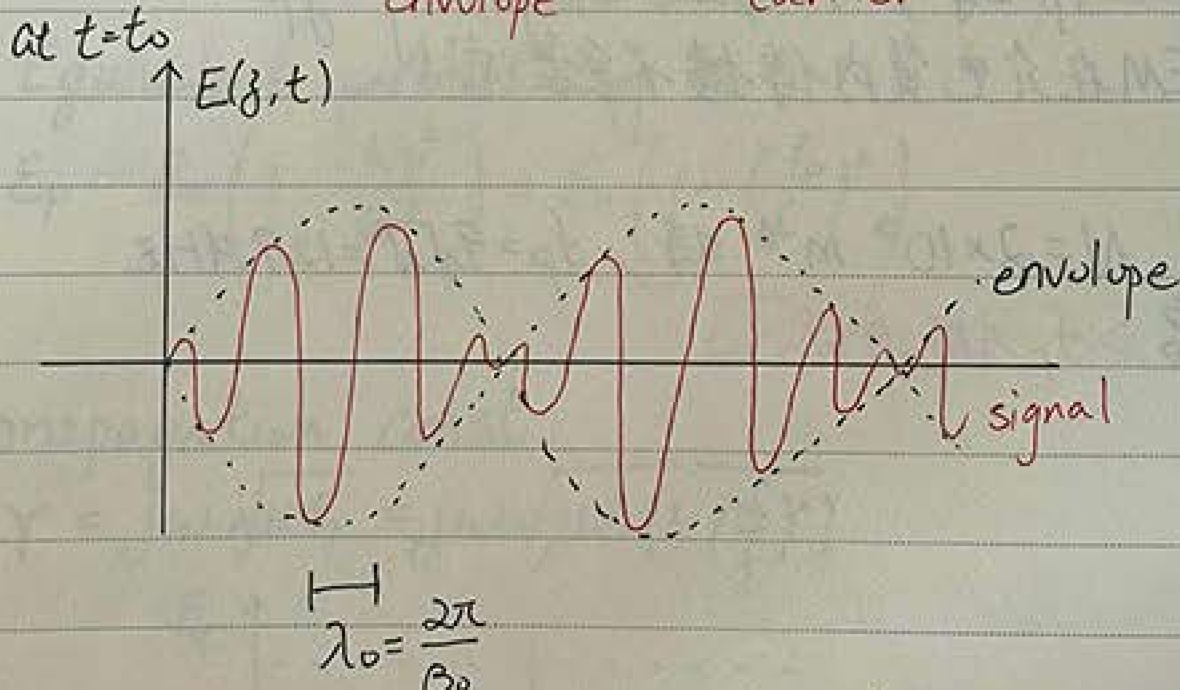
⇒ 不同 f 之波以不同 u_p 傳播

⇒ 訊号波失真，稱色散 (dispersion)

3. 考慮一波包，由兩成份之 ω, β 組成

$$\begin{cases} \omega = \omega_0 \pm \Delta\omega, & \Delta\omega \ll \omega_0 \\ \beta = \beta_0 \pm \Delta\beta, & \Delta\beta \ll \beta_0 \end{cases}$$

$$\begin{aligned} E(z,t) &= E_0 \cos[(\omega_0+\Delta t)t-(\beta_0+\Delta\beta)z] \\ &\quad + E_0\cos[(\omega_0-\Delta\omega)t-(\beta_0-\Delta\beta)z] \\ &= 2E_0 \underbrace{\cos(\Delta\omega t-\Delta\beta z)}_{\text{envolope}} \underbrace{\cos(\omega_0 t-\beta_0 z)}_{\text{carrier}} \end{aligned}$$



(a) u_p of carrier

$$u_p = \left.\frac{dz}{dt}\right|_{\omega_0 t - \beta_0 z = \text{const}} = \frac{\omega_0}{\beta_0}$$

(b) u_p of envelope $\triangleq$ group velocity

$$\triangleq u_g$$

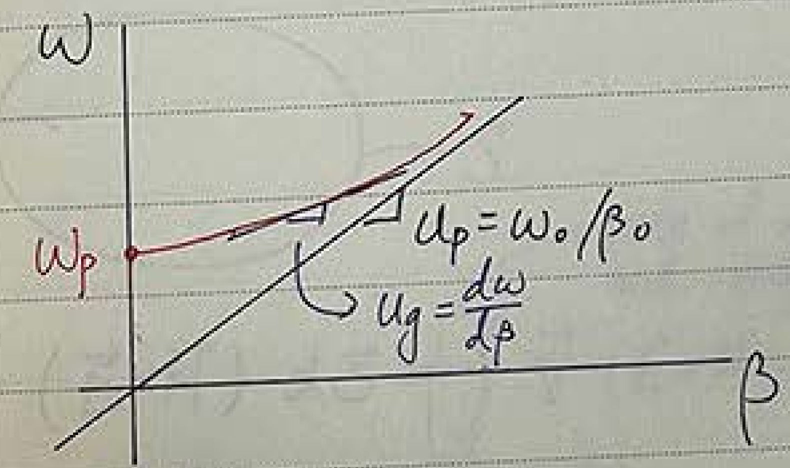
$$= \left.\frac{dz}{dt}\right|_{\Delta\omega t - \Delta\beta z = \text{const.}}$$

inverse function thm

$$= \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega(\beta)}{d\beta} = 1 \Big/ \frac{d\beta(\omega)}{d\omega}$$

4. Plasma $\omega-\beta$ diagram

$$\beta = \omega\sqrt{\mu\varepsilon_0}\left(1-\left(\frac{\omega_p}{\omega}\right)^2\right)^{\frac{1}{2}}$$



ex 8-6 在 $f_0 = 550$ KHz (carrier freq.) 且 loss tangent $\tan\delta_c = \varepsilon''/\varepsilon' = 0.2$, 介电常数 $\varepsilon_r = 2.5 = \varepsilon/\varepsilon_0$. 求

(a) γ, α, β?

(b) u_p, u_g?

(a) $\gamma = \alpha + j\beta$

$= jk_c$

$= j\omega\sqrt{\mu\varepsilon'}\left(1 - j\dfrac{\varepsilon''}{\varepsilon'}\right)^2$

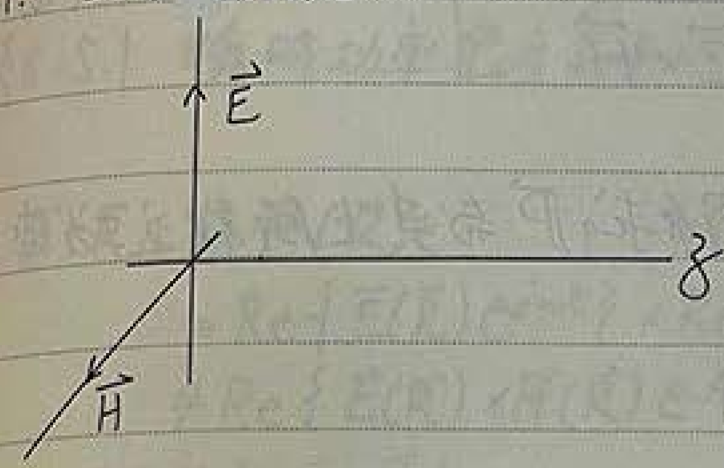
$$\Rightarrow \begin{cases} \alpha = \dfrac{\omega\varepsilon''}{2}\sqrt{\dfrac{\mu}{\varepsilon'}} \approx 1.82\times10^{-3}\ 1/m \\ \beta = \omega\sqrt{\mu\varepsilon'} = \left(1+\dfrac{1}{8}\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2\right) = \cdots = 0.018\ rad/m \end{cases}$$

(b) $$\begin{cases} u_p = \dfrac{\omega_0}{\beta_0} = \cdots = 1.8\times10^8\ m/s \\ u_g = 1\Big/\dfrac{d\beta}{d\omega} = \cdots = u_p \end{cases}$$ #

→ 非色散!

§8-5 Poynting vector

1. EM wave

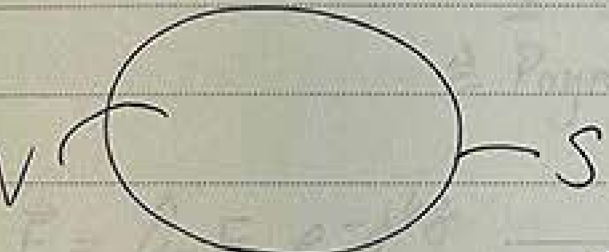


$$
\begin{cases}
\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\
\vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}
\end{cases}
$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$= \cdots$$

$$= -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2}\mu|\vec{H}|^2\right) - \sigma|\vec{E}|^2$$



rate of decrease of stored electric and magnetic energy.

$$\Rightarrow \oint (\vec{E}\times\vec{H})\cdot d\vec{s} = \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\, dv$$

$$= -\frac{\partial}{\partial t}\int_V \left(\underbrace{\frac{1}{2}\varepsilon|\vec{E}|^2}_{} + \underbrace{\frac{1}{2}\mu|\vec{H}|^2}_{}\right) dv - \int_V \underline{\sigma|\vec{E}|^2}\, dv$$

volume density of electric energy

" magnetic "

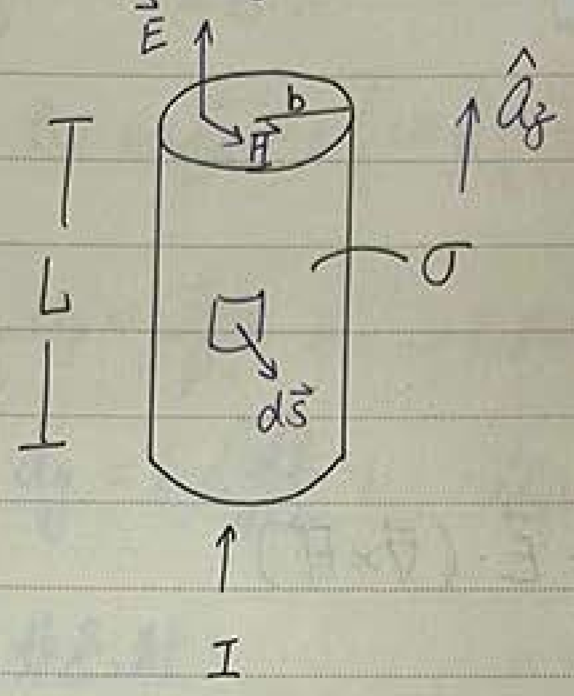
$\vec{E} \rightarrow J$ 造成之熱損耗

定義 Poynting vector $\vec{\mathbb{P}}$ = power density vector

= power flow per unit area (W/m²)

$\triangleq \vec{E} \times \vec{H}$

ex.8-7. 一長直導線如下圖所示，求其上所載之功率？



-sol-. $\because \vec{J} = \hat{a}_z \frac{I}{\pi b^2} = \sigma \vec{E}$

$\therefore \vec{E} = \hat{a}_z \frac{I}{\pi b^2 \sigma}$

On the surface, $\vec{H} = \hat{a}_\phi \frac{I}{2\pi b}$

$$\Rightarrow -\oint_S \vec{\mathbb{P}}\, d\vec{s} = -\left(\int_{side} + \cancel{\int_{top}}^{0} + \cancel{\int_{bottom}}^{0} \right)$$

$$= -\int (\hat{a}_z \times \hat{a}_\phi) \frac{I^2}{2\pi^2 b^3 \sigma} \cdot \hat{a}_r\, ds$$

$R = \frac{1}{\sigma}\frac{l}{A}$

$= I^2 R$ #

Q: $\vec{\mathbb{P}}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$, how about phasor?

§8.5.1 瞬時功率密度與平均功率密度

1. $\vec{\mathbb{P}}(\vec{R},t)=\vec{E}(\vec{R},t)\times\vec{H}(\vec{R},t)$

$$=\mathrm{Re}\{\vec{E}(\vec{R})e^{j\omega t}\}\times\mathrm{Re}\{\vec{H}(\vec{R})e^{j\omega t}\}$$

$$\neq\mathrm{Re}\{\vec{E}(\vec{R})\times\vec{H}(\vec{R})e^{j\omega t}\}$$

→不可定為 $\vec{E}\times\vec{H}$!

conjugate

2. $\because \mathrm{Re}\{\vec{A}\}\times\mathrm{Re}\{\vec{B}\}=\frac{1}{2}\mathrm{Re}\{\vec{A}\times\vec{B}+\vec{A}\times\vec{B}^*\}$

$\therefore \vec{\mathbb{P}}(t)=\frac{1}{2}\mathrm{Re}\{\vec{E}e^{j\omega t}\times\vec{H}e^{j\omega t}+\vec{E}e^{j\omega t}\times\vec{H}^*e^{-j\omega t}\}$

$$=\frac{1}{2}\mathrm{Re}\{\vec{E}\times\vec{H}^*+\vec{E}\times\vec{H}e^{j2\omega t}\}$$

time average $\vec{\mathbb{P}}_{av}\triangleq\frac{1}{T}\int_0^T\vec{\mathbb{P}}(t)\,dt$

$$=\frac{1}{2}\mathrm{Re}\{\vec{E}\times\vec{H}^*\}$$

$\triangleq$ Poynting vector in phasor form

3. $\vec{E}=\hat{a}_x E_0 e^{-jkz}$

$\vec{\mathbb{P}}_{av}=\frac{1}{2}\mathrm{Re}\{[\hat{a}_x E_0 e^{-\alpha z}e^{-j\beta z}]\times[\hat{a}_y\frac{E_0}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\theta_\eta}]^*\}$

$$=\hat{a}_z\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta$$

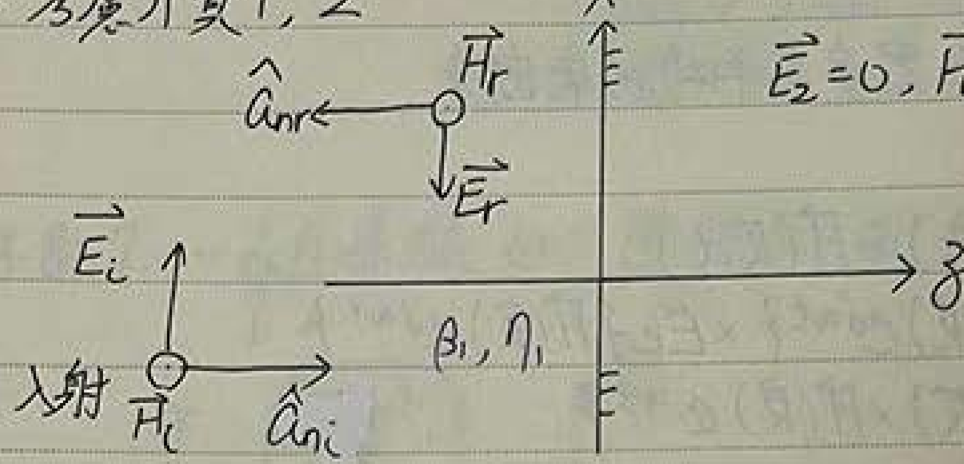
$$\propto\begin{cases}\cos\theta_\eta\\ e^{-2\alpha z}\\ \frac{E_0^2}{2|\eta|}\end{cases}$$

⇒ $\theta_\eta=\pm 90°$時, $\vec{\mathbb{P}}_{av}=0$ ⇒ 电磁場穿不过!能量無法傳輸!

稱"reactive load/medium" ex. $f<f_p$ 之 plasma

§8-6 平坦導電性介面之垂直入射

1. 考慮介質1, 2



(1) incident wave $(+z)$

$$\begin{cases} \vec{E}_i(z) = \hat{a}_x E_{i0} e^{-j\beta_1 z} \\ \vec{H}_i(z) = \hat{a}_y \dfrac{E_{i0}}{\eta_1} e^{-j\beta_1 z} \end{cases}$$

(2) reflected wave $(-z)$

$$\rightarrow \vec{E}_r = \hat{a}_x E_{r0} e^{j\beta_1 z}$$

(3) total $\vec{E}$ in medium 1

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z)$$
$$= \hat{a}_x (E_{i0} e^{-j\beta_1 z} + E_{r0} e^{j\beta_1 z})$$

(4) B.C. at $z=0$: $\vec{E}_1(0) = \hat{a}_x (E_{i0} + E_{r0}) = \vec{E}_2(0) = 0$

$$\Rightarrow E_{r0} = -E_{i0}$$
$$\Rightarrow E_1(z) = \hat{a}_x E_{i0} (e^{-j\beta_1 z} - e^{j\beta_1 z})$$
$$= -\hat{a}_x j E_{i0} \sin\beta_1 z$$

(5) reflected wave

$$\begin{cases} \vec{E}_r = \hat{a}_x E_{r0} e^{j\beta z} \\ \vec{H}_r = \hat{a}_{nr} \times \dfrac{E_r(z)}{\eta_1} = \cdots = \hat{a}_y \dfrac{E_{i0}}{\eta_1} e^{j\beta_1 z} \end{cases}$$

$$\Rightarrow \begin{cases} \vec{E}_1 = \vec{E}_i + \vec{E}_r = -\hat{a}_x j 2E_{i0} \sin\beta_1 z \\ \vec{H}_1 = \vec{H}_i + \vec{H}_r = \hat{a}_x \dfrac{2E_{i0}}{\eta_1} \cos\beta_1 z \end{cases}$$

2. Power

(1) incident power

$$(\vec{\mathbb{P}}_{av})_i = \frac{1}{2} Re\{\vec{E}_i \times \vec{H}_i^*\}$$

$$= \cdots$$

$$= \hat{a}_z \frac{E_{i0}}{2\eta_1}$$

(2) reflected power

$$(\vec{\mathbb{P}}_{av})_r = \frac{1}{2} Re\{\vec{E}_r \times \vec{H}_r^*\} = -\hat{a}_z \frac{E_{i0}^2}{2\eta_1}$$

(3) Total power

$$(\vec{\mathbb{P}}_{av})_1 = (\vec{\mathbb{P}}_{av})_i + (\vec{\mathbb{P}}_{av})_r = 0.$$

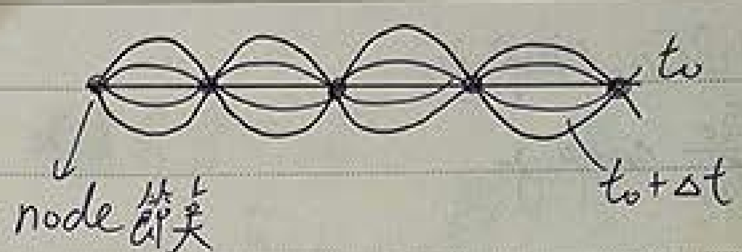
$\rightarrow$ intrinsic impedence $\sqrt{\frac{\mu_1}{\epsilon_1}}$

(4) $\begin{cases} \eta_{total} = \dfrac{E_{1x}}{H_{1y}} = -j\eta_1 \tan\beta_1 z \\ \theta_{\eta_{total}} = -\dfrac{\pi}{2} \end{cases}$ $\Rightarrow (\vec{\mathbb{P}}_{av})_{1,total} = 0 \ (\propto \cos\theta_\eta)$

3. 隨時間变化

$$\begin{cases} \vec{E}_1(z,t) = Re\{\vec{E}_1(z) e^{j\omega t}\} = \hat{a}_x 2E_{i0} \sin\beta_1 z \sin\omega t \\ \vec{H}_1(z,t) = \quad \cdots \quad = \hat{a}_x \dfrac{2E_{i0}}{\eta_1} \cos\beta_1 z \cos\omega t \end{cases}$$

$\Rightarrow$ 為駐波! (standing wave)



$\begin{cases} 駐波: \sin\beta z \sin\omega t \\ 行進波: \cos(\beta z-\omega t) \end{cases}$

4. $\begin{cases} \text{Max of } E_1 \\ \text{Zero of } H_1 \end{cases} \Rightarrow \sin\beta_1 z = \pm 1$

$\Rightarrow z = -(2n+1)\dfrac{\lambda}{4}$

$\Rightarrow$ 時間上 $\vec{E}, \vec{H}$ 差 $90°$

$\Rightarrow$ 能量不傳播!

ex 8-9. 設 $E_{i0} = 6\ mV/m$, $f = 100\ MHz$. 在 air 中

$\begin{cases} \beta_1 = k_0 = \omega/c = \frac{2}{3}\pi \\ \eta_1 = \eta_0 = 120\pi\ \Omega. \end{cases}$

(a) 求 incident wave $\vec{E}_i$? $\vec{H}_i$?

(b) 求 reflected wave $\vec{E}_r$? $\vec{H}_r$?

(c) 求 total field $\vec{E}_1, \vec{H}_1$?

-sol-. (a) $\because \begin{cases} \vec{E}_i(x) = \hat{a}_y E_{i0} e^{-j\beta_1 x} \\ \quad = \hat{a}_y 6\times10^{-3} e^{-j\frac{2}{3}\pi x}\ (V/m) \\ \vec{H}_i(x) = \dfrac{1}{\eta_1}\hat{a}_{ni}\times\vec{E}_i(x) \\ \quad = \hat{a}_z \dfrac{10^{-4}}{2\pi} e^{-j\frac{2}{3}\pi x}\ (A/m) \end{cases}$

$\therefore$ instantaneous form

$\vec{E}_i(x,t) = Re\{\hat{a}_y 6\times10^{-3} e^{-j\frac{2}{3}\pi x} e^{j\omega t}\}$

$$= \hat{a}_y\, 6\times10^{-3}\cos\left(2\pi\times10^{8}t - \frac{2\pi}{3}x\right)$$

$$\vec{H}_i(x,t) = \cdots$$

$$= \hat{a}_z \frac{10^{-4}}{2\pi}\cos\left(2\pi\times10^{8}t - \frac{2\pi}{3}x\right)$$

(b)

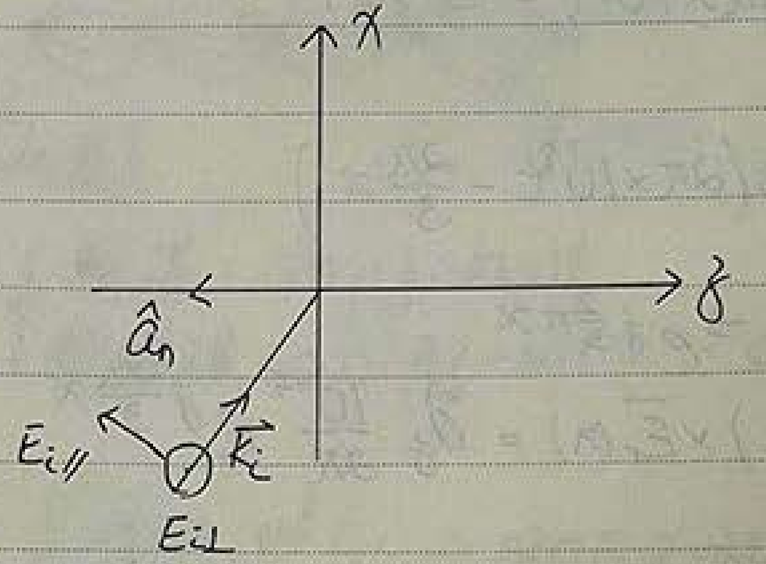
$$\begin{cases} \vec{E}_r(x) = -\hat{a}_y\, 6\times10^{-3} e^{j\frac{2}{3}\pi x} \\ \vec{H}_r(x) = \frac{1}{\eta_1}(-\hat{a}_x)\times\vec{E}_r(x) = \hat{a}_z \frac{10^{-4}}{2\pi} e^{j\frac{2\pi}{3}x} \end{cases}$$

$$\Rightarrow \begin{cases} \vec{E}_r(x,t) = \cdots = -\hat{a}_y\, 6\times10^{-3}\cos\left(2\pi\times10^{8}t + \frac{2}{3}\pi x\right) \\ \vec{H}_r(x,t) = \cdots = \hat{a}_z \frac{10^{-4}}{2\pi}\cos\left(2\pi\times10^{8}t + \frac{2\pi}{3}x\right) \end{cases}$$

(c) total field

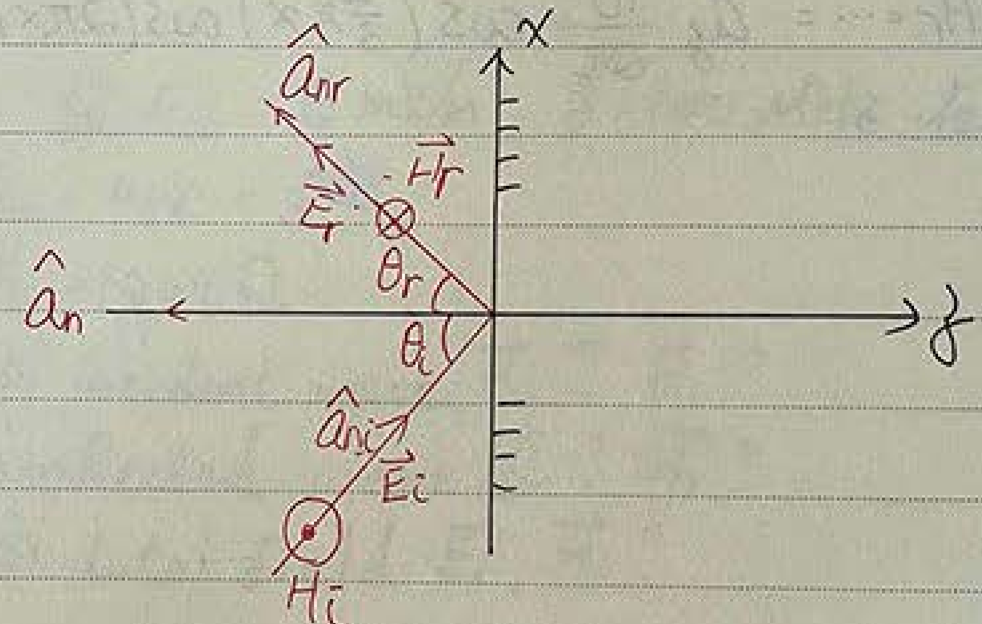
$$\begin{cases} \vec{E}_1 = \vec{E}_i + \vec{E}_r = \cdots = \hat{a}_y\, 1.2\times10^{-2}\sin\left(\frac{2}{3}\pi x\right)\sin\left(2\pi\times10^{8}t\right) \\ \vec{H}_1 = \vec{H}_i + \vec{H}_r = \cdots = \hat{a}_z \frac{10^{-4}}{2\pi}\cos\left(\frac{2}{3}\pi x\right)\cos\left(2\pi\times10^{8}t\right) \end{cases} \quad \#$$

§8.7 在平坦的導电性介面上之斜向入射



Plane of incidence: $\hat{a}_n$ 与波前前進方向 $\vec{k}_i$ 所決定之平面

§8.7.1. Perpendicular polarization



1. 設 $E_{i\perp}=0$，只有 $\vec{E}_{i//}=\vec{E}_i$ 且

$$\begin{cases} \hat{a}_{ni} = \hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i \\ \theta_i = \text{angle of incidence (入射角)} \end{cases}$$

(1) incident wave 之 phasor form 為

$$\begin{aligned} \vec{E}_i(x,z) &= \vec{E}_0 e^{-j\vec{k}_i\cdot\vec{R}} \\ &= (\hat{a}_y E_{i0}) e^{-j\hat{a}_{ni}\beta_1\cdot\vec{R}}, \quad \vec{R}=(x,y,z) \\ &= \hat{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \end{aligned}$$

$$\vec{H}_i = \frac{1}{\eta_1}\hat{a}_{ni}\times\vec{E}_i$$

$$= \frac{E_{i0}}{\eta_1}(-\hat{a}_x\cos\theta_i+\hat{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i+z\cos\theta_i)}$$

(2) reflected wave

$$\begin{cases}\hat{a}_{nr} = \hat{a}_x\sin\theta_r - \hat{a}_z\cos\theta_r \\ \theta_r = \text{angle of reflection (反射角)}\end{cases}$$

$$\Rightarrow \vec{E}_r(x,z) = \hat{a}_y E_{r0}e^{-j\vec{k}_r\cdot\vec{R}}$$

$$= \cdots$$

$$= \hat{a}_y E_{r0}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

(3) B.C. (tangential $\vec{E}$)

$$E_1(x,0) = E_i(x,0)+E_r(x,0)$$

但∵phase matching, $E_1(x,0)=0$

$$\Rightarrow E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_r} = 0$$

$$\Rightarrow \beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r$$

$$\Rightarrow \begin{cases}\theta_r = \theta_i \\ E_{r0} = -E_{i0}\end{cases}$$

物理意義：phase matching，在x軸(介面上)相位之分佈要一樣！

$$\Rightarrow \vec{E}_r(x,z) = -\hat{a}_y E_{i0}e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

2. total field

$$\vec{E}_1(x,z) = \vec{E}_i(x,z)+\vec{E}_r(x,z)$$

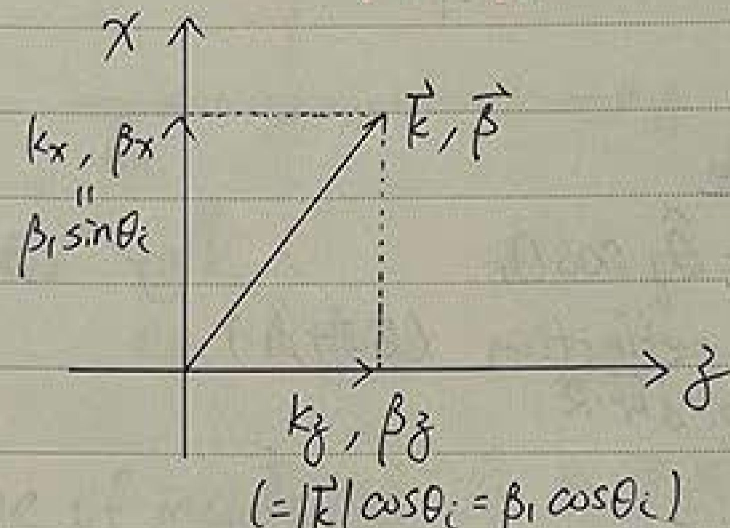
$$= \hat{a}_y E_{i0}e^{-j\beta_1 x\sin\theta_i}\left(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i}\right)$$

$$= -\hat{a}_y j\boxed{2E_{i0}\underbrace{\sin(\beta_1 z\cos\theta_i)}_{z\text{方向是駐波}}}\underbrace{e^{-j\beta_1 x\sin\theta_i}}_{\text{往}x\text{軸正向傳播}}$$

amplitute

(1) compare to plane wave

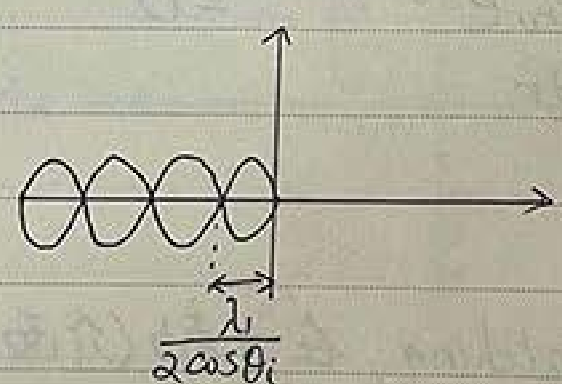
$$e^{-j\vec{k}\cdot\vec{R}} = e^{-j(\underset{\substack{\shortparallel \\ \beta_x}}{k_x} x + \underset{\substack{\shortparallel \\ \beta_z}}{k_z} z)}$$



(2) Zero of $\vec{E}_1 \Rightarrow \sin(\beta_1 z \cos\theta_i) = 0$

$$\Rightarrow z = \frac{-m\pi}{\beta_1 \cos\theta_i} = -\frac{m\lambda_1}{2\cos\theta_i}$$

$\Rightarrow$ 在z軸上產生干涉條紋!



ex 8-10. ⊥以θ_i入射導体平面，求

(a) $\vec{J}_s(x,t)$?

(b) Poynting vector?

-sol-. (a) $\because$ at $z=0$, $\vec{H}(x,0) = -\frac{E_{i0}}{\eta_0}(\hat{a}_x 2\cos\theta_i)e^{-j\beta_0 x \sin\theta_i}$

but $\vec{H}_2(x,0) = 0$

$\therefore$ by (7-68)(b),

$$\vec{J}_s(x) = \hat{a}_{n2} \times \vec{H}_1(x,0)$$

$$= (-\hat{a}_z) \times \hat{a}_n(\quad \cdots \quad)$$

$$= \hat{a}_y \frac{E_{i0}}{60\pi} \cos\theta_i \, e^{-j\frac{\omega}{c} x \sin\theta_i}$$

$$\Rightarrow \vec{J}_s(x,t) = \hat{a}_y \frac{E_{i0}}{60\pi} \cos\theta_i \cos\left(\omega\left(t - \frac{x}{c}\sin\theta_i\right)\right)$$

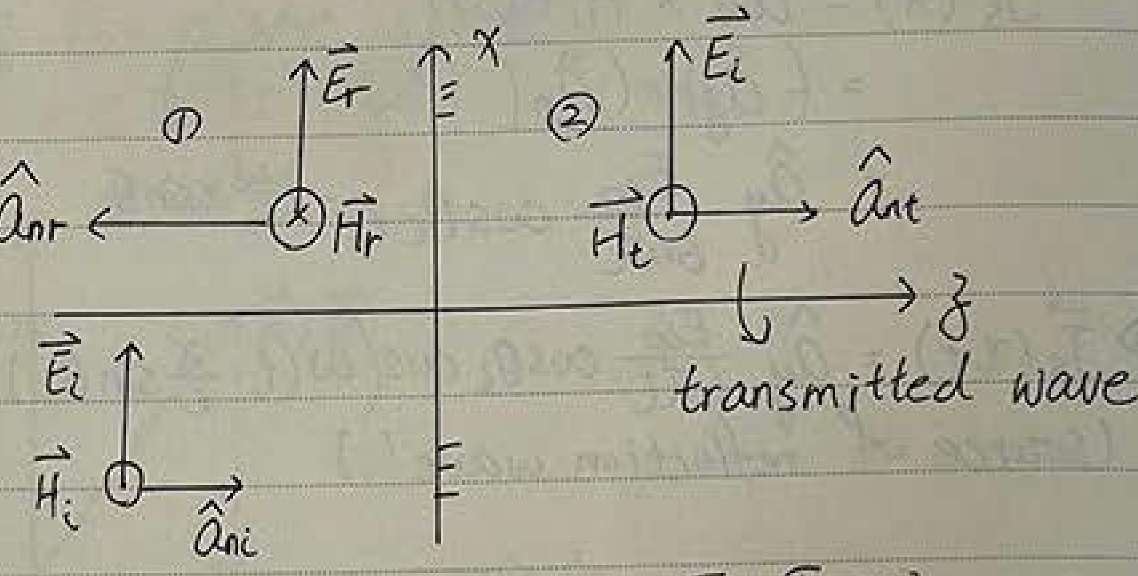
(source of reflection wave!)

(b) Poynting vector

$$(\vec{\mathbb{P}}_{av})_1 = \frac{1}{2} \mathrm{Re}\{\vec{E}_1 \times \vec{H}_1^*\}$$

$$= \hat{a}_x \frac{2E_{i0}^2}{\eta_1} \sin\theta_i \sin^2(\beta_i \cos\theta_i z)$$

§8.8 在平坦的介电質界面上之垂直入射



$\sigma_1 = \sigma_2 = 0$

1. incidence wave

$$\begin{cases} \vec{E}_1(z) = \hat{a}_x E_{i0} e^{-j\beta_1 z} \\ \vec{H}_1(z) = \hat{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z} \end{cases}$$

2. reflected wave

$$\begin{cases} \vec{E}_r(z) = \hat{a}_x E_{r0} e^{j\beta_1 z} \\ \vec{H}_r(z) = -\hat{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z} \end{cases}$$

3. transmitted wave

$$\begin{cases} \vec{E}_t(z) = \hat{a}_x E_{t0} e^{-j\beta_2 z} \\ \vec{H}_t(z) = \hat{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z} \end{cases}$$

4. B.C. (tangential)

$$\begin{cases} E_{1t}(0) = E_{ix}(0) + E_{rx}(0) = E_{2t}(0) = E_{tx}(0) \\ H_{1t}(0) = H_{iy}(0) + H_{ry}(0) = H_{2t}(0) = H_{ty}(0) \end{cases}$$

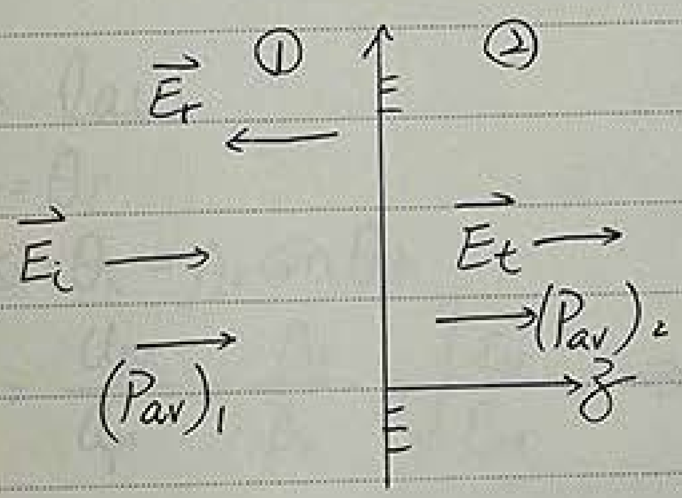
E_{t0}, E_{r0} 為未知数

$$\Rightarrow \begin{cases} E_{i0} + E_{r0} = E_{t0} \\ \dfrac{E_{i0}}{\eta_1} - \dfrac{E_{r0}}{\eta_1} = \dfrac{E_{t0}}{\eta_2} \end{cases}$$

$$\Rightarrow \begin{cases} E_{r0} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0} \\ E_{t0} = \dfrac{2\eta_2}{\eta_2 + \eta_1} E_{i0} \end{cases}$$

定義 $\begin{cases} \text{反射係數 } \Gamma \triangleq \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\ \text{穿透係數 } \tau \triangleq \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1} \end{cases}$

ex8-11. 試證：$1-\Gamma^2 = \dfrac{\eta_1}{\eta_2}\tau^2$ 及求 $(\vec{P}_{av})_1$, $(\vec{P}_{av})_2$



$$\begin{cases} E_i = \hat{a}_x E_{i0} e^{-j\beta_1 z} \\ \vec{E}_r = \hat{a}_x \Gamma E_{i0} e^{+j\beta_1 z} \\ E_t = \hat{a}_x \tau E_{i0} e^{-j\beta_2 z} \end{cases}$$

-sol-. $\because \begin{cases} \vec{E}_1 = \vec{E}_i + \vec{E}_r \\ \vec{E}_2 = \vec{E}_t \end{cases}$

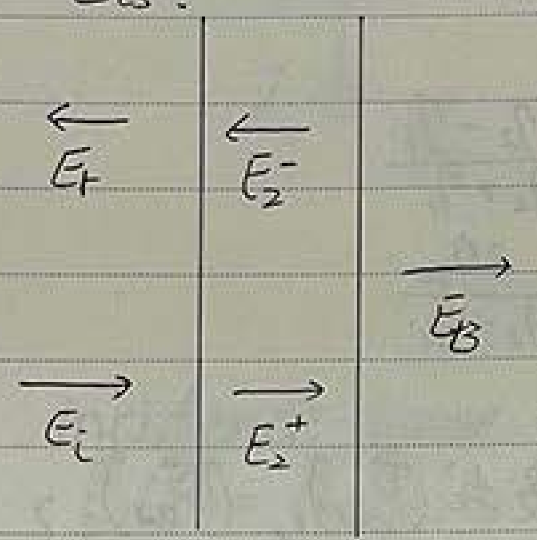
$\therefore$ 可得 $\vec{H}_1$, $\vec{H}_2$

$$\Rightarrow \begin{cases} (\vec{P}_{av})_1 = \cdots = \hat{a}_z \dfrac{E_{i0}}{2\eta_1}(1-\Gamma^2) \\ (\vec{P}_{av})_2 = \cdots = \hat{a}_z \dfrac{\tau^2 E_{i0}^2}{2\eta_2} = (\vec{P}_{av})_1 \end{cases}$$

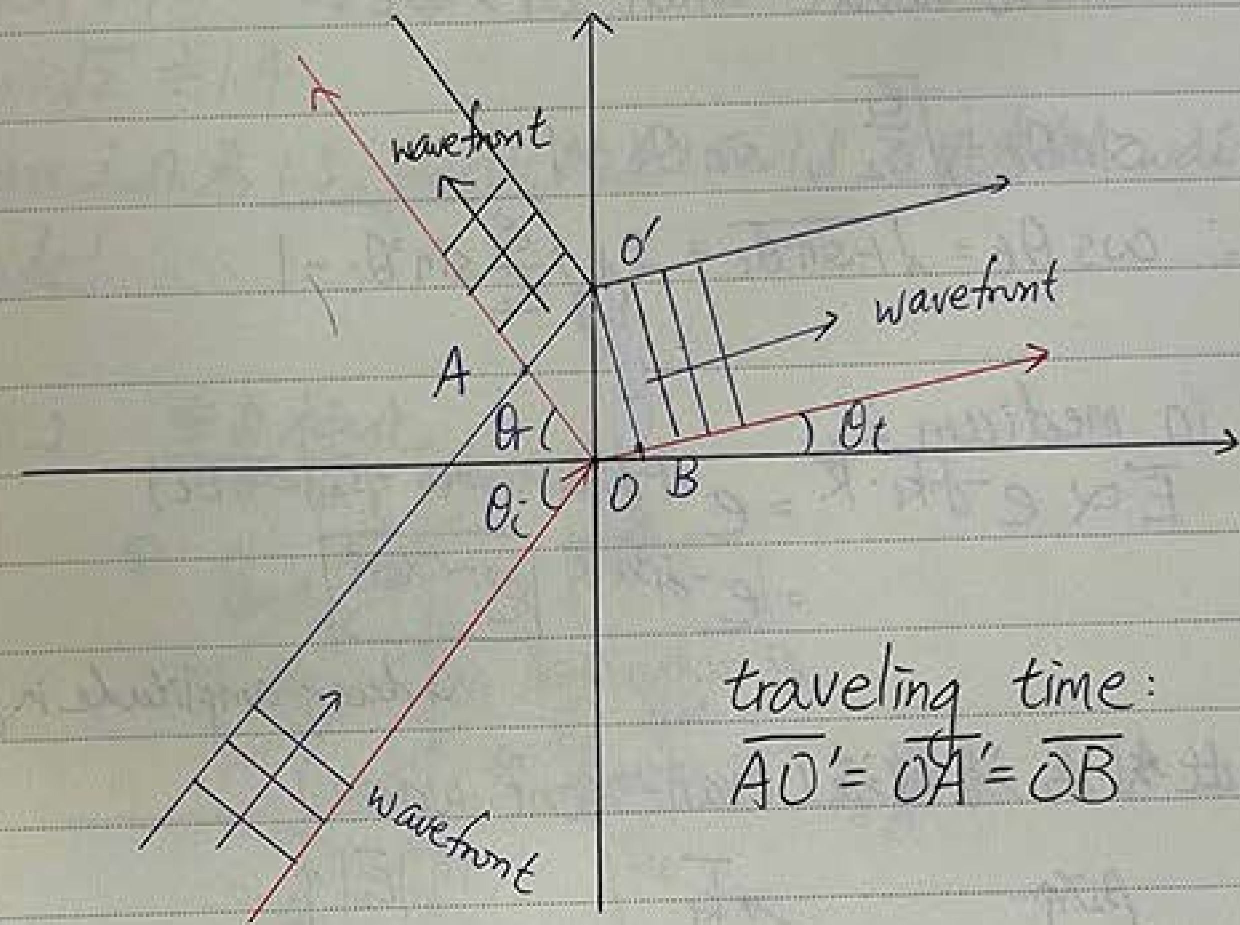
in lossless media

$\Rightarrow 1-\Gamma^2 = \frac{\eta_1}{\eta_2}\tau^2$

5. 若有多个介质 ⇒ 可能经多次透射、反射才会形成 $\vec{E}_r$、$\vec{E}_{t3}$.



§8-10 在平坦介電質界面上之斜向入射



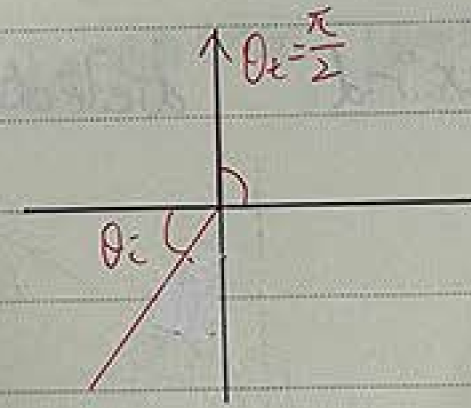
traveling time:
$\overline{AO'} = \overline{OA'} = \overline{OB}$

1. Snell's law

$$\begin{cases} \theta_r = \theta_i \\ n_1 \sin\theta_i = n_2 \sin\theta_t \end{cases}$$

$$\Rightarrow \frac{n_1}{n_2} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \sqrt{\frac{\varepsilon_{1r}}{\varepsilon_{2r}}} \Rightarrow n = \sqrt{\varepsilon_r}$$

§8.10.1 total reflection



1. If $n_1 > n_2$ ($\varepsilon_1 > \varepsilon_2$)

$$\Rightarrow \sin\theta_t = \frac{n_1}{n_2} \sin\theta_1$$

定義 critical angle $\theta_c \triangleq$ 使 $\theta_t = \frac{\pi}{2}$ 之入射角 θ_i

$$\Rightarrow \theta_c = \sin^{-1}\frac{n_2}{n_1}$$

Q: How about when $\theta_i > \theta_c$?

∵ $\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$, $\sin\theta_i > 1$

∴ $\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \pm j\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$

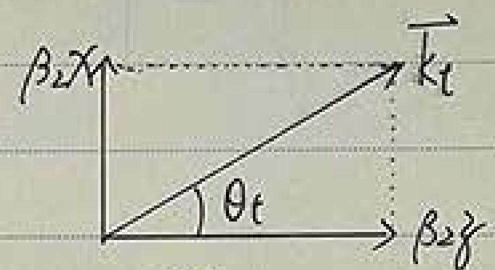
in medium 2

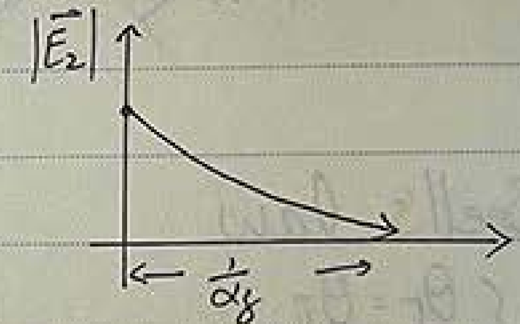
$\vec{E} \propto e^{-j\vec{k}_t\cdot\vec{R}} = e^{-j(\beta_2 x\sin\theta_t + \beta_2 z\cos\theta_t)}$

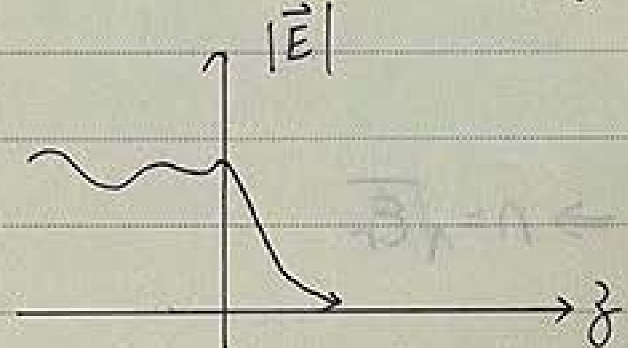
$= \boxed{e^{-j\beta_x x}} \cdot \boxed{e^{-\alpha_z z}}$

往+x方向傳播 ↳ decay amplitude in z direction

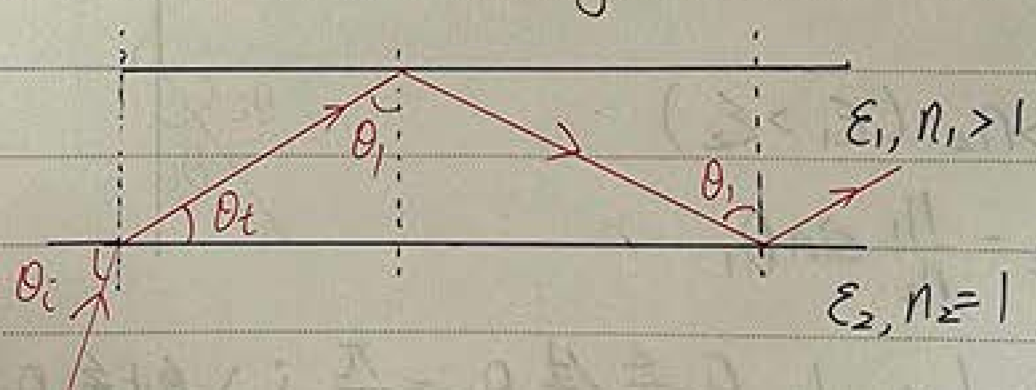
此稱為消散波(evanescent wave)







ex 8-14. dielectric waveguide



when $\theta_1 \geq \theta_c$, $\sin\theta_c \leq \sin\theta_1 = \cos\theta_t$

but $\sin\theta_t = \frac{1}{\sqrt{\varepsilon_{r1}}}\sin\theta_i$

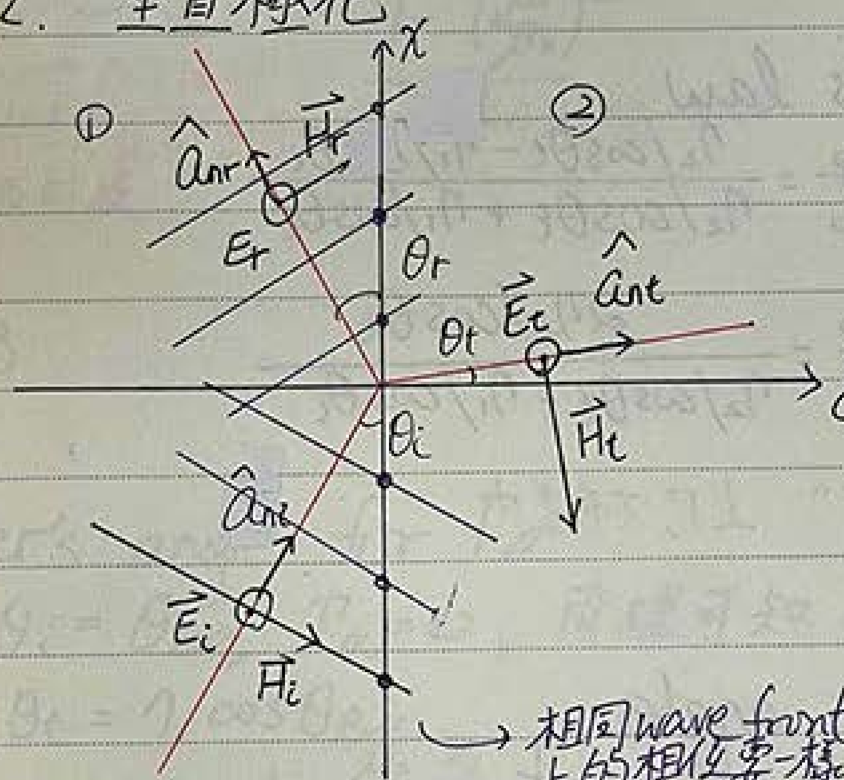
∵ $\cos\theta_t = \sqrt{1-\sin^2\theta_t}$

$\therefore \ \varepsilon_{r1} \geq 1+\sin^2\theta_i \geq 2$

$\Rightarrow n_1 \gtrsim \sqrt{2} \doteqdot 1.4$

$\Rightarrow$ glass之n_1為1.5, plastic約為1.6~1.8, 可用來做 wave guide!

§8.10.2. 垂直極化



→ 相同wave front在介面上的相位要一樣!
(phase matching condition)

1. incident wave

$$\vec{E}_i(x,z) = \hat{a}_y E_{i0} e^{-j\vec{k}_i\cdot\vec{R}}$$
$$= \hat{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$
$$H_i(x,z) = \frac{E_{i0}}{\eta_1}(-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i+z\cos\theta_i)}$$

2. B.C. (tangential $\vec{E}$、$\vec{H}$)

$$\begin{cases} E_{iy}(x,0) + E_{ry}(x,0) = E_{ty}(x,0) \\ H_{ix}(x,0) + H_{rx}(x,0) = H_{tx}(x,0) \end{cases}$$

phase matching condition

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t$$

$$\Rightarrow \begin{cases} \theta_r = \theta_i \\ \dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{\beta_1}{\beta_2} = \cdots = \dfrac{\eta_1}{\eta_2} \end{cases}$$ (Snell's law)

$$\Rightarrow \begin{cases} E_{i0} + E_{r0} = E_{t0} \\ \dfrac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \dfrac{E_{t0}}{\eta_2}\cos\theta_t \end{cases} \quad -(*)$$

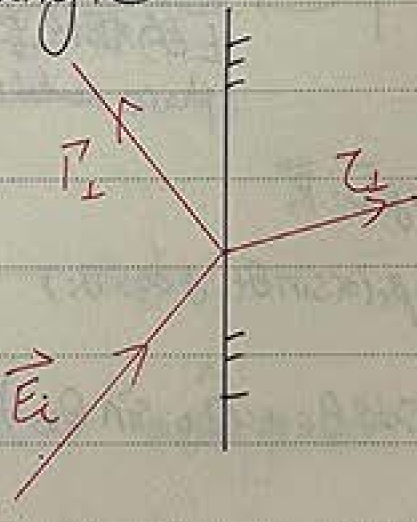
解(*)

$\Rightarrow$ Fresenel's law

$$\begin{cases} \Gamma_{\perp} \triangleq \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \\ \tau_{\perp} \triangleq \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \end{cases}$$

垂直偏極化、平行”之Γ, τ 有差!

△ Brewster's angle



Q: 已推出

$$\Gamma = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

則入射波以哪個角度入射時会沒有反射波?

A: 假設此角度為 $\theta_{B\perp}$ (Brewster's angle)

$(\hat{\theta_i} =)$

$\Rightarrow \eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t$ 時，$\Gamma_\perp = 0 \rightarrow$ no reflection!

By snell's law,

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{B\perp}} = \frac{\eta_2}{\eta_1}\cos\theta_{B\perp}$$

$$\Rightarrow \sin^2\theta_{B\perp} = \frac{1-\frac{\mu_1}{\varepsilon_1}\frac{\varepsilon_2}{\mu_2}}{\underbrace{1-\left(\frac{\mu_1}{\mu_2}\right)^2}_{\text{when 这项} \rightarrow 0}}$$

$\rightarrow \infty$，此時任意角度都会有反射波

§8.10.3

Brewster's angle for $\Gamma_{/\!/}$

$\rightarrow$ at $\theta_i = \theta_{B/\!/}$，$\Gamma_{/\!/} = 0$，同理可知

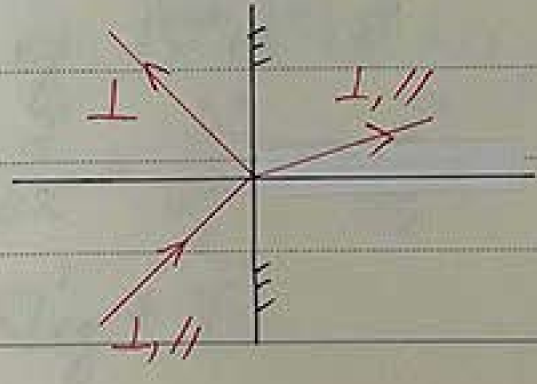
$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B/\!/}$$

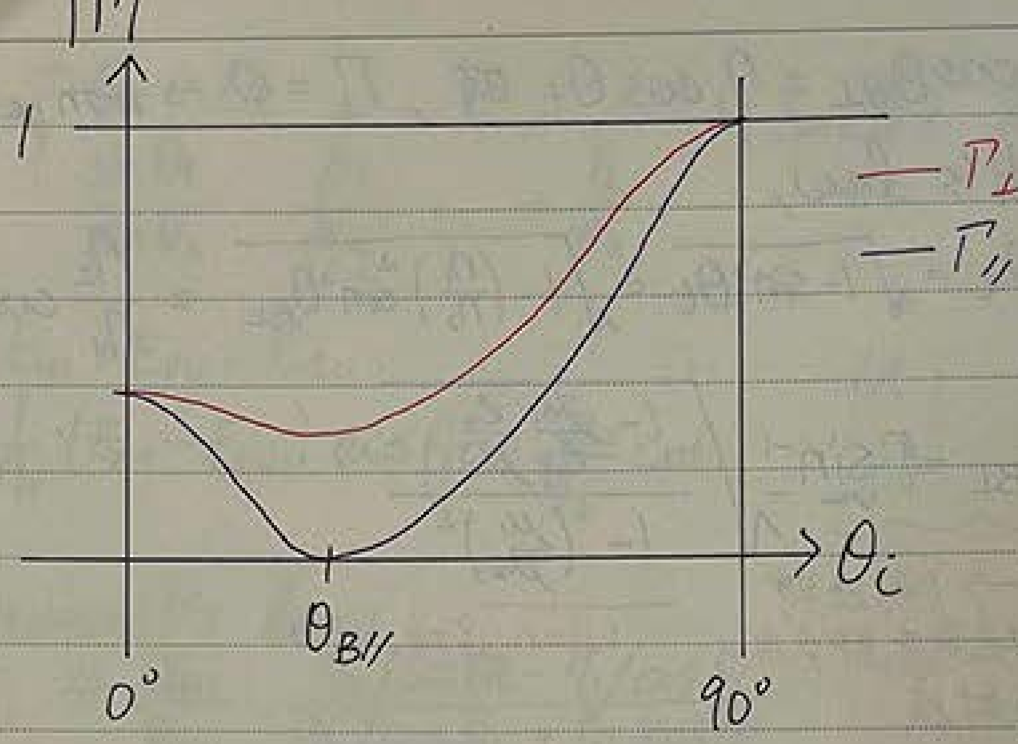
$$\Rightarrow \sin^2\theta_{B/\!/} = \frac{1-\frac{\mu_2}{\varepsilon_2}\frac{\varepsilon_1}{\mu_1}}{1-\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}$$

For $\mu_1 = \mu_2$，$\sin\theta_{B/\!/} = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$，or

$$\theta_{B/\!/} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1}\frac{n_2}{n_1}$$

ex. 若入射光同時有⊥，// 之成份，則以 $\theta_{B/\!/}$ 入射，反射光只剩下⊥的成份 偏極化





|Γ|
1
—Γ⊥
—Γ∥
θ_i
θ_B∥
0°
90°

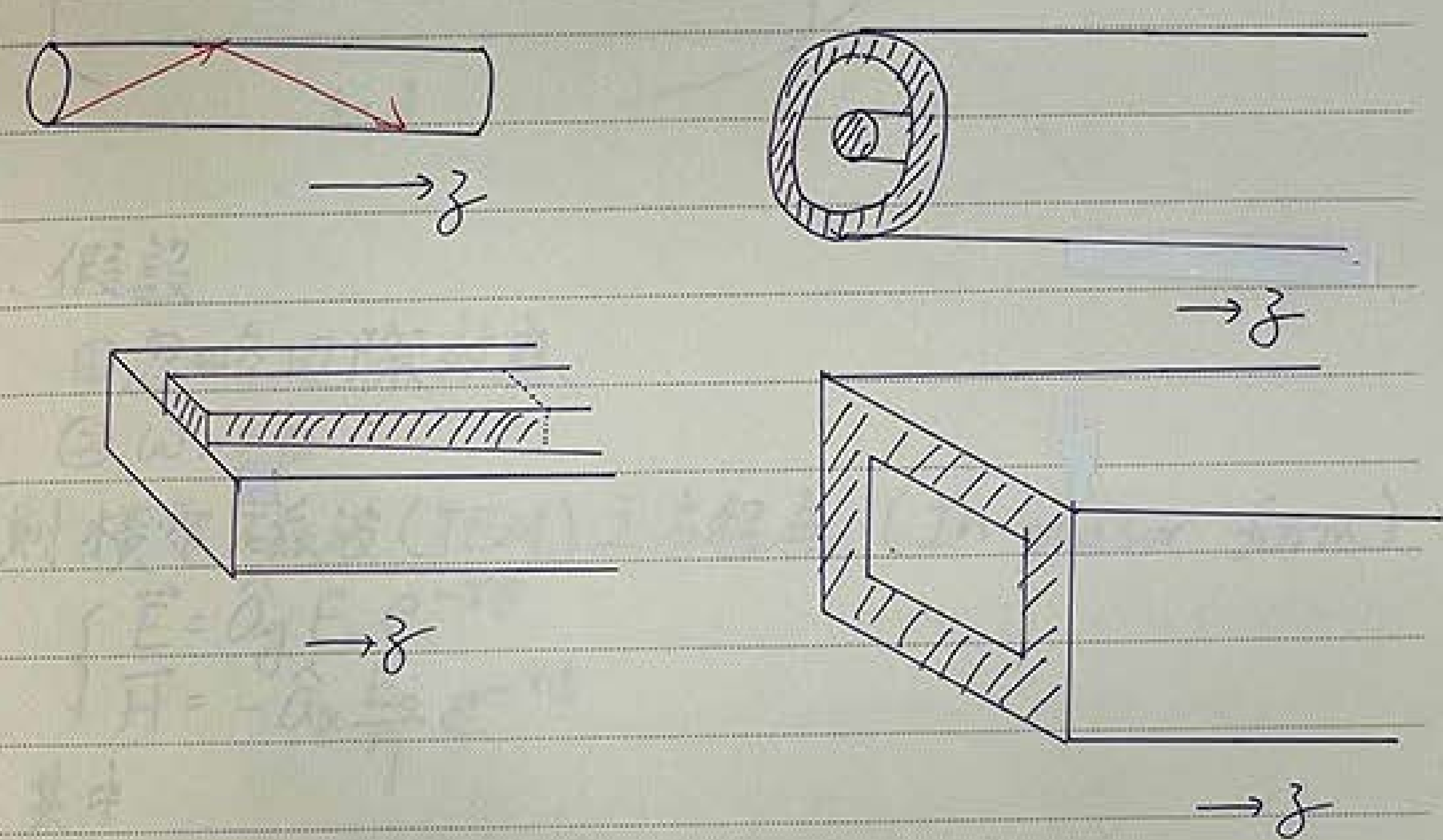
Ch10 波導管

§10-2 均勻波導結構上波之適性

1. 波導：能將EM wave傳播出去之結構

   (1) 特性：若EM wave之頻率 < 截止頻率，則不傳播.

   (2) 常見的波導管：



2. 設EM wave往 $+z$ 傳播，則無論波導長怎樣，

   $\vec{E}(x,y,z,t) = \mathrm{Re}\{\vec{E}(x,y)e^{-\gamma z}e^{j\omega t}\}$

   皆為其上EM wave之形式

$\Rightarrow$ In phasor form, $\frac{\partial}{\partial t} \to j\omega$; $\frac{\partial}{\partial z} \to (-\gamma)$

$\Rightarrow$ wave eq. $(\nabla^2 + k^2)\vec{E} = 0$,

$\because \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$= \nabla_{xy}^2 + \gamma^2$

traverse

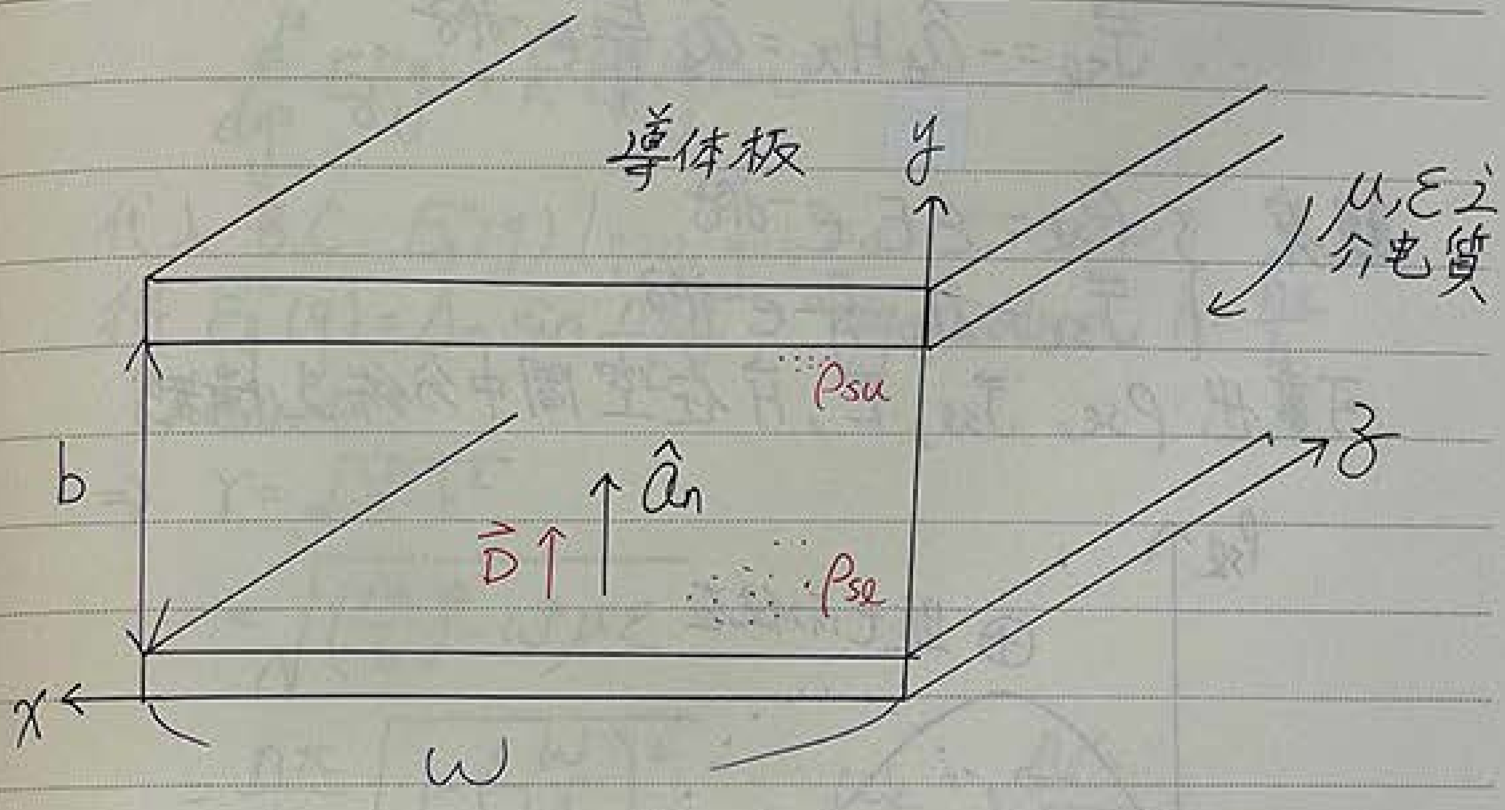
$$\triangleq \nabla_t^2 + \gamma^2$$

$$\therefore\ (\nabla^2 + k^2)\vec{E} = (\nabla_{xy}^2 + \gamma^2 + k^2)\vec{E}$$

let $h^2 = \gamma^2 + k^2$, $\nabla_{xy}^2 = \nabla_t^2$, then

$(\nabla_t^2 + h^2)\vec{E} = 0 \rightarrow$ wave eq.

§10-3 平行板波導管



1. 假設

① 忽略辺際效應

② $w >> b$

則橫電磁波(TEM)之方程為(In phasor form)

$$\begin{cases} \vec{E} = \hat{a}_y E_0 e^{-\gamma z} \\ \vec{H} = -\hat{a}_x \dfrac{E_0}{\eta} e^{-\gamma z} \end{cases}$$

其中

$$\begin{cases} \gamma = j\beta = j\omega\sqrt{\mu\varepsilon} \\ \eta = \sqrt{\mu/\varepsilon} \end{cases}$$

B.C. : at $y=0$, $y=b$, $E_t = 0$, $H_n = 0$

2. At lower plate $(y=0)$

$$\begin{cases} \hat{a}_n = +\hat{a}_y \text{ from B.C.} \\ \hat{a}_n \cdot \vec{D} = \rho_{sl} \end{cases}$$

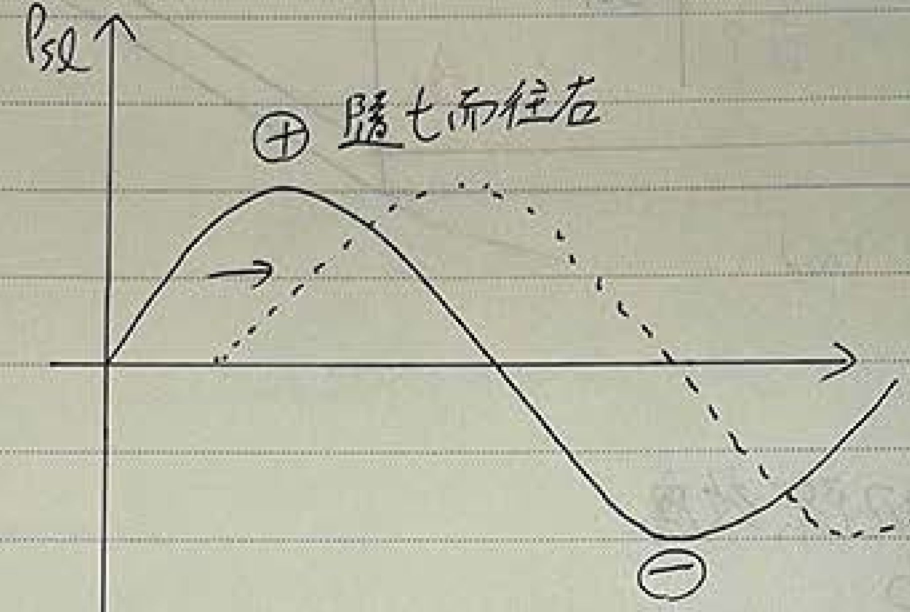
$\Rightarrow \rho_{sl} = \varepsilon E_y = \varepsilon E_0 e^{-j\beta z}$

又，$\because \hat{a}_n \times \vec{H} = \vec{J}_{sl}$

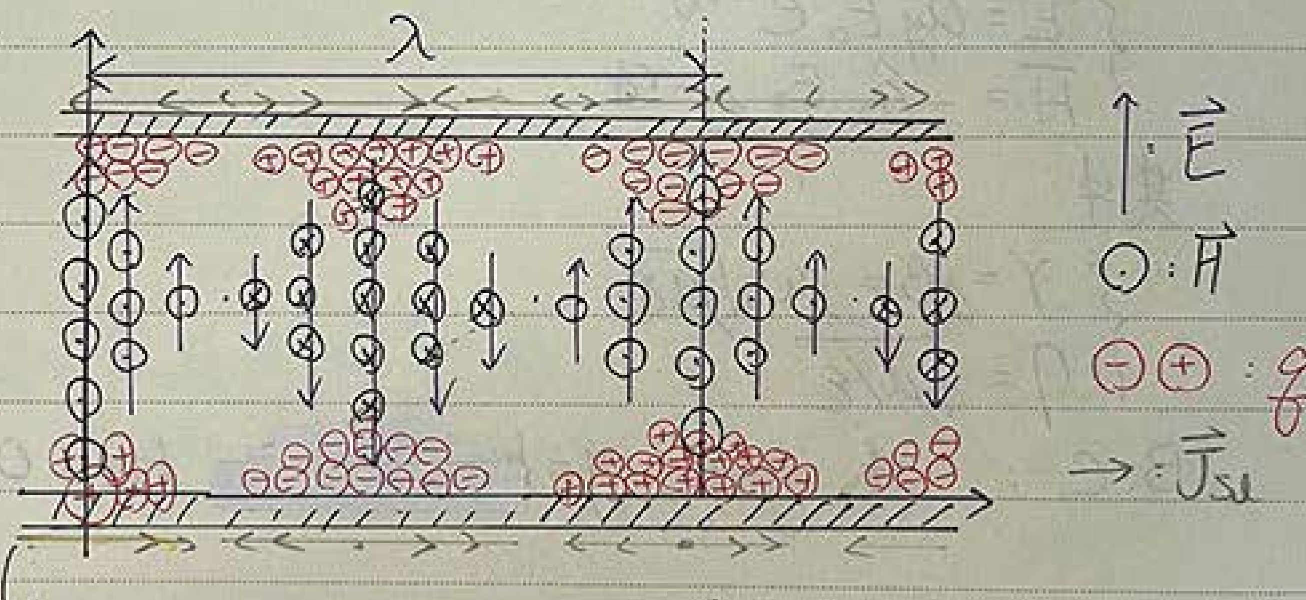
$\therefore \vec{J}_{sl} = -\hat{a}_z H_x = \hat{a}_z \frac{E_0}{\eta} e^{-j\beta z}$

觀察 $\begin{cases} \rho_{sl} = \varepsilon E_0 e^{-j\beta z} \\ \vec{J}_{sl} = \hat{a}_z \frac{E_0}{\eta} e^{-j\beta z} \end{cases}$

可畫出 ρ_{sl}, $\vec{J}_{sl}$, $\vec{E}$, $\vec{H}$ 在空間中分佈之情形



→ 電磁場感應出之电荷会隨時間往右移动



是在plate内但畫不下故畫在外面

3. 但目前仍不知 E_0 是誰，先記為 $E_0 = E_z^0(y) = E_z^0$

$\because E_z(x,y,z) = E_z^0(y) e^{-\gamma z}$，代入wave eq.

$(\nabla_{xy}^2 + k^2)\vec{E} = 0$ 得

$$\frac{d^2}{dy^2}E_z^0(y)+h^2E_z^0(y)=0$$

代入 B.C. $E_z^0(y)\big|_{y=0,\,y=b}=0$ $(\because E_t=0)$

得 $E_z^0(y)=A_n\sin\frac{n\pi y}{b}$, eigenvalue $h=\frac{n\pi}{b}$

(discrete)

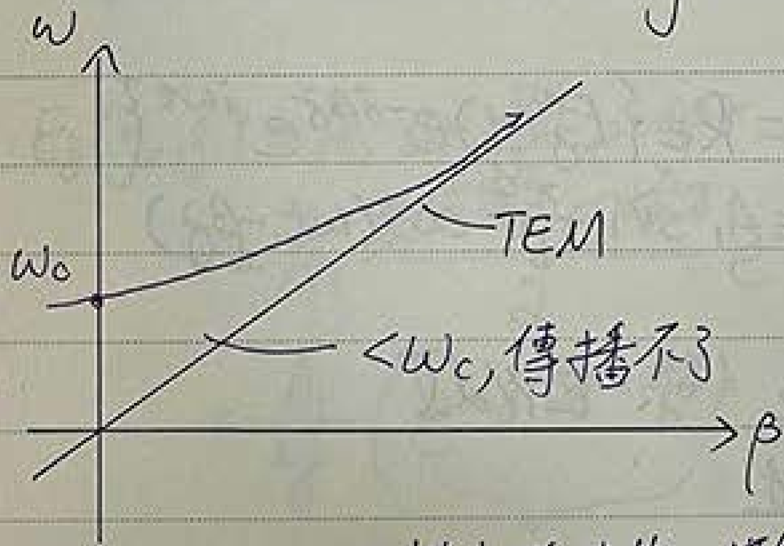
$$\Rightarrow \gamma=\sqrt{h^2-k^2}$$

$$=\sqrt{\left(\frac{n\pi}{b}\right)^2-\omega^2\mu\varepsilon}$$

$$=\frac{n\pi}{b}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}\ ,\quad \omega_c=\frac{n\pi}{b\sqrt{\mu\varepsilon}}$$

(1) Let $f_c=\frac{n}{2b\sqrt{\mu\varepsilon}}$

1. $f>f_c \Rightarrow \gamma=j\beta$ (propagation mode)
2. $f<f_c \Rightarrow \gamma=\alpha$ (decay / evanescent mode)

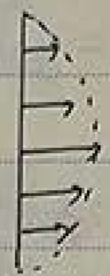


橫电磁波第0模態

(2) $n=0 \Rightarrow TM_0 \Leftrightarrow E_z^0=0$, $f_{c0}=0$

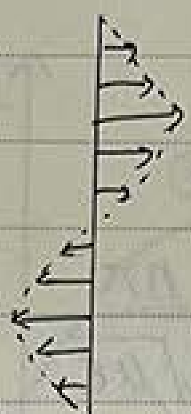
→沒有 cutoff, 都能以此模態傳播

$n=1 \Rightarrow TM_1 \Leftrightarrow E_z^0(y)=A_1\sin\frac{\pi y}{b}$



$$f_{c1} = \frac{1}{2b}\frac{1}{\sqrt{\mu\varepsilon}}$$

$$n=2 \Rightarrow TM_2 \Longleftrightarrow E_z^0(y) = A_2 \sin\frac{2\pi y}{b}$$



$$f_{c2} = \frac{1}{b}\frac{1}{\sqrt{\mu\varepsilon}}$$

依此類推，TM_k 有k个node，会希望EM wave在波導中只以一个mode傳播

ex10-3 (a) 寫出 TM_1 之 $\vec{E}$, $\vec{H}$ (vector form)

(b) 畫电場線&磁場線

-sol- (a). $E_z(y,z,t) = Re\{E_z^0(y)e^{-j\beta z}e^{j\omega t}\}$

$$= A_1 \sin\frac{\pi y}{b}\cos(\omega t - \beta z)$$

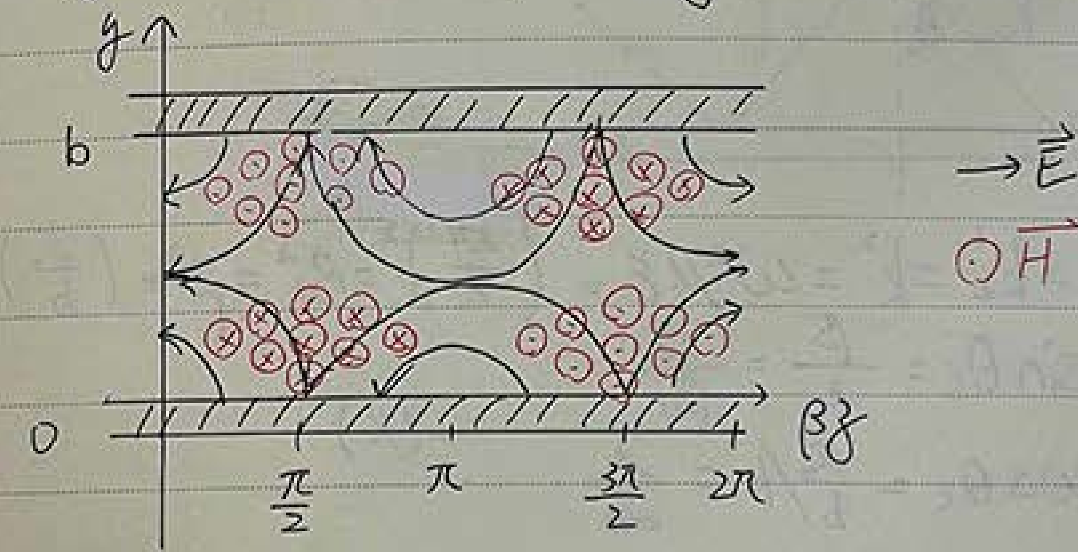
(b) Field line: $\because \vec{E} \parallel d\vec{l}$

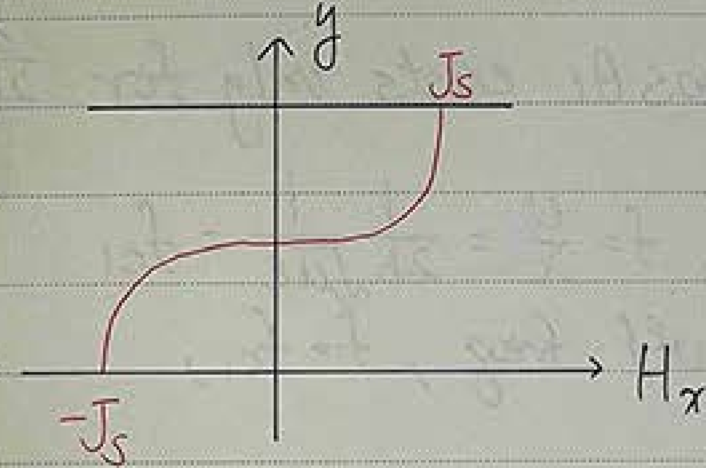
$$\therefore \frac{dy}{E_y} = \frac{dz}{E_z}$$

$$\Rightarrow \frac{dy(z)}{dz} = \frac{E_y}{E_z},$$

$\because$ At $t=0$, $\frac{dy}{dz} = \frac{E_y(y,z,t=0)}{E_z(y,z,t=0)} = -\frac{\beta b}{\pi}\cot\frac{\pi y}{b}\tan\beta z$

⇒ integration, $\cos\frac{\pi y}{b}\cos\beta z = \text{const.}$ 為場線方程





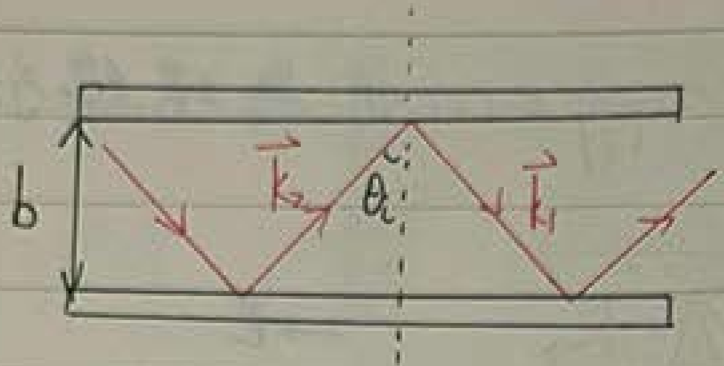
ex10-4. 將 TM_1 之 EM wave 寫成 2 行進波之和

-sol- $E_z(y,z) = A_1 \sin\frac{\pi y}{b} e^{-j\beta z}$

$$= A_1 \frac{e^{j\frac{\pi}{b}y} - e^{-j\frac{\pi}{b}y}}{2} e^{-j\beta z}$$

$$= \frac{A_1}{2j}\Big(\underbrace{e^{-j(\beta z - \frac{\pi y}{b})}}_{\text{plane wave with } \vec{k}_1 = \hat{a}_z\beta - \hat{a}_y\frac{\pi}{b}} - \underbrace{e^{-j(\beta z + \frac{\pi y}{b})}}_{\text{plane wave with } \vec{k}_2 = \hat{a}_z\beta + \hat{a}_y\frac{\pi}{b}}\Big)$$

其中 k_1, k_2 為行進方向.



且 $k_1^2 = k_2^2 = k^2 = \omega^2\mu\varepsilon = \left(\frac{2\pi}{\lambda}\right)^2 = \beta_1^2 = \beta^2 + \left(\frac{\pi}{b}\right)^2$

$$\Rightarrow \begin{cases} \sin\theta_i = \frac{\beta}{k} = \frac{\beta}{\beta_1} \\ \cos\theta_i = \frac{\pi}{b}/k = \frac{\lambda}{2b} \end{cases} \quad -(*)$$

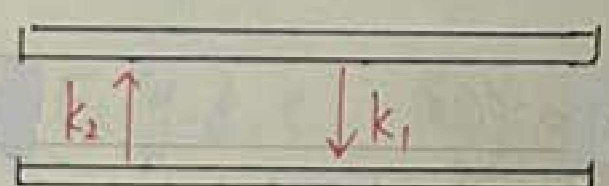
① From $(*)$, $\cos\theta_i$ exists only for $\frac{\lambda}{2b} \leq 1$

② at $b = \frac{\lambda}{2}$, $f = \frac{u}{\lambda} = \frac{1}{2b}\frac{1}{\sqrt{\mu\varepsilon}} = f_{c1}$

$\Rightarrow$ at cut-off freq., $f = f_{c1}$,

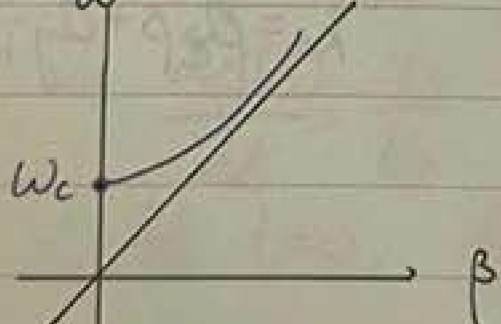
$$\begin{cases} \cos\theta_i = 1 \\ \sin\theta_i = 0 \end{cases}$$

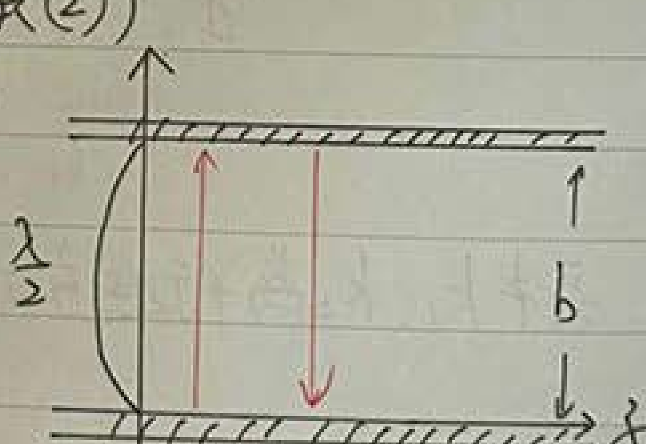
$\Rightarrow \frac{\lambda}{2b} = 1$, $\lambda = 2b$



$\Rightarrow$ 物理意義：沒有往 +z 方向傳播之現象

③ $\beta = \beta_1 \sin\theta_i = 0$ (得(②))

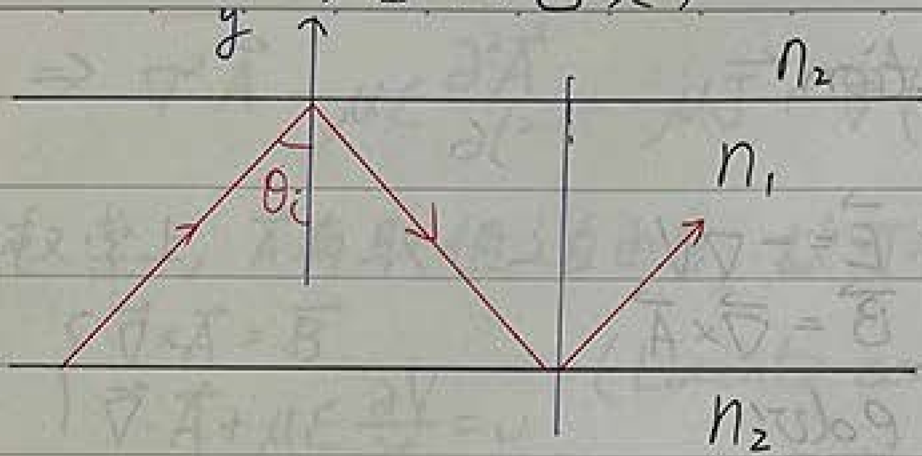




→ 物理意義：形成駐波！

Date . . No.

§10-6 波導管(介電質)



$n_1 > n_2$

$\theta_i > \theta_c$

→ 即：解 $\frac{d^2 E_z}{dz^2} + h^2 E_z = 0$ 在 plate 內、外之情形

→ 解之，plate 外應形如 $e^{-\alpha y}$

　　　　　　內 " $\vec{E} \propto \sin k_y y$ or $\cos k_y y$

Date . . No.

Ch11. 天線与輻射系統

§11-1 Introduction

1. $\begin{cases} \vec{\nabla}\times\vec{E}=0 \\ \vec{\nabla}\cdot\vec{B}=0 \end{cases} \Rightarrow \begin{cases} \vec{E}=-\nabla V \\ \vec{B}=\vec{\nabla}\times\vec{A} \end{cases}$

$$\Rightarrow \begin{cases} V=\int_{V'}\frac{1}{4\pi\varepsilon_0}\frac{\rho\,dv'}{R} \\ \vec{A}=\int_{V'}\frac{\mu_0}{4\pi}\frac{\vec{J}\,dv'}{R} \end{cases}$$

2. 但如果是時变电場，由 Maxwell eq.,

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

$$\Rightarrow \vec{E}+\frac{\partial\vec{A}}{\partial t}=-\nabla V$$

$$\Rightarrow \vec{E}=\underbrace{-\vec{\nabla}V}_{\text{charge}}\underbrace{-\frac{\partial\vec{A}}{\partial t}}_{\text{time varying B field}}$$

3. From $\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$

$$\Rightarrow \vec{\nabla}\times\left(\frac{\vec{\nabla}\times\vec{A}}{\mu}\right)=\vec{J}+\frac{\partial}{\partial t}\left(\varepsilon\left(-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}\right)\right)$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\mu\vec{J}+\mu\epsilon\frac{\partial}{\partial t}\left(-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}\right)$$

$$=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2 A$$

$$\Rightarrow \overbrace{\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}}^{\text{wave eq.}} = -\mu \vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

数学上，$\vec{A}$有取值之自由度，故可取

$$\begin{cases} \vec{\nabla}\times\vec{A} = \vec{B} \\ \vec{\nabla}\cdot\vec{A} + \mu\varepsilon \dfrac{\partial V}{\partial t} = 0 \end{cases} \quad \text{(Lorentz condition)}$$

則 $\nabla^2 \vec{A} - \mu\varepsilon \dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$

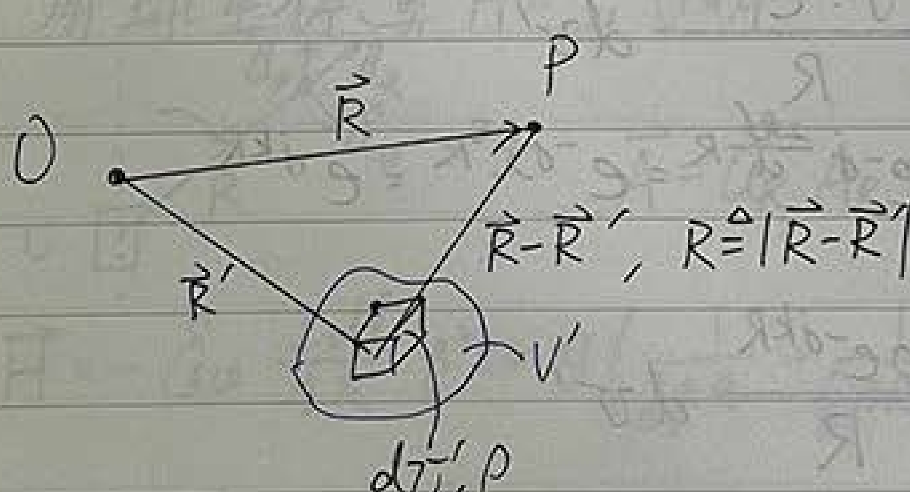
Similarly, from $\vec{\nabla}\cdot\vec{D} = \rho$, we have

$$\vec{D} = \varepsilon\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right)$$

4. The wave eq. of V

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

(1) Static $\rightarrow$ Poisson eq. $\nabla^2 V = -\dfrac{\rho}{\varepsilon}$



$$\Rightarrow V(\vec{R}) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho(\vec{R}')\, d\tau'}{|\vec{R} - \vec{R}'|}$$

(2) Time-varying case:

$$V(\vec{R},t)=\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(\vec{R}',t-\frac{|\vec{R}-\vec{R}'|}{u})}{|\vec{R}-\vec{R}'|}dv',$$

where $u=\frac{1}{\sqrt{\mu\varepsilon}}$ retarded potential 可当成 $t-\Delta t$

For $\Delta t \ll t$,

$$V(\vec{R},t)\approx\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(\vec{R}',t)}{R}dv'$$

→知瞬間电荷分佈 ⇒ 可知瞬間位的分佈

5. $\vec{A}(\vec{R},t)=Re\{\vec{A}(\vec{R})e^{j\omega t}\}$

$$=\frac{\mu}{4\pi}\int_{V'}\frac{Re\{\vec{J}(\vec{R}')e^{j\omega(t-\Delta t)}\}}{R}dv'$$

$$=\frac{\mu}{4\pi}\int_{V'}\frac{Re\{\vec{J}(\vec{R}')e^{-j\omega\frac{R}{u}}e^{j\omega t}\}}{R}dv'$$

→ $\vec{A}$ 之 phasor form 為

$$\vec{A}(\vec{R})=\frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}\cdot e^{-jkR}}{R}dv'$$

其中 $e^{-j\omega\frac{R}{u}}=e^{-j\frac{2\pi f}{u}R}=e^{-j\frac{2\pi}{\lambda}R}=e^{-jkR}$

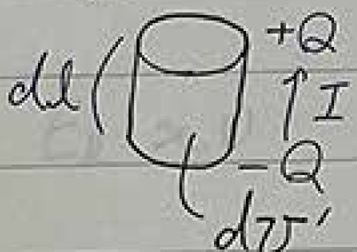
同理,

$$V=\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho e^{-jkR}}{R}dv'$$

Date . . No.

§11-2 Hertizen Dipole

1. 考慮一电偶极(极短)



$$\begin{cases} i(t) = \dfrac{dq(t)}{dt} \\ I = j\omega Q \\ \vec{P} = \hat{a}_z\, dl \cdot Q = \hat{a}_z \dfrac{I\,dl}{j\omega} \end{cases}$$

(电偶极向量)

vector potential due to $\vec{P}$, I

$$\vec{A} = \frac{\mu}{4\pi}\frac{\vec{J}e^{-jkR}}{R}$$

$\because dv = A\,dl$

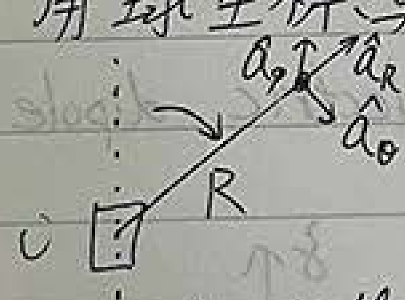
$\therefore J\,dv = I\,dl \quad (J = \frac{I}{A})$

$$\Rightarrow \vec{A} = \hat{a}_z \frac{\mu}{4\pi}\frac{I\,dl\,e^{-j\beta R}}{R} e^{j\omega t}$$

$\Rightarrow$ wave front 為 $R = $ const

$\Rightarrow$ spherical wave!

2. 用球坐標算 $\vec{H} = \frac{1}{\mu}\vec{\nabla}\times\vec{A}$



$$\vec{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta$$

$$\vec{H} = -\hat{a}_\phi \frac{I\,dl}{4\pi}\beta^2 \sin\theta \left(\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right) e^{-j\beta R}$$

$$\Rightarrow \vec{E} = \frac{1}{j\omega\varepsilon_0}\vec{\nabla}\times\vec{H}$$

$= \cdots$ (略)

(a) Near field (近場)

$\rightarrow \beta R = 2\pi \frac{R}{\lambda} \ll 1$, $e^{-j\beta R} \approx 1$

$$\Rightarrow H_\phi = -\frac{Hdl}{4\pi}\beta^2 \sin\theta \frac{1}{(j\beta R)^2}$$

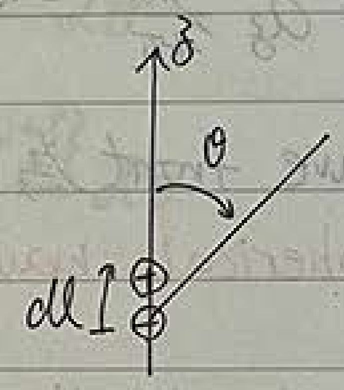
$$= \frac{Idl}{4\pi R^2}\sin\theta$$

= quasi static H field due to Idl

⇒ 靠近天線的磁場,可忽略 delay,电流所生之磁場会瞬間跟天線上之磁場会瞬間跟天線上之磁場会一樣,如同静磁場

此外,

$$\because \begin{cases} E_R \approx \dfrac{P}{4\pi\varepsilon_0 R^3} 2\cos\theta \\ E_\theta \approx \dfrac{P}{4\pi\varepsilon_0 R^3}\sin\theta \end{cases}$$

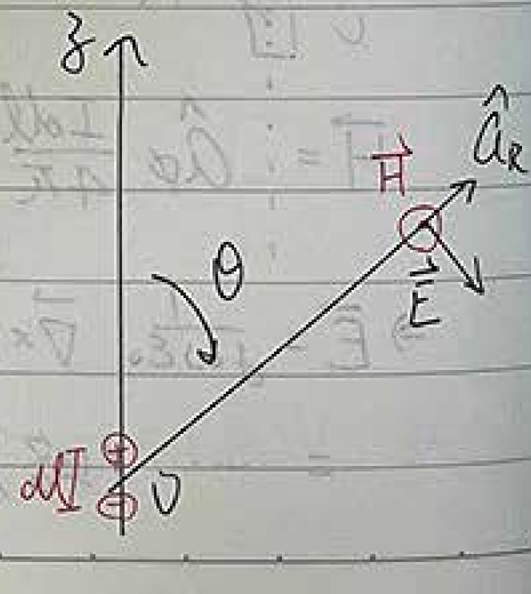


→ quasi static E field due to electric dipole $\vec{P}$

(b) Far Field ($\beta R \gg 1$)

$$H_\phi \approx -\frac{Idl}{4\pi}\beta^2 \sin\theta \frac{e^{-j\beta R}}{j\beta R}$$

$$= j\frac{Idl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\beta \sin\theta$$



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$$E_\theta \approx j\frac{I\,dl}{4\pi}\underbrace{\left(\frac{e^{-j\beta R}}{R}\right)}_{\text{sphirical wave!}}\eta_0\beta\sin\theta$$

$E_R \approx 0$

其中の $\vec{E}$, $\vec{H}$ 等相位!

②很遠 → 球面波可看成平面波 ex. 陽光

* 結論:

遠場 → 球面波, 能量往遠方傳播

近" → 算 Poynting vector 可知能量会留在天線附近之空間中, 不傳出去

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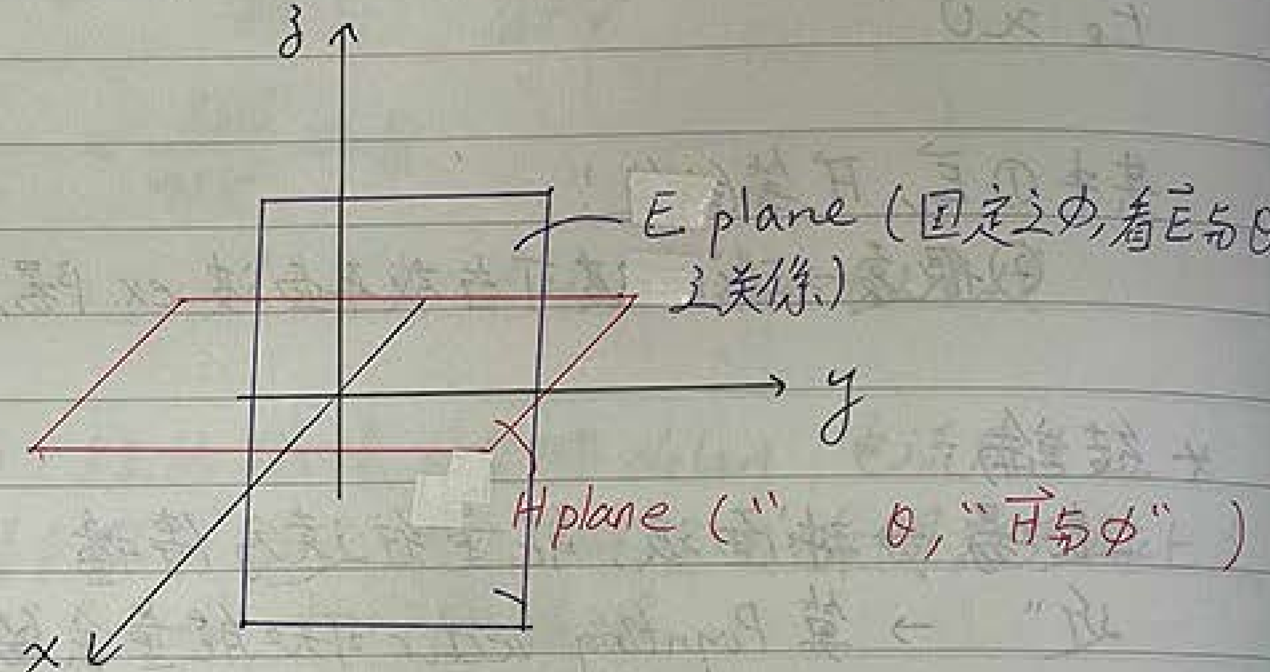
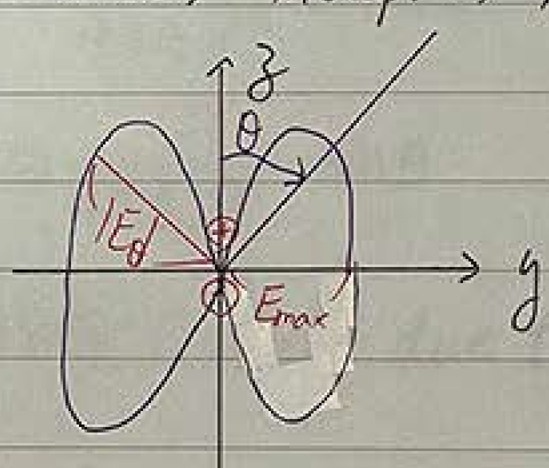
§11-3 Antenna Pattern

1. 輻射圖騰：描述遠場強度对方向之变化，分 E plane 与 H plane



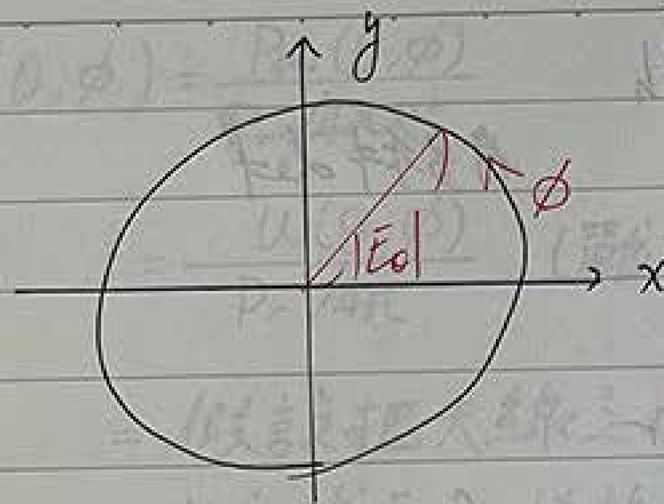
ex 11-1 (a) E-plane pattern

$|E| = |E_\theta| \propto \sin\theta$, indep. of ϕ



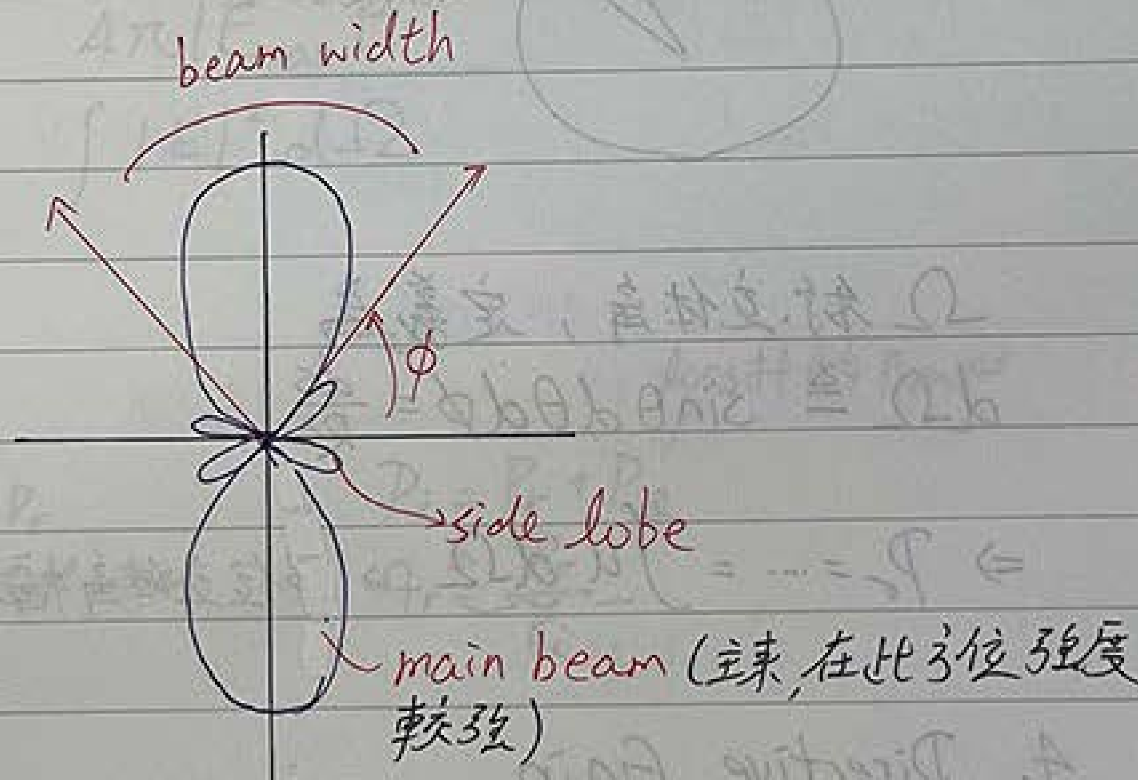
电場振幅隨角度θ之变化

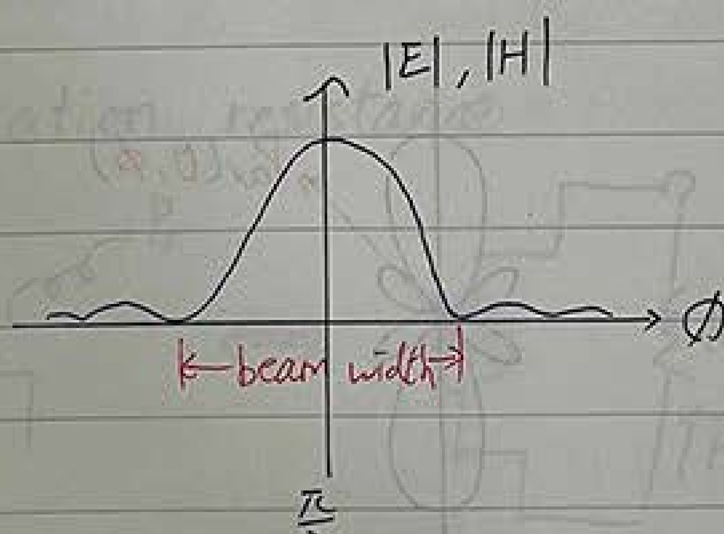
(b) H plane : $|E_\theta| = const$



→ Isotropic omnidirectional

2. 各种術語

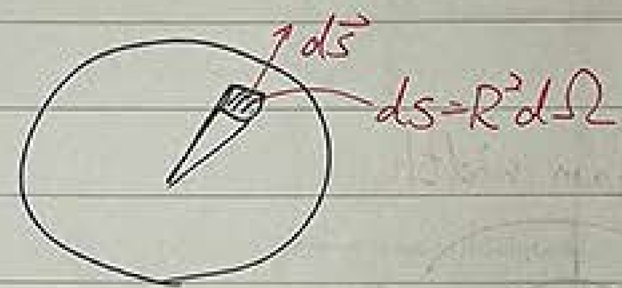




$\left\{\begin{array}{l}\text{beam width}\\ \text{side lobe}\\ \text{directivity}\end{array}\right.$ 有多种 def

3. Total Power

$$P_r = \oint \vec{P}_{av}\cdot d\vec{s} = \int P_{av}\cdot \underbrace{R^2 \sin\theta\, d\theta\, d\phi}_{ds}$$

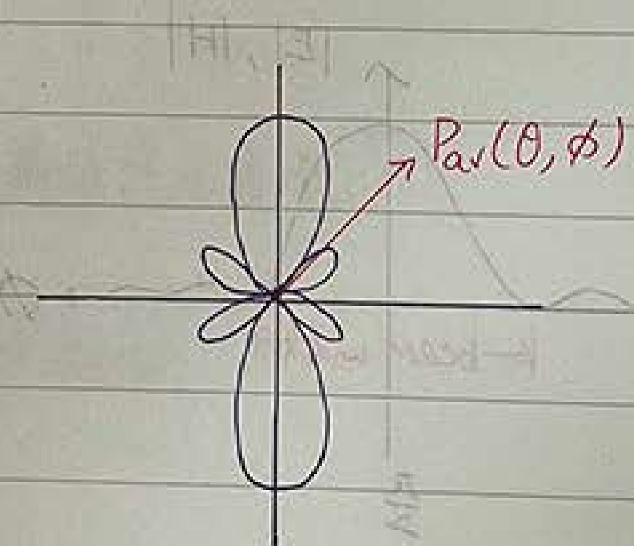


Ω 稱立体角，定義為

$$d\Omega \triangleq \sin\theta\, d\theta\, d\phi = \frac{ds}{R^2}$$

$\Rightarrow P_r = \cdots = \int \underbrace{U}\cdot \underbrace{d\Omega}$ → 單位立体角輻射出去多少功率

4. Directive Gain



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$$G_D(\theta,\phi) = \frac{P_{av}(\theta,\phi)}{P_r/4\pi R^2}$$

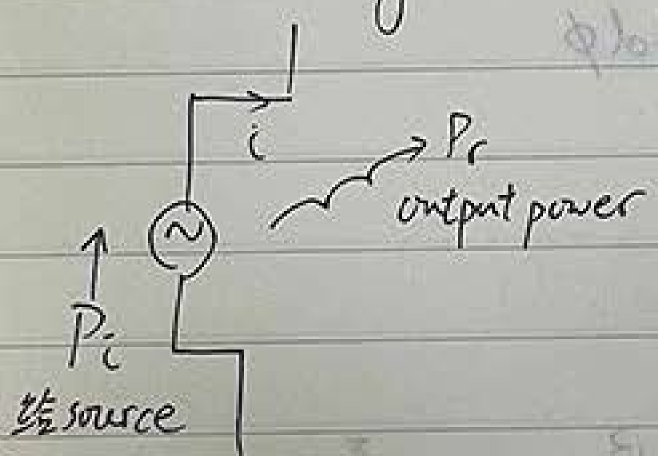
$$= \frac{U(\theta,\phi)}{P_r/4\pi} \quad \text{(單位立体角上總功率)}$$

= 假設把天線之功率平分在一半徑為 r 之球面上，以此情形之功率為基準，與實際上某方向之功率比較.

$$= \frac{U_{max}}{P_r/4\pi}$$

$$= \frac{4\pi|E_{max}|^2}{\iint |E|^2 \, d\Omega}$$

(d) efficiency

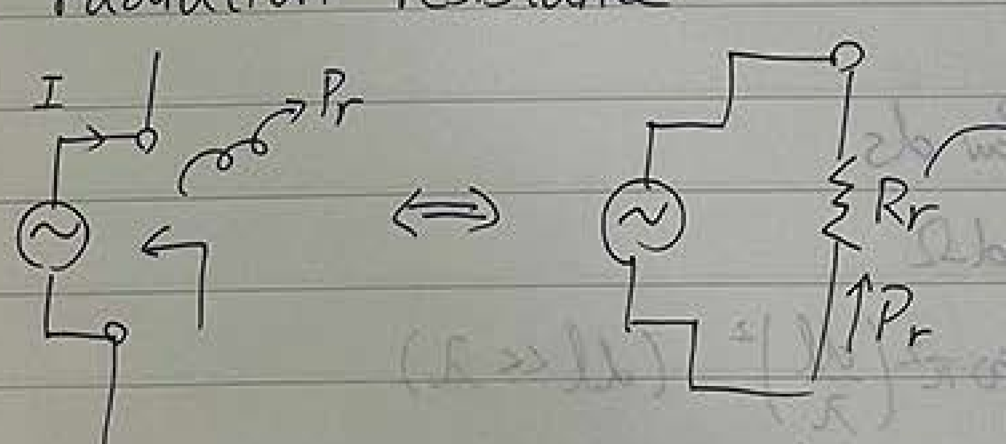


loss掉的 Power

$$P_i = P_r + P_l$$

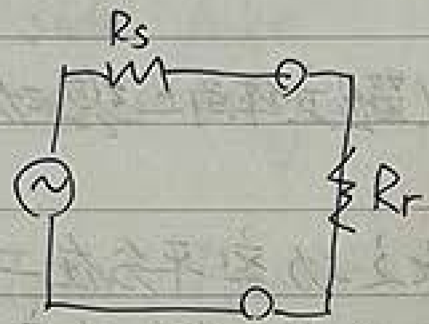
$$\eta_r \triangleq \frac{P_r}{P_i}$$

(e) radiation resistance



假設有一了load 使有進去消耗 P_r

$\Rightarrow P_r = \frac{I^2}{2} R_r$, $R_r = \frac{2P_r}{I^2}$ (amplitude)



可能有阻抗匹配之情形

ex11-3. Hertizen Dipole 之輻射电阻 R_r

$$E_\theta \approx j \frac{I\,dl}{4\pi}\frac{e^{-j\beta R}}{R}\eta_0 \beta \sin\theta$$

$P_{av} \propto |E_\theta|^2$

$$\Rightarrow U = P_{av} R^2 = \cdots = \frac{(I\,dl)^2}{32\pi^2}\eta_0\beta^2\sin^2\theta$$

$$\Rightarrow G_D = \frac{4\pi\sin^2\theta}{\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\cdot\sin\theta\,d\theta\,d\phi}$$

$$= \cdots$$

$$= \frac{3}{2}\sin^2\theta$$

Directivity Gain $G_{D,max} = \frac{3}{2}$ at $\theta = \frac{\pi}{2}$

$\frac{3}{2} = 10\log_{10} 1.5 = 1.76\,dB$

$$\Rightarrow P_r = \oint P_{av}\,ds$$

$$= \int U\,d\Omega$$

$$= \frac{I^2}{2}\underbrace{80\pi^2\left(\frac{dl}{\lambda}\right)^2}_{R_r} \quad (dl \ll \lambda)$$

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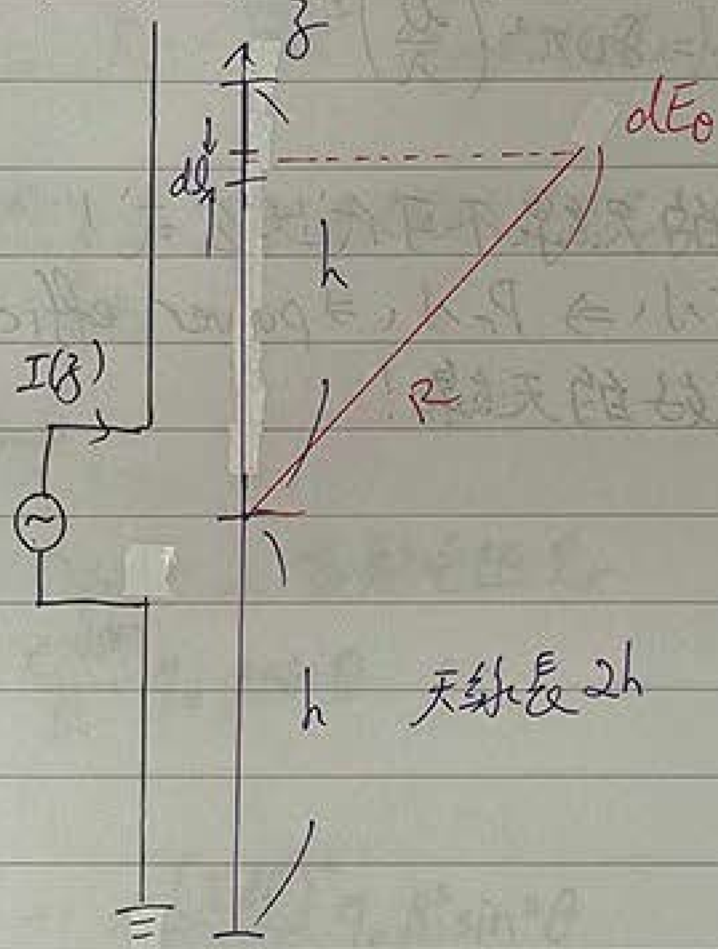
$$\Rightarrow R_r = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

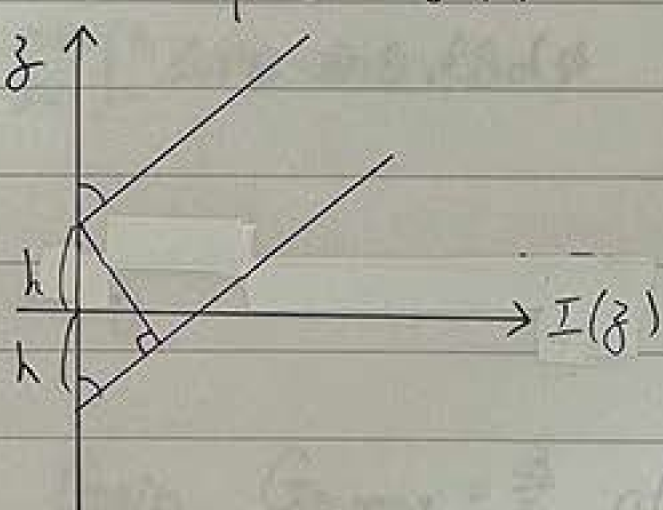
* ① 很長的天線不可代此公式!
② R_r 太小 $\Rightarrow P_r$ 小 $\Rightarrow$ power efficiency 低
$\Rightarrow$ 不是好的天線!

§11.4 薄型直線天線



1. 導線上 I 之 phasor 為：

$$I(z) = I_m \sin\beta(h-|z|)$$



B.C.：$I|_{z=\pm h} = 0$

$dE_\theta = \eta_0 \, dH_\phi$

$\Rightarrow E_\theta = \eta_0 H_\phi$

$= \int_{-h}^{h} dE_\theta$

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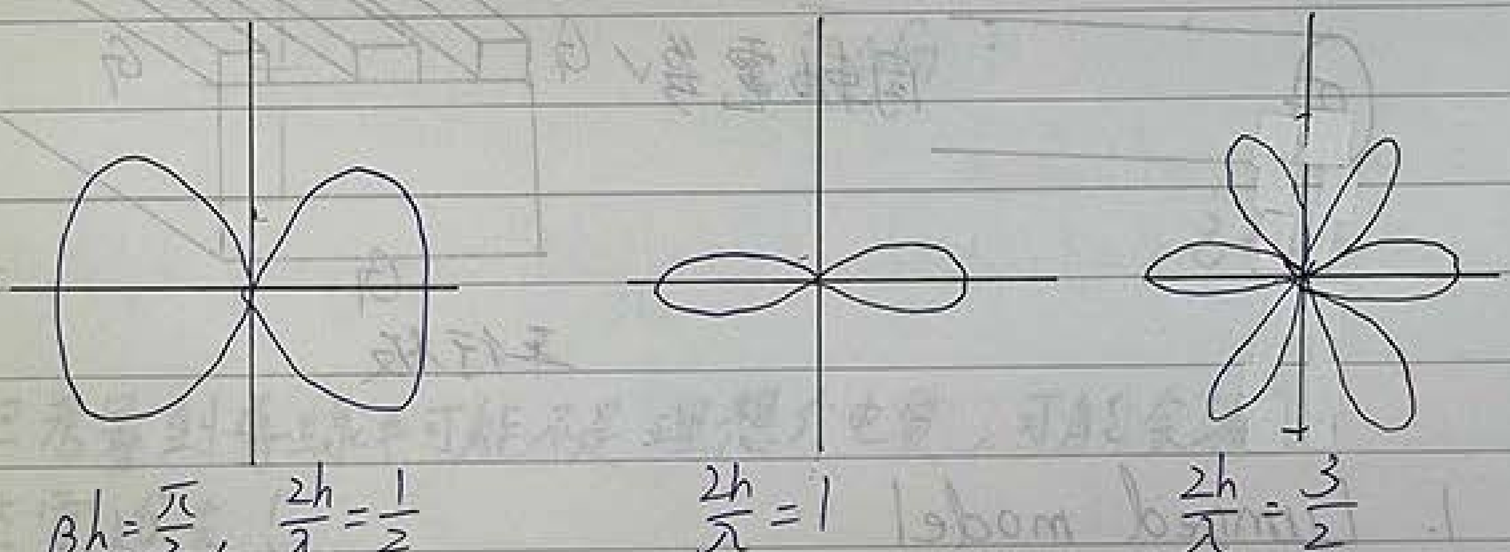
$$= \int_{-h}^{h} j \frac{I(z)}{4\pi} e^{-\frac{j\beta R'(z)}{R'(z)}} \eta_0 \beta \sin\theta \, dz$$

$$= j\,60\, I_m \frac{e^{-j\beta R}}{R} F(\theta)$$

其中 $F(\theta) = \frac{\cos(\beta h \cos\theta) - \cos\beta h}{\sin\theta}$ = pattern function

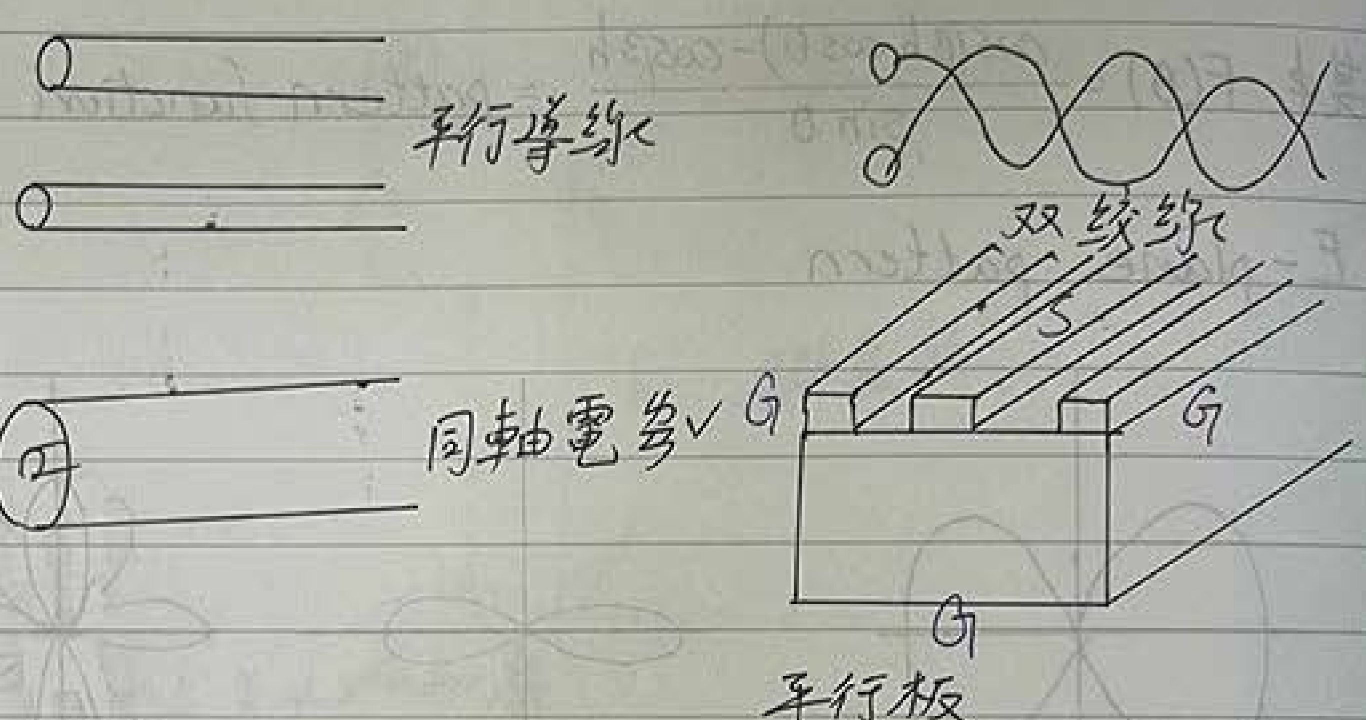
E-plane pattern



Ch9 傳輸線

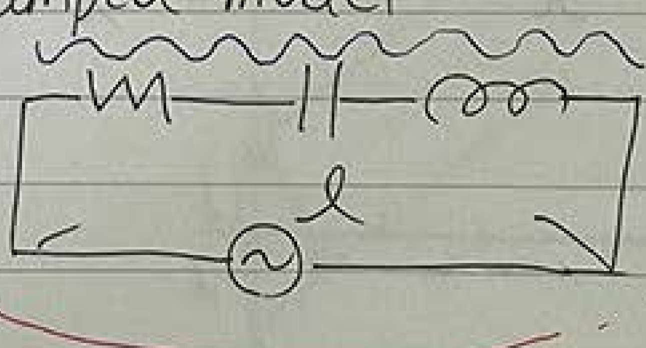
§9.1 Introduction

1. 常見的傳輸線



1. Lumped model

λ很小 → distributed model



λ很大 → lumped model

KCL → 假設 I處處相同，但這是基於 $l \ll \lambda$

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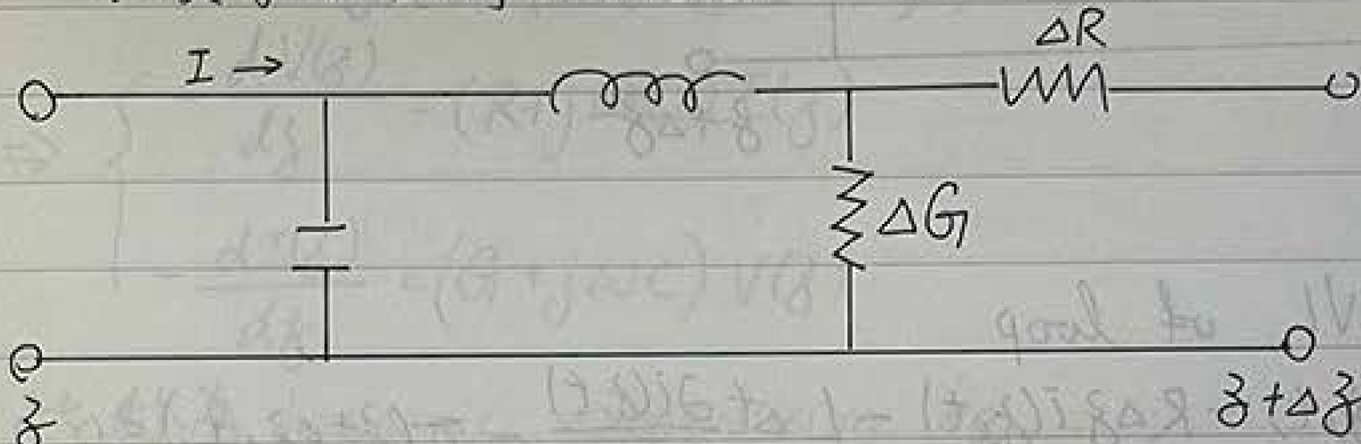
§9.3 一般的傳輸線方程式

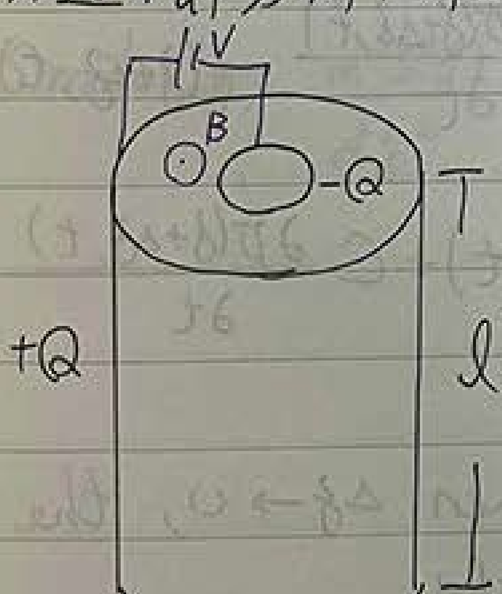
1. 如上一節所述，lumped model 是建立在 $l \ll \lambda$ 時用
$\Rightarrow$ 取一小段導線 Δl，若 $\Delta l \ll \lambda$，則可用 lumped model 分析！

2. 一小段導線的等效 model



但考量到導線中可能不是理想介电質，可能会漏一些電進去！

不过本節分析同軸電纜：



C' = capacitance per unit length

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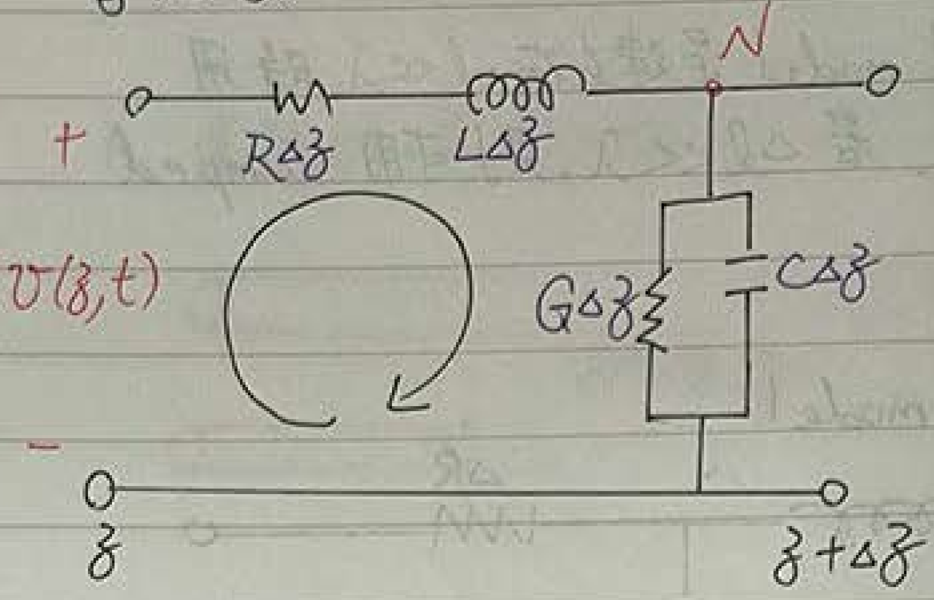
等效於:



3. KVL of loop

$$v(z,t) - R\Delta z\, i(z,t) - L\Delta t \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z, t) = 0$$

$$\Rightarrow -\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R i(z,t) + \frac{L\partial i(z,t)}{\partial t}$$

4. KCL at N

$$i(z,t) - G\Delta z\, v(z+\Delta z, t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z, t) = 0$$

$$\Rightarrow -\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = G v(z+\Delta z, t) + C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Let $v = v(z,t)$, $i = (z,t)$, then when $\Delta z \to 0$, the general transmission eq. is given by

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$$\begin{cases} -\dfrac{\partial v}{\partial z} = Ri + L\dfrac{\partial i}{\partial t} \\ \dfrac{\partial i}{\partial z} = Gv + C\dfrac{\partial v}{\partial t} \end{cases}$$

→ special case! V是弦波

5. When $v(z,t) = Re\{V(z)e^{j\omega t}\}$, $i(z,t) = Re\{I(z)e^{j\omega t}\}$

$$\Rightarrow \begin{cases} -\dfrac{dV(z)}{dz} = (R+j\omega L)I(z) \\ -\dfrac{dI(z)}{dz} = (G+j\omega C)V(z) \end{cases}$$

此称为 harmonic transmission eq.

$$\Rightarrow \frac{d^2V(z)}{dz^2} = -(R+j\omega L)\frac{dI(z)}{dz}$$

$$= (R+j\omega L)(G+j\omega C)V(z)$$

$$\triangleq \gamma^2 V(z)$$

Similarly, $\dfrac{d^2I(z)}{dz^2} = \gamma^2 I(z)$

> wave eq!

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§9.3.1 ∞長 wire

1. 定義 $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} \triangleq \alpha + j\beta$

$$\begin{cases} \gamma : \text{propogation const.} \\ \alpha : \text{attenuation const.} \quad (Np/m) \\ \beta : \text{phase const.} \quad (rad/m) \end{cases}$$

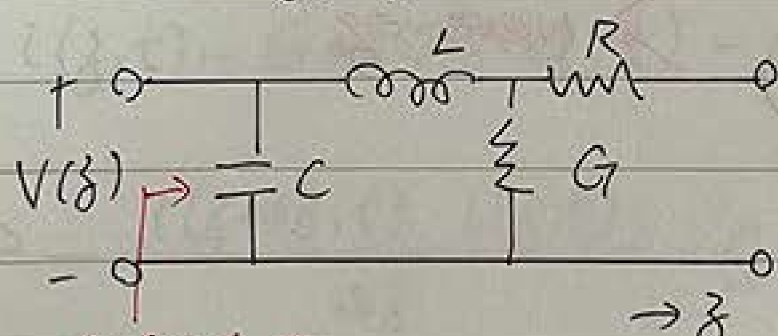
2. 解 V、I 之 wave eq, 得

$$V_0(z) = \underbrace{V_0^+ e^{-\gamma z}}_{+z \text{ wave}} + \underbrace{V_0^- e^{\gamma z}}_{-z \text{ wave}}, \quad \because V(\infty) = 0, \therefore$$

$$\Rightarrow \begin{cases} V(z) = V_0^+ e^{-\gamma z} \\ I(z) = I_0^+ e^{-\gamma z} \end{cases}$$

$$\Rightarrow \frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+} \triangleq Z_0 = \text{const}$$

Z_0 稱 特徵阻抗 (characteristic impedence),
物理意義為:



看進去為 Z_0

(1) $Z_0 = \dfrac{R+j\omega L}{\gamma} = \dfrac{\gamma}{Gj\omega C} = \sqrt{\dfrac{R+j\omega L}{G+j\omega C}} \triangleq R_0 + jX_0$

(2) 此現象好比 free space 中

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$$-\frac{dE_x(z)}{dz} = (\omega\mu'' + j\omega\mu')H_y(z)$$

$E_x \longleftrightarrow V$ $\quad R \longleftrightarrow \omega\mu'$ $\quad G \longleftrightarrow \omega\varepsilon''$

$H_y \longleftrightarrow I$ $\quad L \longleftrightarrow \mu'$ $\quad C \longleftrightarrow \varepsilon'$

3. Special case

(1) lossless $(R=0, G=0)$

$\Rightarrow \gamma = j\omega\sqrt{LC} \doteq j\beta, \quad \beta = \omega\sqrt{LC}$

$\Rightarrow u_p = \frac{\omega}{\beta} = \sqrt{\frac{1}{LC}} = \text{const}$

$\Rightarrow$ dispersionless, distortionless

$\Rightarrow Z_0 = \sqrt{\frac{L}{C}} \triangleq R_0 + jX_0$

$$\begin{cases} X_0 = 0 \\ R_0 = \sqrt{L/C} \end{cases}$$

(2) lowloss

$$\gamma = \sqrt{j\omega L}\sqrt{j\omega C}\left(1+\frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1+\frac{G}{j\omega C}\right)^{\frac{1}{2}}$$

$$\doteq j\omega\sqrt{LC}\Big(\underbrace{1}_{\text{for }\beta} + \underbrace{\frac{1}{2}\frac{1}{j\omega}\left(\frac{R}{L}+\frac{G}{C}\right)}_{\text{for }\alpha}\Big)$$

$$\Rightarrow \begin{cases} Z_0 \approx \sqrt{\frac{L}{C}}\Big(\underbrace{1}_{\text{for }R_0} + \underbrace{\frac{1}{2j\omega}\left(\frac{R}{L}-\frac{G}{C}\right)}_{\text{for }X_0\text{, but small}}\Big) \\ R_0 = \sqrt{\frac{L}{C}} \\ X_0 \to 0 \end{cases}$$

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(c) distortionless $\left(\frac{R}{L}=\frac{G}{C}\right)$

$$\gamma=\sqrt{\frac{C}{L}}\sqrt{(R+j\omega L)\left(\frac{GL}{C}+j\omega C\cdot\frac{L}{C}\right)}$$

$$=\sqrt{\frac{C}{L}}(R+j\omega L)$$

$$\Rightarrow\begin{cases}\alpha=\sqrt{\frac{C}{L}}R\\ \beta=\omega\sqrt{LC}\end{cases}$$

① $\Rightarrow u_p=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}}=\text{const.}$

② $Z_0=\sqrt{\frac{L}{C}}=R_0+jX_0$

$$\begin{cases}X_0=0\\ R_0=\sqrt{\frac{L}{C}}\end{cases}$$

ex9-3. 有一 distortionless 導線，衰減為 0.01 dB/m，

$C=0.1\ nF/m$，求

(a) 每公尺 R, L, G

(b) EM wave 在其上之傳播速度

(c) 在 1km, 5km 處之 V 降低之百分率

-sol-. 可知：$\because e^{-\alpha z}=\frac{|V_2|}{|V_1|}$，

$\therefore$ attenuation/decay: $-20\log_{10}e^{-\alpha\Delta z}=\alpha\Delta z\,20\log_{10}e$

$=8.69\alpha\Delta z\ (dB)=0.01\Delta z\ (dB/m)$

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$\Rightarrow \alpha = \frac{0.01}{8.69} \approx 1.5\times10^{-3}$ (1/m or Np/m)

(a) $\because Z_0 = R_c = \sqrt{\frac{L}{C}} = 50\,\Omega$

$\therefore \begin{cases} R = R_0\alpha = 0.057\ \Omega/m \\ L = C\cdot R_0^2 = \cdots = 0.25\ \mu H/m \\ G = \frac{RC}{L} = \cdots = 22.8\ \mu s/m \end{cases}$

(b) $u_p = \sqrt{\frac{1}{LC}} = \cdots = 2\times10^8\ m/s$

(c) attenuation in 1km

$\Rightarrow \frac{V_2}{V_1} = e^{-\alpha\Delta z} = 31.7\%$

5km：同理，0.32% #

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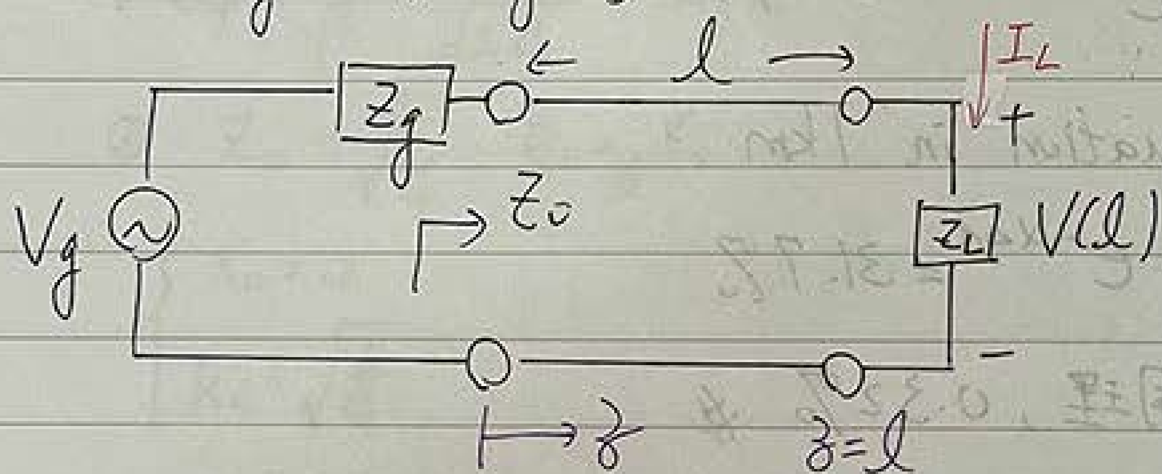
§9.4 有限長 wire

1. $V(z)$、$I(z)$ 之 wave eq 之 sol

$$\begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \end{cases}$$

$$\Rightarrow Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-}$$

2. 考虑一有限長 wire，長 l，load 為 Z_L，有一电压源 V_g（内阻 Z_g）与之相連



$z'=l$ $\leftarrow z'$ → 考虑另一組坐標

$\Rightarrow$ net 正/负向波相对大小？

at $z=l$, $\dfrac{V(l)}{I(l)} = \dfrac{V_L}{I_L} = Z_L$

3. $$\begin{cases} V(l) = V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l} \\ I(l) = I_L = I_0^+ e^{-\gamma l} + I_0^- e^{\gamma l} = \dfrac{V_0^+}{Z_0} e^{-\gamma l} - \dfrac{V_0^-}{Z_0} e^{\gamma l} \end{cases}$$

解联立求 V_0^+, $V_0^- = \cdots$

$$\Rightarrow \quad V(z) = \frac{I_L}{2}\Big(\underbrace{(Z_L+Z_0)e^{\gamma(l-z)}}_{-z\ \text{wave}} + \underbrace{(Z_L-Z_0)e^{-\gamma(l-z)}}_{+z\ \text{wave}}\Big)$$

$$I(z) = \frac{I_L}{2Z_0}\Big((Z_L+Z_0)e^{\gamma(l-z)} - (Z_L-L_0)e^{-\gamma(l-z)}\Big)$$

Let $z' = l - z$, then

$$V(z') = \frac{I_L}{2}\Big((Z_L+Z_0)e^{\gamma z'} + (Z_L-Z_0)e^{-\gamma z'}\Big)$$

$$= I_L(Z_0 \cosh\gamma z' + Z_0 \sinh\gamma z')$$

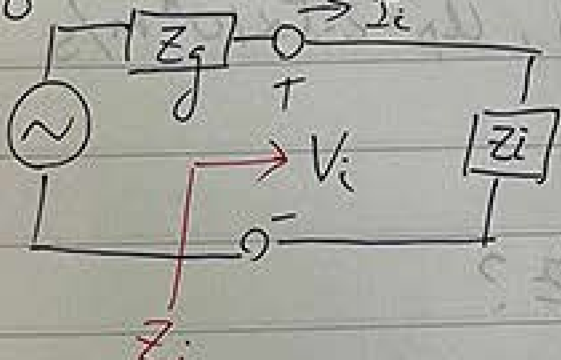
$$\Rightarrow I(z') = \frac{I_L}{Z_0}(Z_0 \cosh\gamma z' + Z_L \sinh\gamma z')$$

$$\Rightarrow Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L + Z_0 \tanh\gamma z'}{Z_0 + Z_L \tanh\gamma z'}$$

4. At source $z' = l$

$$Z(z'=l) = Z_i = Z_0 \frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l}$$

eq. circuit 為



可化簡為 lumped model!

$$\Rightarrow \begin{cases} V_i = V_g \dfrac{Z_i}{Z_g + Z_i} \\[2ex] I_i = \dfrac{V_g}{Z_g + Z_i} \end{cases}$$

端

5. source 给 input 端之平均功率

$$(P_{av})_i = \frac{1}{2} Re\{V_i I_i^*\}$$

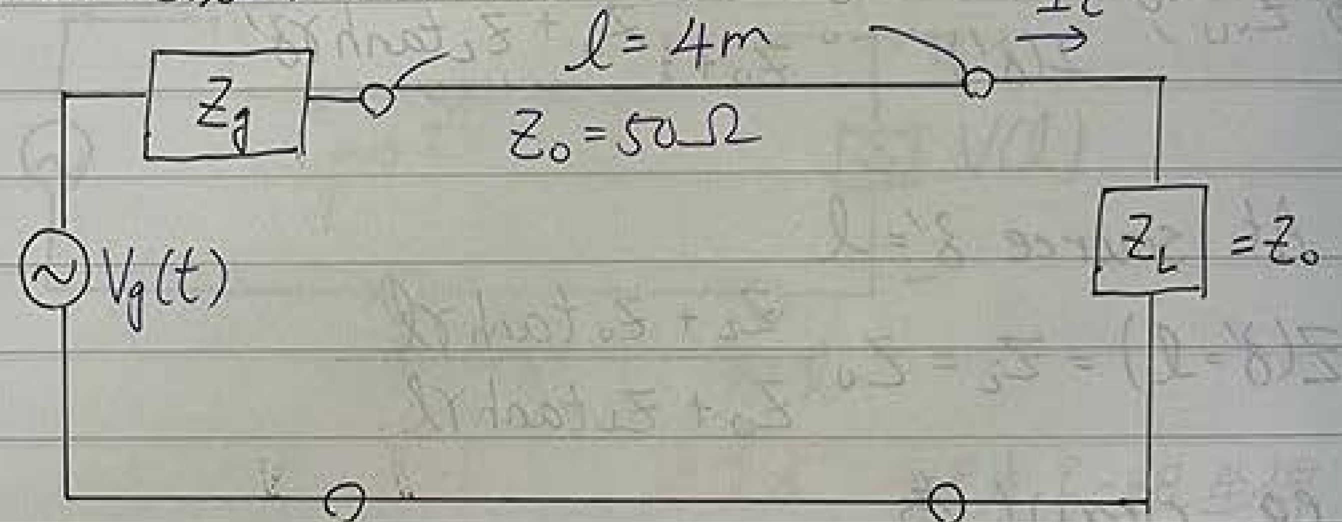
$$(P_{av})_L = \frac{1}{2} Re\{V_L I_L^*\}$$

6. For $Z_L = Z_0 \Rightarrow Z(z') = Z_0,\ \forall z'$

$\Rightarrow Z_i = Z_0$

$\Rightarrow V(z) = V_i e^{-\gamma z},\ V_i = V(z=0)$

ex9-5. 考虑以下之 wire



令 $V_g(t) = 0.3\cos(2\pi\times10^8 t)$, $u_p = 2.5\times10^8\ m/s$

求 (a) $v(z,t)$, $i(z,t)$

(b) Z_L 上之 v, i

(c) 送到 Z_L 之平均功率?

-sol-. $\because V_g(t) = 0.3\angle 0°$, $\omega = 2\pi\times10^8$

$\therefore \beta = \dfrac{\omega}{u_p} = 0.8\pi\ rad/m$

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(a) $v(z,t) = Re\{V(z)e^{j\omega t}\}$

$$V_i = V_g \cdot \frac{Z_0}{Z_0 + Z_g} = 0.3(\angle 0^\circ)\frac{50}{50+1} = 0.294\angle 0^\circ$$

$\Rightarrow V(z) = V_i e^{-j\beta z} = 0.294 e^{-j0.8\pi z}$

$\Rightarrow v(z,t) = 0.294\cos(2\pi\times10^8 t - 0.8\pi z)$

同理, $i(z,t) = 0.0059\cos(2\pi\times10^8 t - 0.8\pi z)$

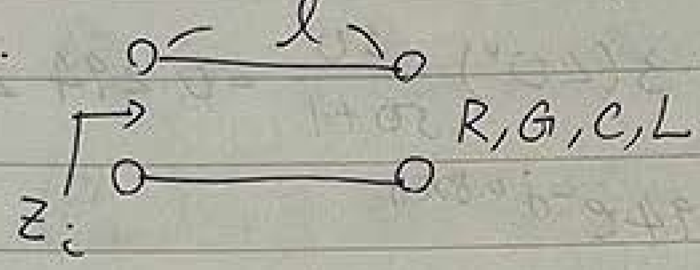
(b) $v(4,t) = 0.294\cos(2\pi\times10^8 t - 3.2\pi)$

$i(4,t) = 0.0059\cos(2\pi\times10^8 t - 0.8\pi z)$

(c) $(P_{av})_L = (P_{av})_i = \frac{1}{2}Re\{VI^*\} = \cdots$

$= 0.87\,mW$ #

§9.4.1 視為电路元件之 wire

1. 

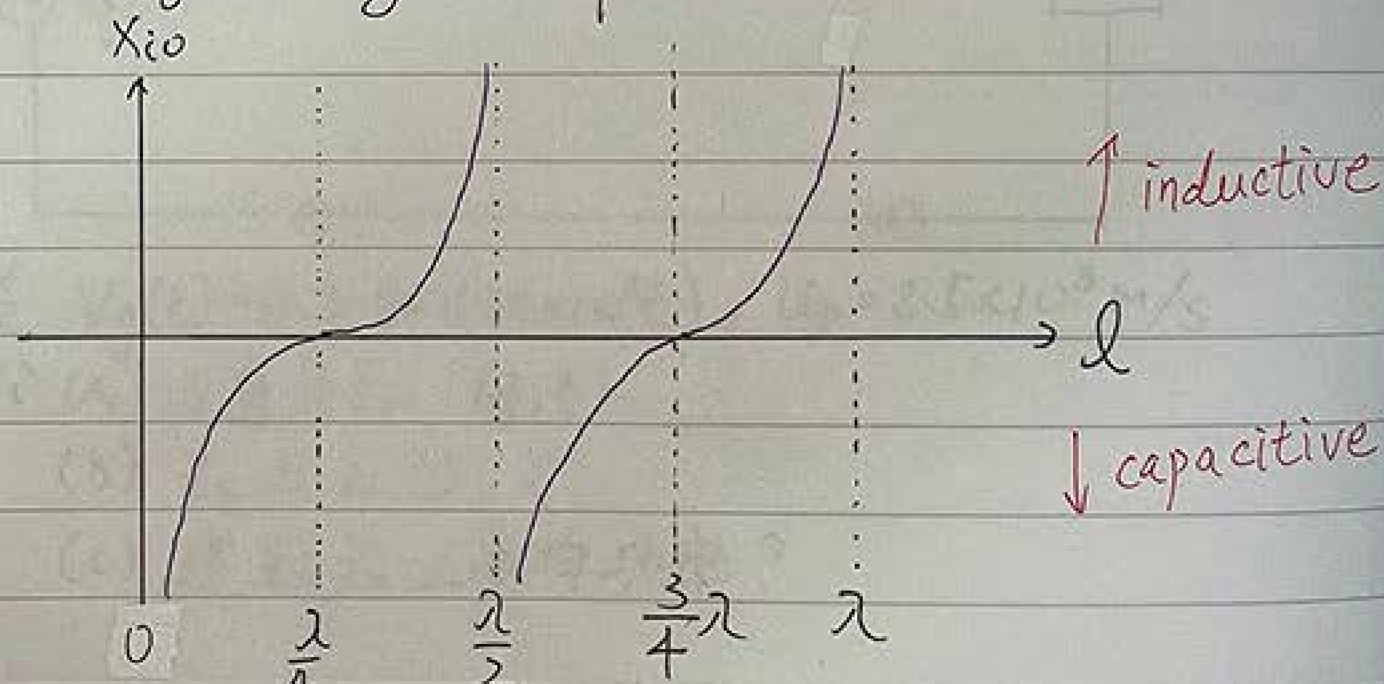
$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'} , \gamma = j\beta , Z_0 = R_0$$

$\because \tanh \gamma l = \tanh(j\beta l) = j \tan \beta l$

$$\therefore Z_i = Z(z' = l) = R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l}$$

(1) $Z_L \to \infty$ (open circuit termination)

$Z_{io} = jX_{io} = -jR_0 \cot \beta l$ (令此時 $Z_i \triangleq Z_{io}$)



When $l \ll \lambda$, $\beta l \ll 1$,

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$$Z_{io} = jX_{io}$$
$$\approx -jR_0\frac{1}{\beta l}$$
$$= -j\frac{\sqrt{L/C}}{\omega\sqrt{LC}\,l}$$
$$= \frac{1}{j\omega CL}$$

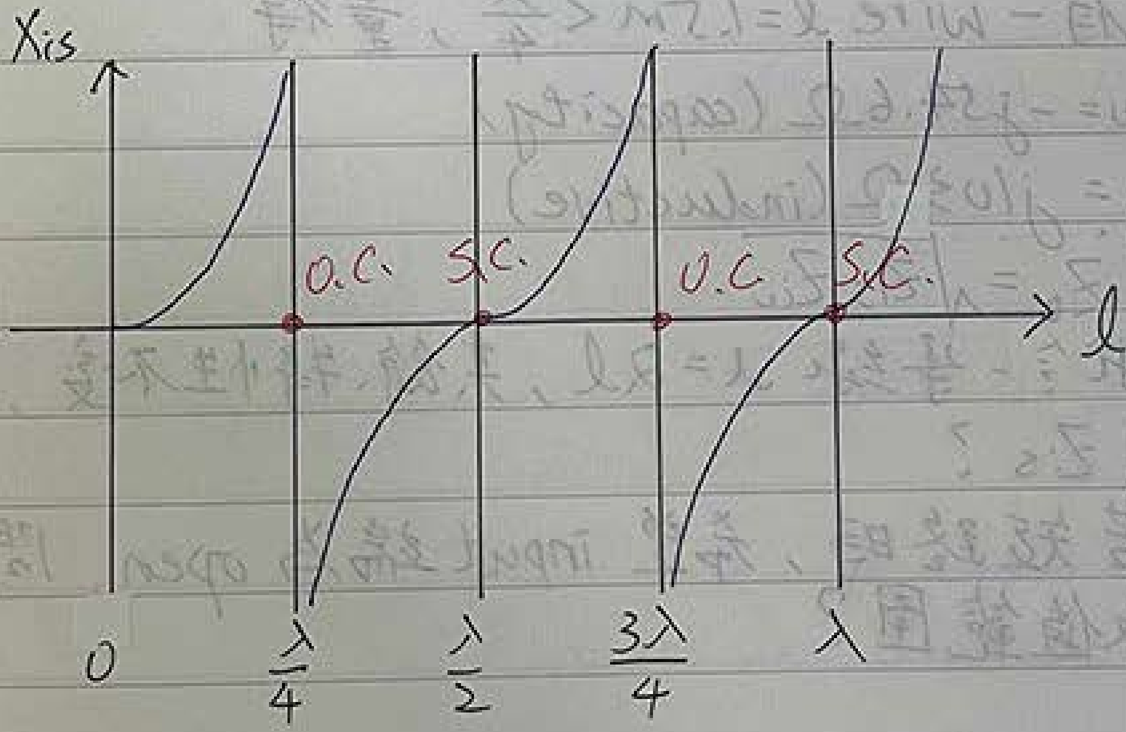
⇒ 線短時，看到的阻抗大

(2) $Z_L \to 0$ (Short circuit)，令此時 $Z_i = Z_{is}$，則

$$Z_{is} = R_0\frac{Z_L + R_0 j\tan\beta l}{R_0 + Z_L j\tan\beta l}\Bigg|_{Z_L\to 0}$$

$$= jX_{is}$$

$$= jR_0\tan\beta l$$

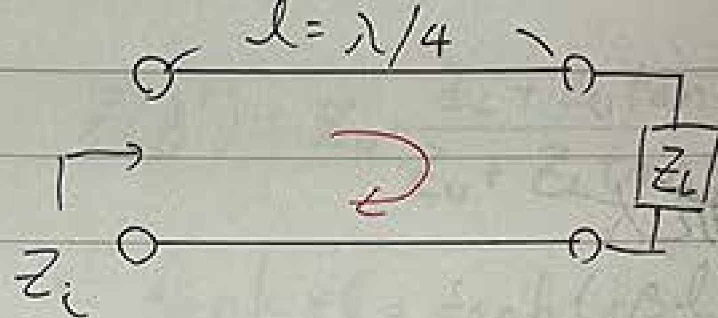


When $\beta l \ll 1 \Rightarrow Z_{is} \approx \cdots = j\omega Ll$

(3) quarterwave sedion

$l = \lambda/4$, $\beta l = \pi/2$, $\tan\beta l = \pm\infty$

$\Rightarrow Z_i = \dfrac{R_0^2}{Z_L}$ = quarter wave transfer



e.g. $\begin{cases} Z_L = 0 \rightarrow Z_i = \infty \\ Z_L = \infty \rightarrow Z_i = 0 \end{cases}$

(4) Half wave

$l = \lambda/2$, $\beta l = \pi$, $Z_i = Z_L$

ex9-6. 有一wire $l = 1.5m < \frac{\lambda}{4}$，量得

$\begin{cases} Z_{io} = -j54.6\Omega \text{ (capacity)} \\ Z_{is} = j103\Omega \text{ (inductive)} \end{cases}$

(a) 求 Z_0、γ？

(b) 若另一導線 $l' = 2l$，其餘特性不變，求其 Z_{is}？

(c) 若短路時，希望 input 端為 open，問 l 之取值範圍？

-sol-. (a) $\begin{cases} Z_0 = \sqrt{Z_{is} Z_{io}} = 75\Omega \\ \gamma = \cdots = j0.628 \text{ rad/m} \end{cases}$

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(b) for $l' = 2l = 3$ (m),

$$Z_{is} = Z_0 \tanh \gamma l'$$
$$= Z_0 j \tan \beta l'$$
$$= -j231\,\Omega \text{ (capacitive)}$$

(c). For $Z_L = 0$ (short circuit) and $Z_{is} = \infty$,

$$l = (2n+1)\frac{\lambda}{4}$$
$$= (2n+1)\frac{2\pi}{4\beta}$$
$$= \dots$$
$$= 2.5 + 5(n-1) \text{ (m)}$$

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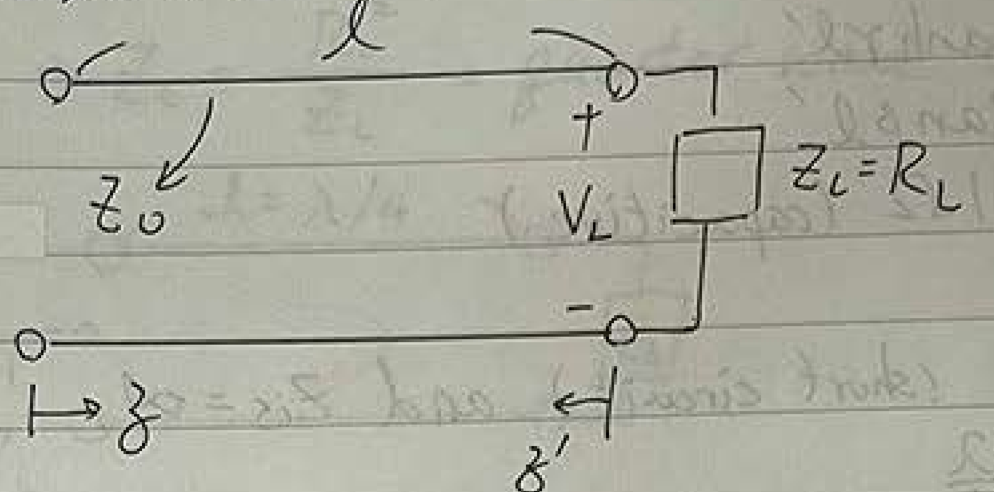
§9.4.2. 連接电阻性負載之wire

1. 考慮 wire



1. $V(z)$ 之 wave eq 之 sol.

$$V(z') = \frac{I_L}{2}\Big(\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{\text{incident}} + \underbrace{(Z_L + Z_0)e^{-\gamma z'}}_{\text{reflected}}\Big)$$

$$= \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\Big(1 + \frac{Z_L Z_0}{Z_L + Z_0}e^{-2\gamma z'}\Big)$$

$$= \cdots$$

$$\triangleq 1 + \Gamma e^{-2\gamma z'}$$

Γ 稱為 voltage reflection coeff.

$$\Gamma = |\Gamma| e^{j\theta_\Gamma} \triangleq \frac{\text{reflected}}{\text{incident}} \text{ at } z' = 0$$

$$= \frac{Z_L - Z_0}{Z_L + Z_0}$$

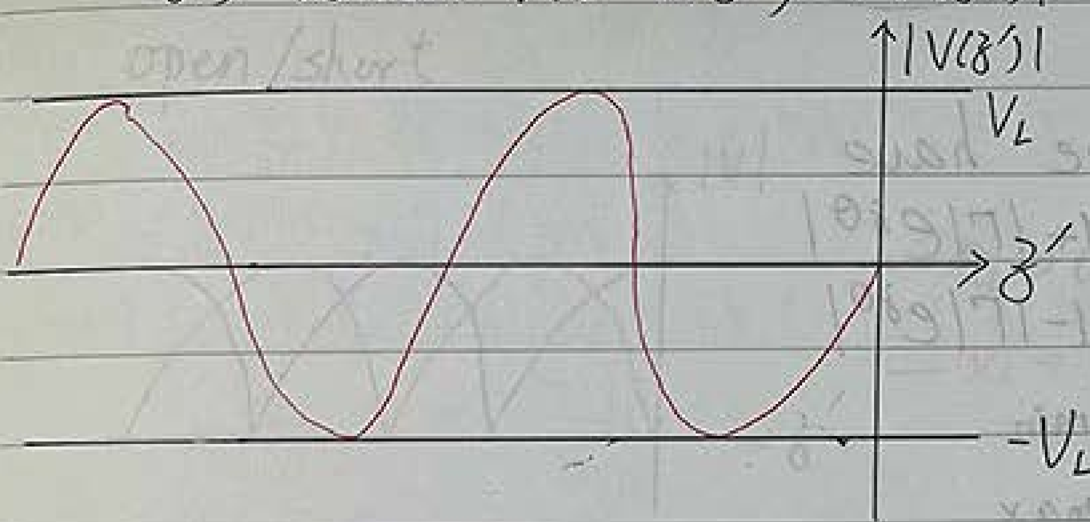
2. For lossless line, $Z_0 = R_0$, $\gamma = j\beta$

$$\Rightarrow V(z') = \frac{I_L}{2}(Z_0 + R_0)\, e^{j\beta z'} \left(1 + |\Gamma| e^{j(\theta_r - 2\beta z')}\right)$$

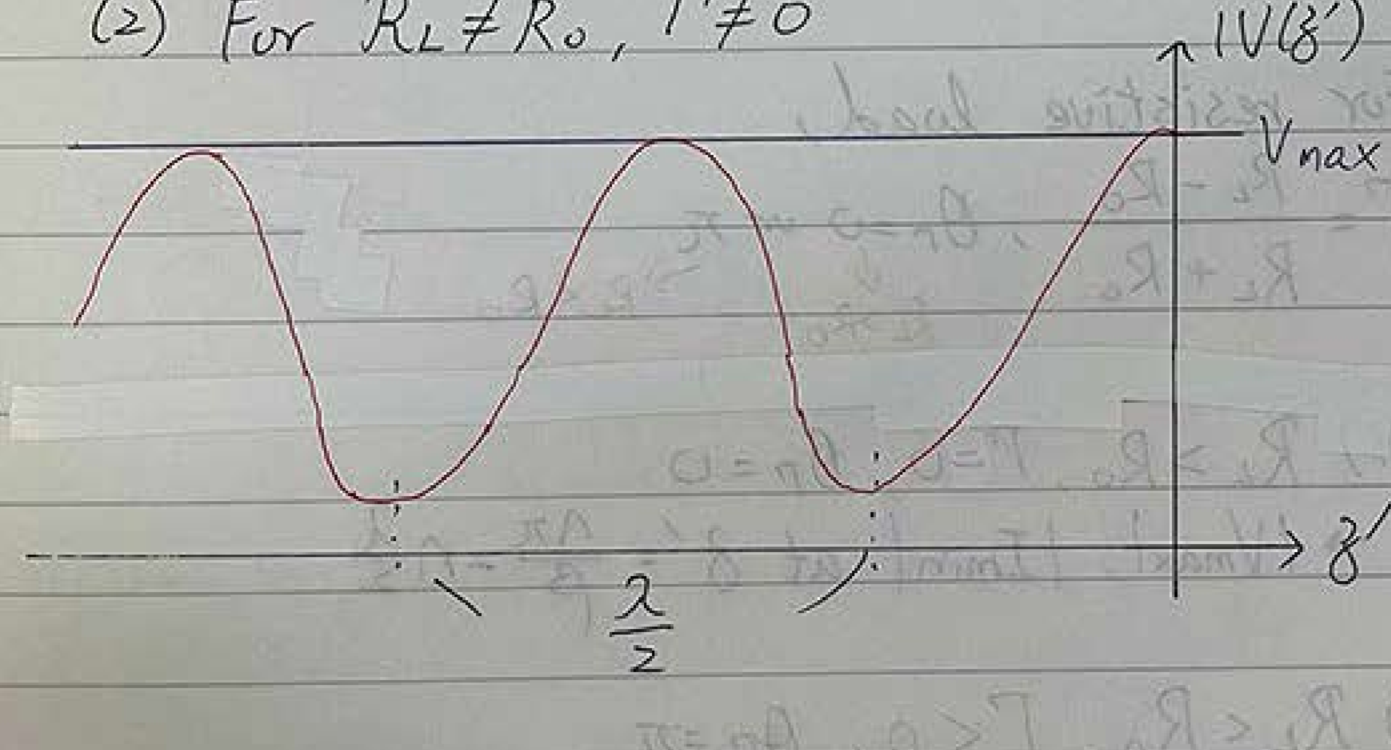
$$|V(z')| = V_L \sqrt{\cos^2\beta z' + \left(\frac{R_0}{R_L}\right)\sin^2\beta z'}$$

3. Standing wave

(1) When $R_L = R_0$, $|V(z')| = V_L = \text{const}$



(2) For $R_L \neq R_0$, $\Gamma \neq 0$



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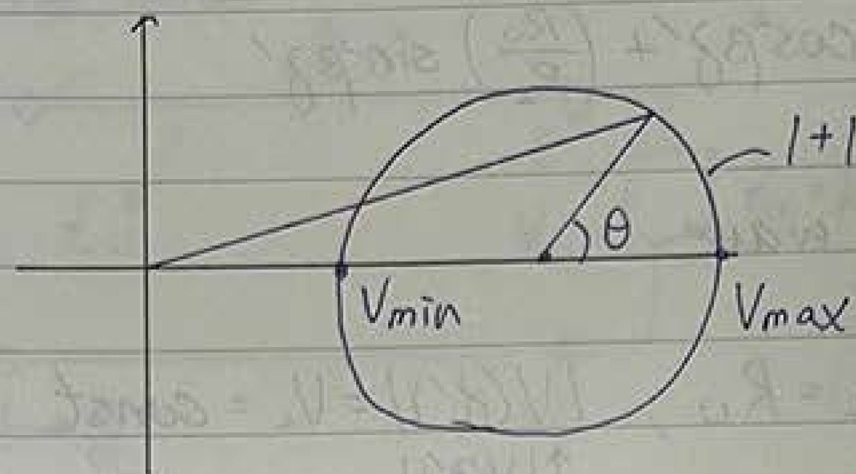
(3) $SNR \triangleq \frac{|V_{max}|}{|V_{min}|} = \frac{1+|\Gamma|}{1-|\Gamma|}$



4. From 1, we have

$$\begin{cases} |V(z')| \propto |1+|\Gamma|e^{j\theta}| \\ |I(z')| \propto |1-|\Gamma|e^{j\theta}| \end{cases}$$

$$\Rightarrow \begin{cases} V_{max} \rightarrow I_{min} \\ V_{min} \rightarrow I_{max} \end{cases}$$

V_{max} at $\theta = \theta_\Gamma - 2\beta z' = -2n\pi$

For resistive load,

$\Gamma = \frac{R_L - R_0}{R_L + R_0}$, $\theta_\Gamma = 0$ or π

$0 \downarrow R_L > R_0$, $\pi \rightarrow R_L < R_0$

(1) $R_L > R_0$, $\Gamma > 0$, $\theta_\Gamma = 0$

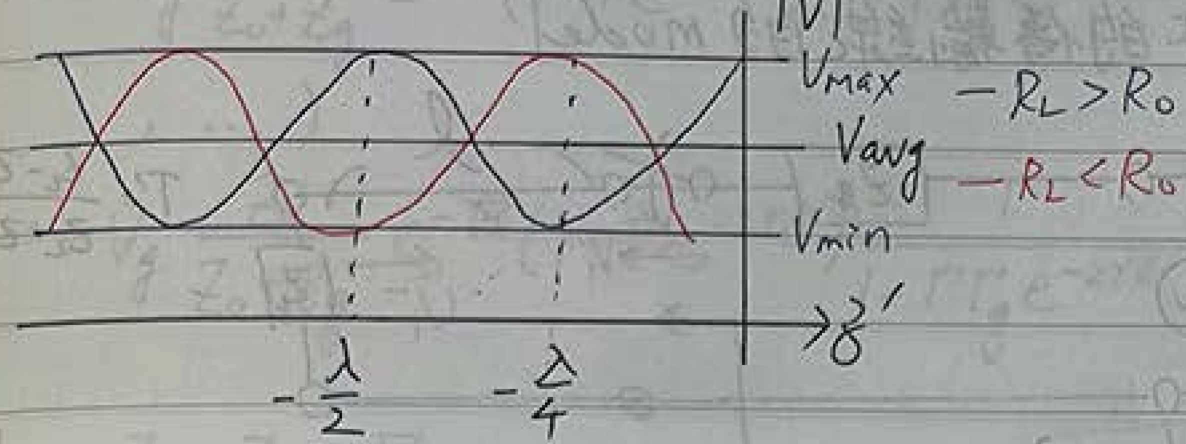
$\rightarrow |V_{max}|, |I_{min}|$ at $z' = \frac{n\pi}{\beta} = n\frac{\lambda}{2}$

(2) $R_L < R_0$, $\Gamma < 0$, $\theta_\Gamma = \pi$

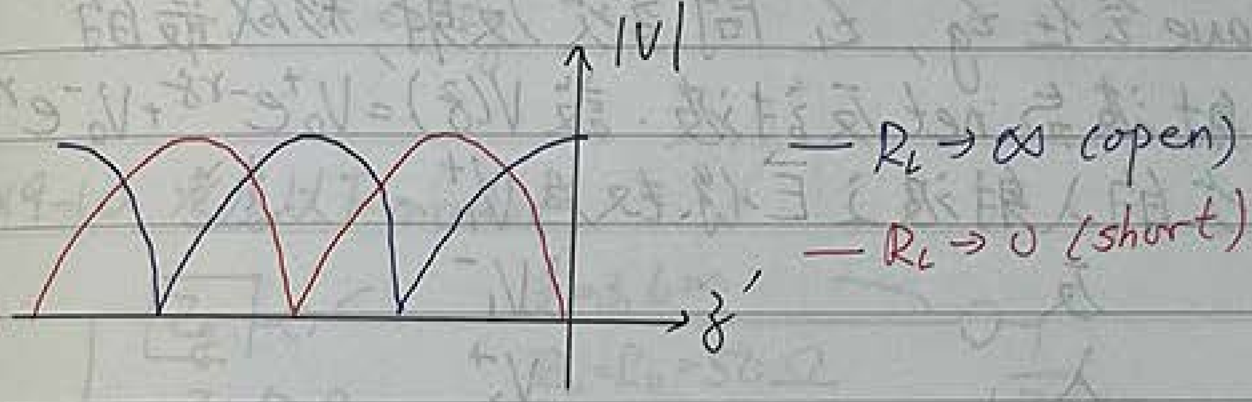
$\rightarrow |V_{min}|, |I_{max}|$ at $z' = n\frac{\lambda}{2}$

$\hookrightarrow \theta = (2n+1)\pi$

一般情形

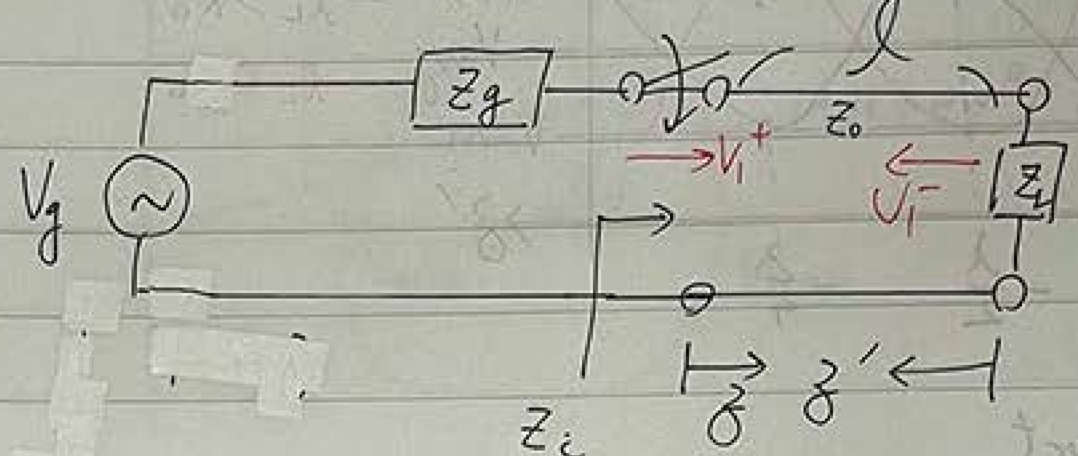


open / short



§9.4.4 傳輸線电路

1. 考慮以下的傳輸線的model



$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$

EM wave 会在 Z_g, Z_L 間多次反射, 形成最的 net 入射波与 net 反射波. 設 $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$, 則
設第1次的入射波之V係数為 V_1^+

1	反	V_1^-
2	入	V_2^+
2	反	V_2^-
⋮		

$$\Rightarrow V(z') = V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots$$

$$= V_g \cdot \frac{Z_0}{Z_0 + Z_g} e^{-\gamma l} + V_g \frac{Z_0}{Z_0 + Z_g} e^{-\gamma l} \cdot \Gamma e^{-\gamma z'}$$

$$+ V_g \frac{Z_0}{Z_0 + Z_g} e^{-\gamma l} \cdot \Gamma e^{-\gamma l} \cdot \Gamma_g e^{-\gamma z} + \cdots$$

$$= V_g \frac{Z_0}{Z_0 + Z_g} e^{-\gamma z} \Big(1 + \Gamma \underbrace{e^{-\gamma l} \cdot e^{-\gamma z'} e^{\gamma z}}_{= -\gamma(l + z' - z)} + \Gamma e^{-2\gamma l} \Gamma_g^{+} + \cdots \Big)$$

$$= V_g \frac{Z_0}{Z_0 + Z_g} e^{-\gamma z} \big(\underbrace{1 + \Gamma e^{-2\gamma z'}}_{V_1^+ + V_1^-} + \underbrace{\Gamma e^{-2\gamma l} \Gamma_g}_{V_2^+ + V_2^-} + \cdots \big)$$

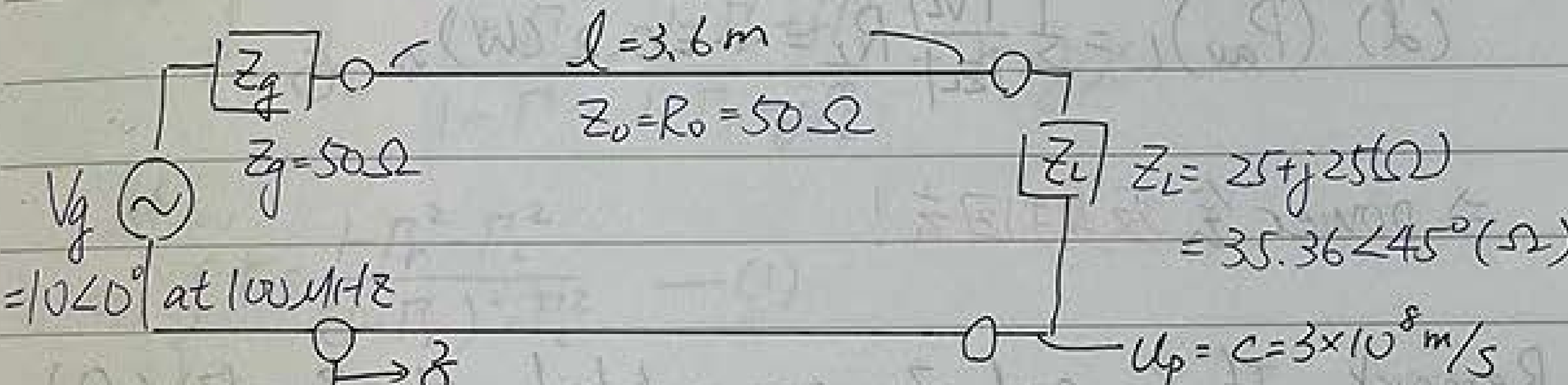
$$= V_g \frac{Z_0}{Z_0+Z_g} e^{-\gamma z}\left(1+\Gamma e^{-2\gamma z'}\right)\left(1+\Gamma\Gamma_g e^{-2\gamma l}+\left(\Gamma\Gamma_g e^{-2\gamma l}\right)^2 + \cdots\right)$$

$$= V_g \frac{Z_0}{Z_0+Z_g} e^{-\gamma z}\left(1+\Gamma e^{-2\gamma z'}\right)\frac{1}{1-\Gamma\Gamma_g e^{-2\gamma l}}$$

When $Z_L = Z_0$, $\Gamma = 0$

$$\Rightarrow V(z') = V_g \frac{Z_0 e^{-\gamma z}}{Z_0+Z_g}$$

ex9-10. 考慮以下model



求：(a) $V(z) = ?$

(b) input端之 $V_i = ?$ load處 $V_L = ?$

(c) wire上之駐压一电波比？

(d) 傳送之負載之平均功率？

-sol- 可知 $\beta = \frac{\omega}{c} = \frac{2\pi}{3}$ rad/m，$\beta l = 2.4\pi$ (rad)

$$\Rightarrow \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.447\angle 116.6°, \quad \Gamma_g = 0$$

(a) $V(z) = V_g \frac{Z_0}{Z_0+Z_g} e^{-j\beta z}\left(1+\Gamma e^{-j2\beta(l-z)}\right)$

$= \cdots$

$= 5\left(e^{-j\frac{2\pi}{3}z} + 0.447 e^{j\left(\frac{4}{3}\pi - 0.152\right)\pi}\right)$

(b) $V_i = V(z=0) = \cdots = 7.06\angle -8.43^\circ$

$V_L = V(z=3.6) = \cdots = 4.47\angle -45.5^\circ$

(c) $S = \frac{1+|\Gamma|}{1-|\Gamma|} = 2.62$

(d) $(P_{av})_L = \frac{1}{2}\frac{|V_L|^2}{|Z_L|} R_L = \cdots = 0.2\ (W)$ #

$\Rightarrow$ power 会反射回去!

Remark. If Z_L and Z_0 are matched, $Z_L = Z_0 = 50\ (\Omega)$

$\Rightarrow \Gamma = 0$

$\Rightarrow V_i = V_L = 5\ (V)$

$\Rightarrow (P_{av})_{L,max} = \frac{V_L^2}{2R_L} = 0.25W$

$\Rightarrow$ ex9-10 中, 被反射的 power 有 $0.25 - 0.2 = 0.05W$

$= 0.25|\Gamma|^2$ (in general)

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§9-6 史密斯图

1. Smith chart ⇒ 求 Z_i，Z_L 好用！

回憶 $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\theta_r}$

2. 以 wire 之 $R_0 = \sqrt{L/C}$ normalize Z_L，

$$z_L \triangleq \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0}$$

$$\triangleq r + jx$$ ，想求 r 與 x！

$$\Rightarrow \begin{cases} \Gamma = \dfrac{z_L - 1}{z_L + 1} \triangleq \Gamma_r + j\Gamma_i \\ z_L = \dfrac{1+\Gamma}{1-\Gamma} = \dfrac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} = r + jx \end{cases}$$

解之，

$$\Rightarrow \begin{cases} r = \dfrac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} & —(1) \\ x = \dfrac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} & —(2) \end{cases}$$

3. 由(1)，可得关係式

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \quad —(3)$$

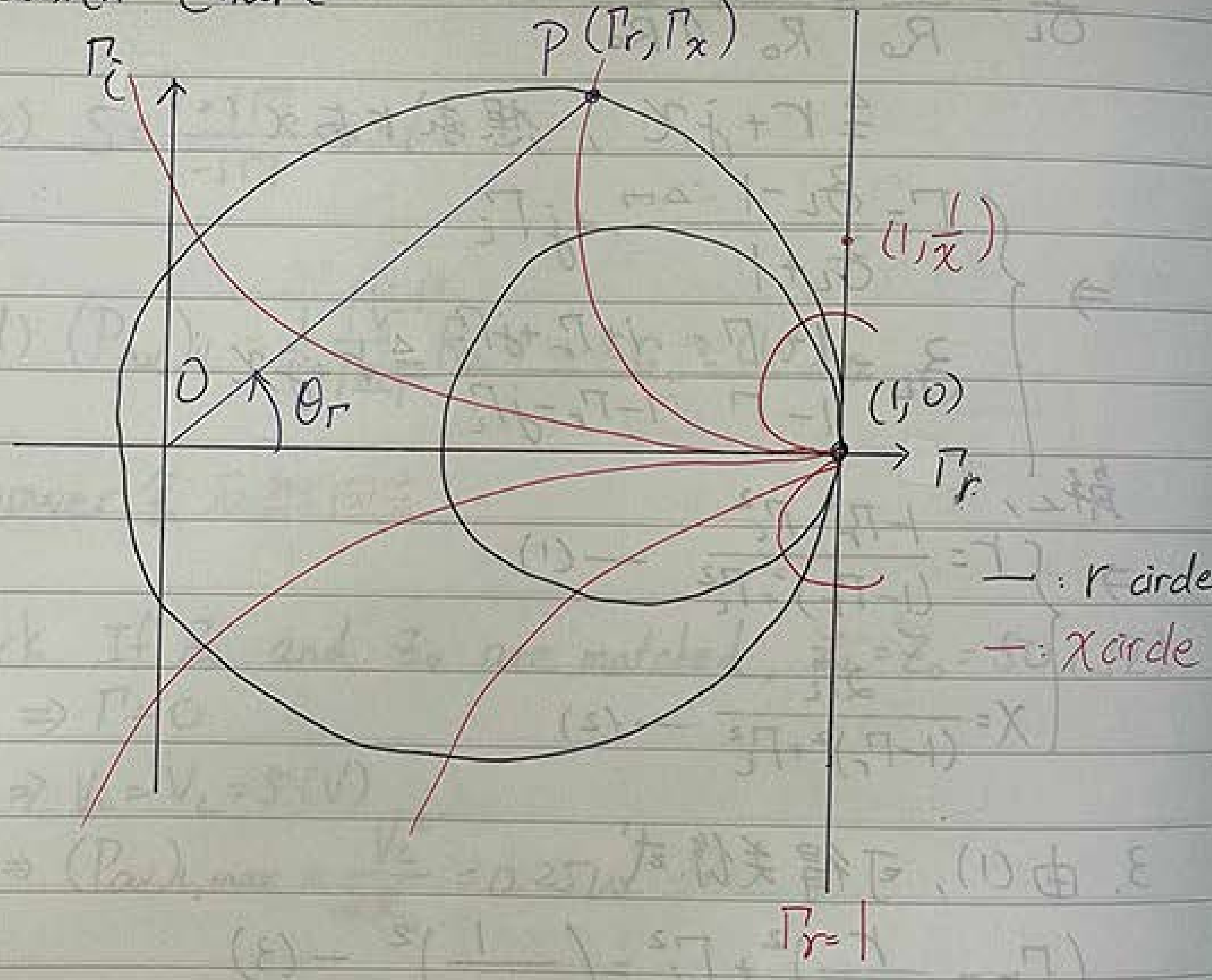
將 Γ_r 看成橫坐標，Γ_i 看成縱坐標，則 (3) 為 Γ-plane ($\Gamma_r - \Gamma_i$ plane) 上之园！园心為 $\left(\frac{r}{1+r}, 0\right)$，半徑為 $\frac{1}{1+r}$

同理，由(2)，

$$(\Gamma_r-1)^2+(\Gamma_i-\frac{1}{x})^2=(\frac{1}{x})^2 \quad -(4)$$

為一圓心$(1,\frac{1}{x})$，半徑$\frac{1}{x}$之圓

將(3)稱為 r circle，(4)稱為 x circle，將不同 r, x 对应之 circle 繪於 $\Gamma_r-\Gamma_i$ plane 上，即為 Smith Chart



4. Smith charts 之 property

(1) 所有圓必过 (1, 0)

(2) r circle 之圓心在 Γ_r(水平)軸上；x circle 之圓心在直線 $\Gamma_r=1$ 上

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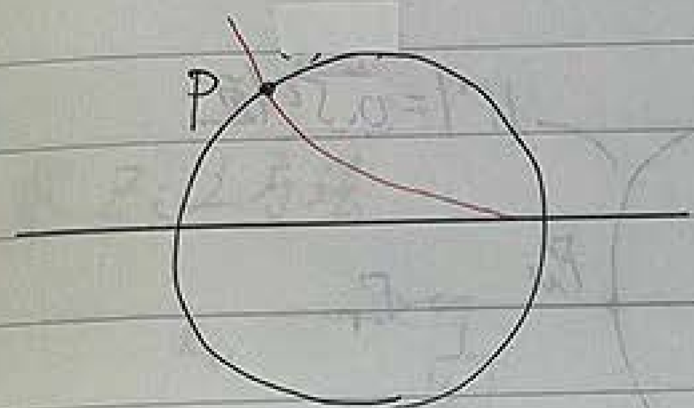
5. Smith chart 之物理意義



(1) 設P点為 $r=2.0$, $x=1.2$ 之园之交点，可從 Chart 边缘之刻度读取 x 值，Γ_r 軸上读取 r 值，即得 z_L

$\Rightarrow$ 用 $Z_L=R_0 z_L$ 解 Z_L

(2) 可用内插法求表上沒有之数值

(3) Smith chart 最左边之点 P_{sc} $(-1, 0)$

$\Rightarrow r+jx=0+j0=0$

$\Rightarrow \left\{\begin{array}{l} x=0 \rightarrow \Gamma_r\ (\text{水平軸}) \\ r=0 \rightarrow \text{最大的园} \end{array}\right\}$ 之交点

$\Rightarrow$ 物理上小為 short circuit

(4) Smith chart " 右 " P_{oc} $(1, 0)$

$\Rightarrow r+jx=\infty+j\infty$

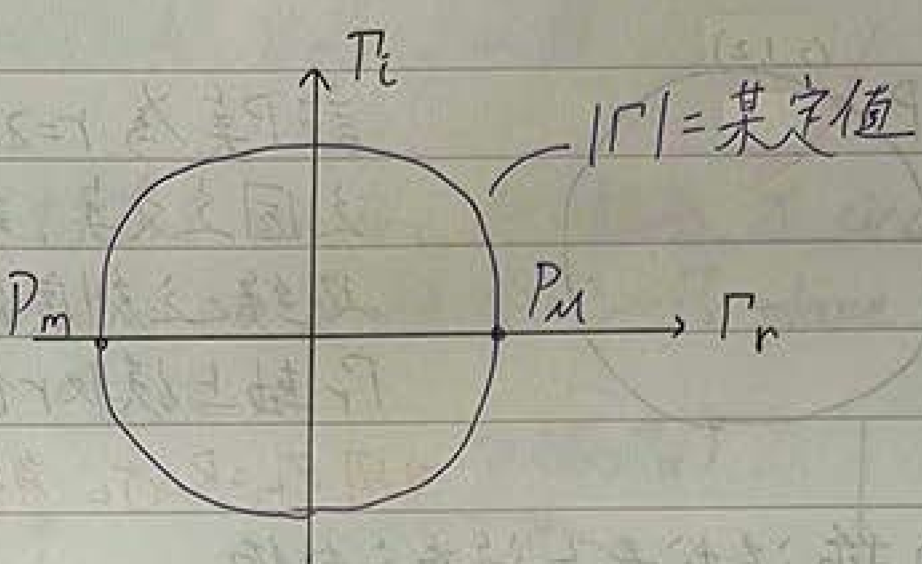
$\left\{\begin{array}{l} r=\infty \rightarrow \Gamma_r \text{ " 最右边} \\ x=\infty \rightarrow \text{最小的元} \end{array}\right\}$ 之交点

$\Rightarrow$ " open circuit

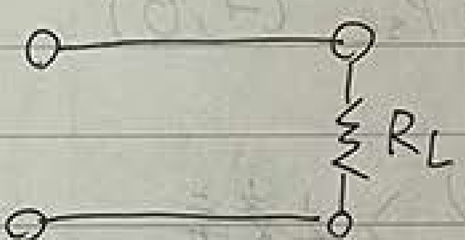
(5) 如左图，設P之坐標為 (Γ_r, Γ_x). 設P点对应之 Γ 值為 $\Gamma_p=|\Gamma_p|e^{j\theta_r}$，則 $|\Gamma_L|=\dfrac{\overline{OP}}{\overline{OP_{oc}}}$

$\Rightarrow$ 用尺量 $\cdots$，θ_r 可用量角器量

(b) $|\Gamma|$ 之等值線為以 0 為圓心，$|\Gamma|$為半徑之圓



在 P_M, P_m, $x=0$ ⇒ resistive load



(a) At P_m, $r=\frac{R_L}{R_0}>1$

$$\Rightarrow S=\frac{1+|\Gamma|}{1-|\Gamma|}$$

S：駐波比

$$=\frac{1+\frac{R_L-R_0}{R_L+R_0}}{1-\frac{R_L-R_0}{R_L+R_0}}$$

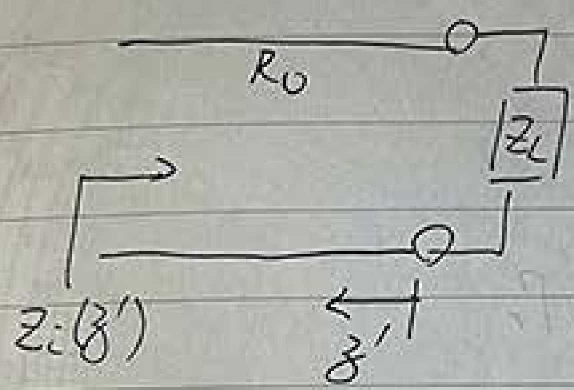
$$=\frac{R_L}{R_0}$$

$$=r$$

(b) 同理, at P_m, $r<1$, $R_L<R_0$

$$\Rightarrow S=\frac{1}{r} \text{ or } r=\frac{1}{S}$$

5. 求 Z_i 之方法

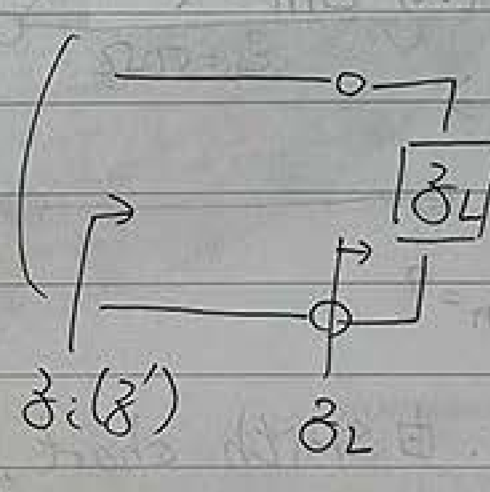


$$\because Z_i(z')=\frac{V(z')}{I(z')}$$

$$=\frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}\cdot Z_0$$

$$\therefore z_i(z')=\frac{Z_i}{Z_0}$$

$$=\frac{1+|\Gamma|e^{j(\theta_\Gamma-2\beta z')}}{1-|\Gamma|e^{j(\theta_\Gamma-2\beta z')}}$$



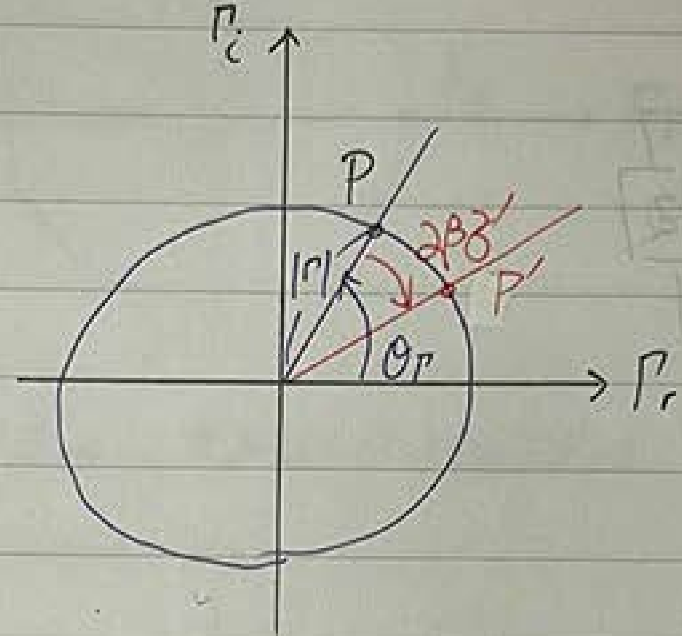
$$=\frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}}$$

$$\Rightarrow z_L=z_i(z'=0)=\frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$$

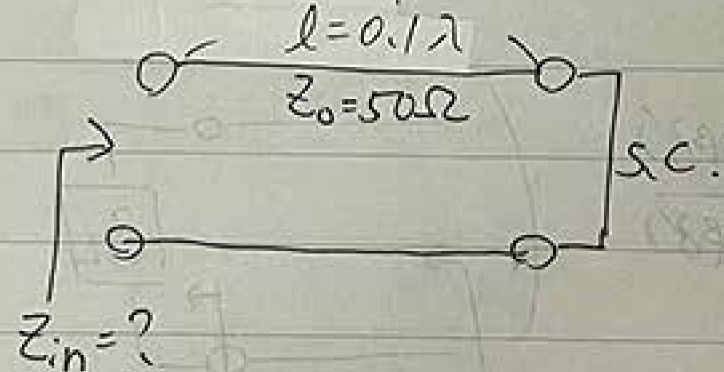
(1) 先由 Smith chart 得 P点之 r, x → 知 z_L

(2) 与 O 连线画一圆，得 $|\Gamma|$ 与 θ_r（用尺&量角器量 $|\Gamma|$, θ_r）

(3) 往回 $2\beta z'$ 角度之 r, x 对应之 z 值 (at P') 即为 z_i



ex9-13，有一 short circuit 之傳輸線长 0.1λ，$Z_0=50\Omega$，求 $Z_{in}=?$



-sol-. 由 Smith chart, rotate from z_L by 0.1λ "toward generator" to P,

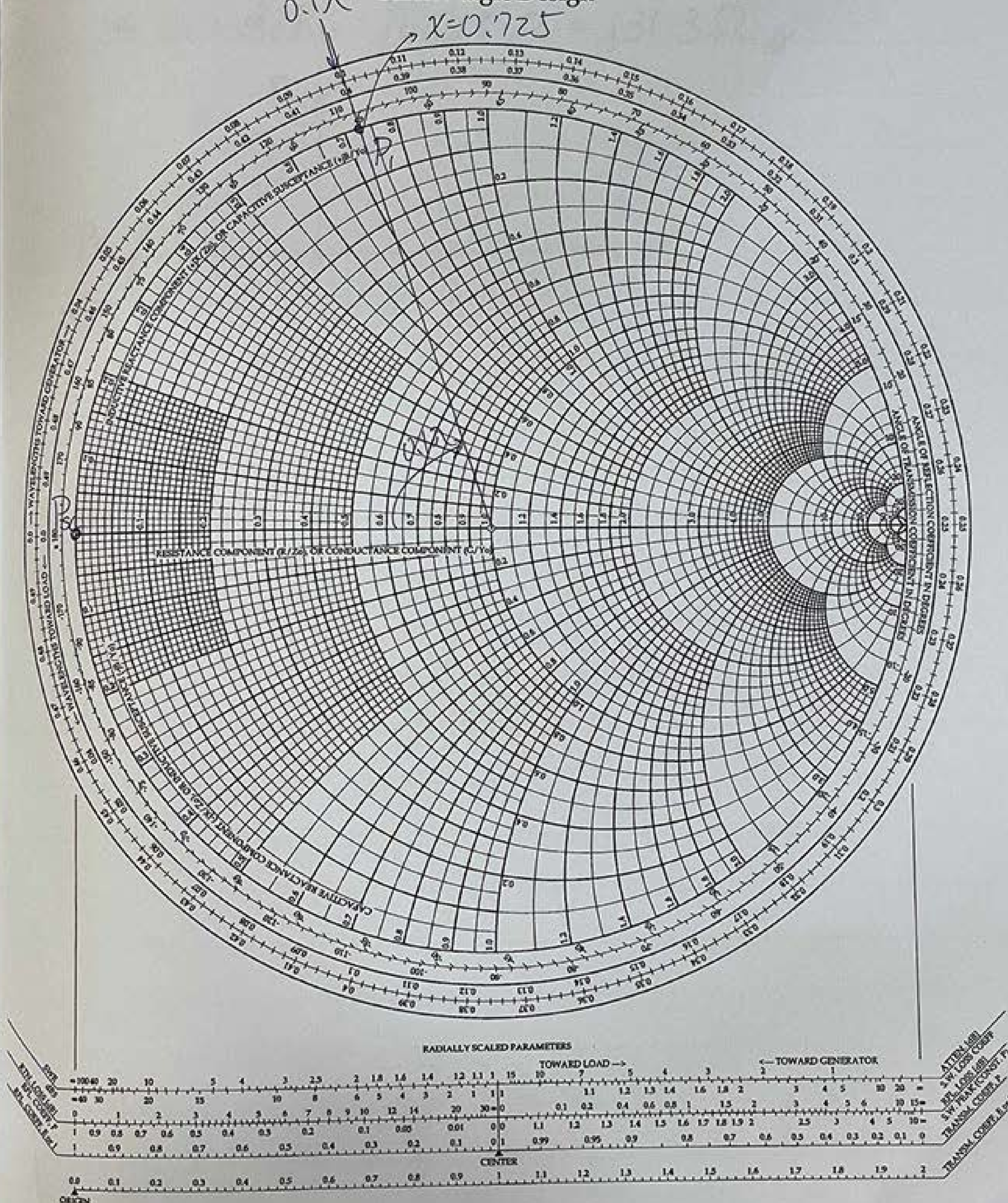
$\Rightarrow z_i = r + jx = 0 + j0.725$

↑ smith chart

ex9-13

The Complete Smith Chart

Black Magic Design



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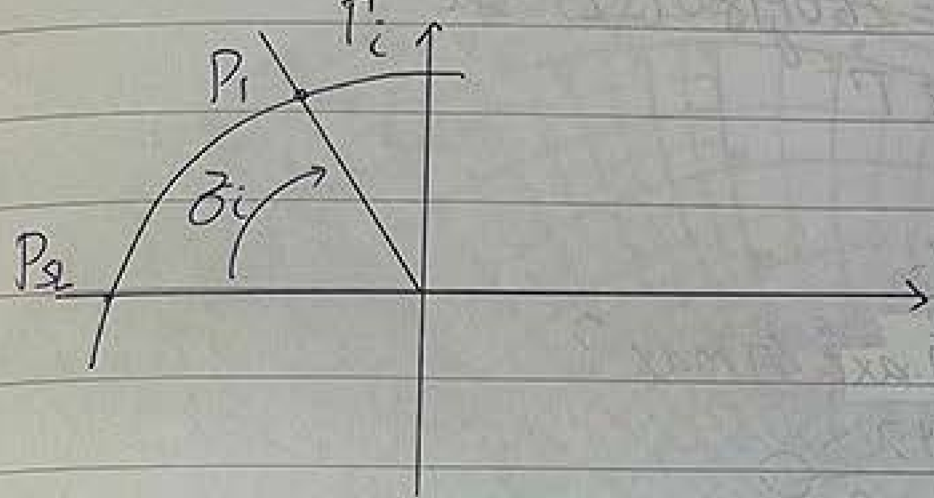
$$\Rightarrow Z_i = z_i R_0 = j0.725 \cdot 50 = j36.3\,\Omega$$ #



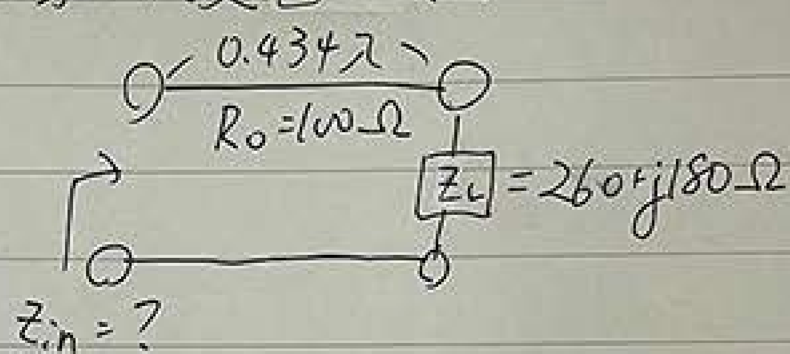
ex9-14. 設一無損耗傳輸線，長0.434λ，特性阻抗為100Ω，$Z_L = 260+j180\,(\Omega)$. 求

(a) 电压反射係数 Γ

(b) 駐波比 S

(c) 輸入阻抗 Z_{in}

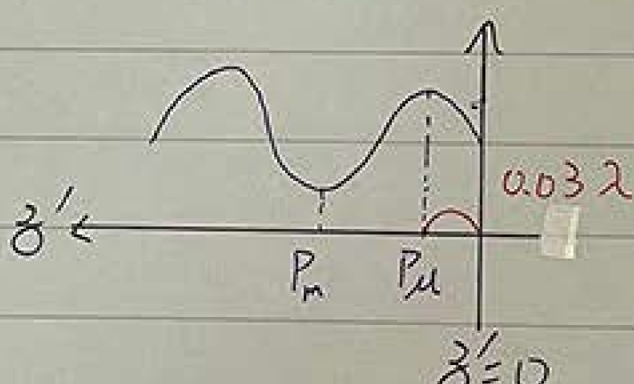
(d) 在線上何處电压会有max？



-sol-. (a) $\because z_L = \frac{Z_L}{R_0} = 2.6 + j1.8$

$\therefore \Gamma = |\Gamma| e^{j\theta_\Gamma} = \frac{\overline{OP_2}}{\overline{OP_{oc}}} \approx 0.6 e^{j21^\circ} = 0.6\angle 21^\circ$

(b) $S = r|_{P_M} = \frac{1}{r}\Big|_{P_m} = 4$



(c) move z_L (P_2) by 0.434λ toward generator to P_3 (z_i)

⇒ 移动 $0.22\lambda + 0.434\lambda - 0.5\lambda = 0.154\lambda$ 到 P_3

一圈只有0.5λ

ex9-14.

The Complete Smith Chart

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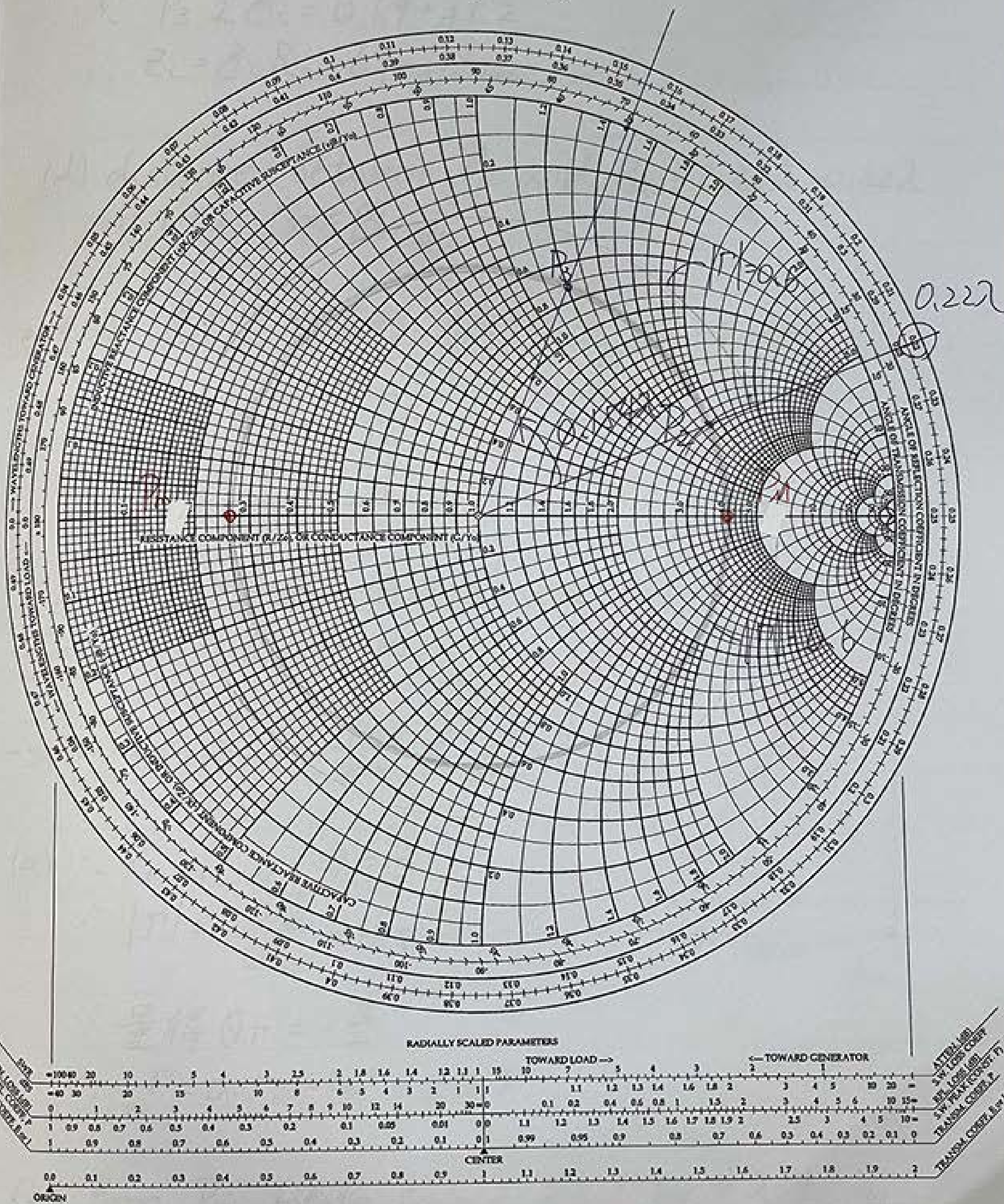
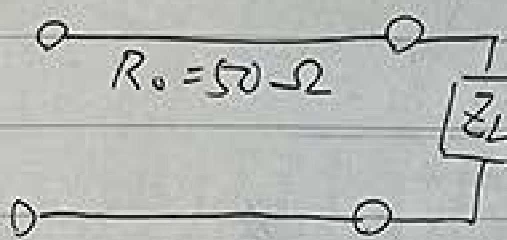


$\because$ P_3 之 $z_i = 0.69 + j1.2$

$\therefore$ $Z_i = z_i R_o = 69 + j120\,\Omega$

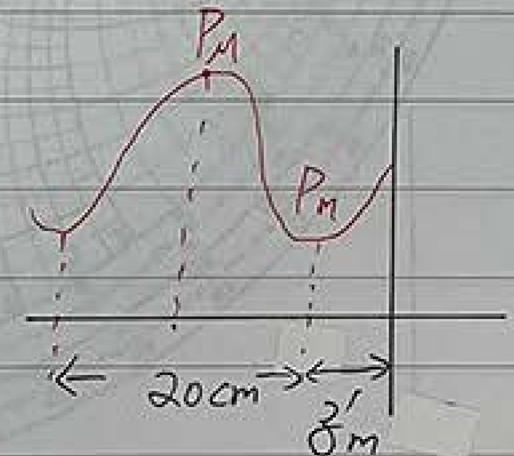
(d) distance between P_2 and P_M $= 0.25\lambda - 0.22\lambda$
$= 0.03\lambda$. (如左图) //

ex9-15. 考虑以下模形，已知 $S=3$，$\lambda = 0.4m$



且第一个电压最小值在 $z_m' = 0.05\,(m)$. 求 (a) Γ
(b) Z_L (c) l_m 和 R_m

-sol-. 可知 $S = 3 = \frac{V_{max}}{V_{min}}$ 及 $z_m' = \frac{1}{8}\lambda$



(a) $\because$ P_M at $(r=3) = 3$

$\therefore$ $|\Gamma| = \frac{\overline{OP_M}}{\overline{OP_{oc}}} = 0.5$

$\because$ 量得 $\theta_\Gamma = -\frac{\pi}{2}$

$\therefore$ $\Gamma = 0.5\angle -90^\circ = -j0.5$

(b) From P_m rotate toward load by $\frac{1}{8}\lambda$ to z_L

$\Rightarrow z_L = r + jx = 0.6 - j0.8$

$\Rightarrow Z_L = R_o z_L = 30 - j40\,\Omega$

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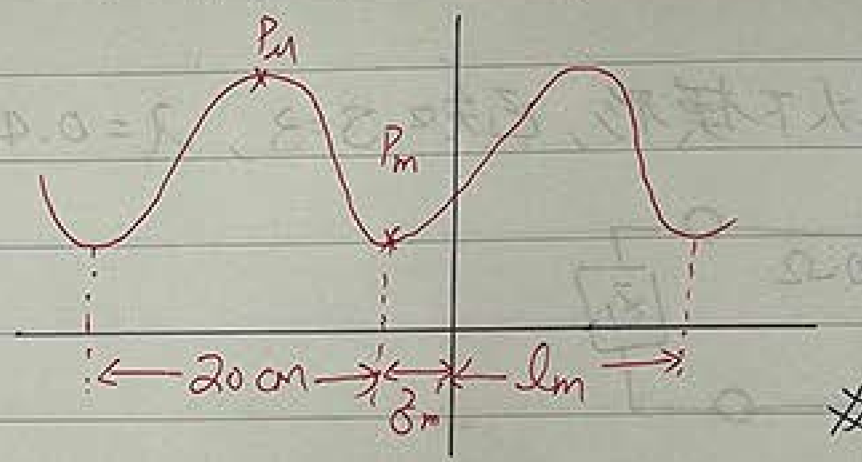
(c) $l_m = \frac{\lambda}{2} - z'_m = 0.15 (m)$

$R_m = R_0 \cdot r / p_m$

$= R_0 \cdot \frac{1}{S}$

$= 16.7 (\Omega)$



※

25 lines / 10mm

ex9-15

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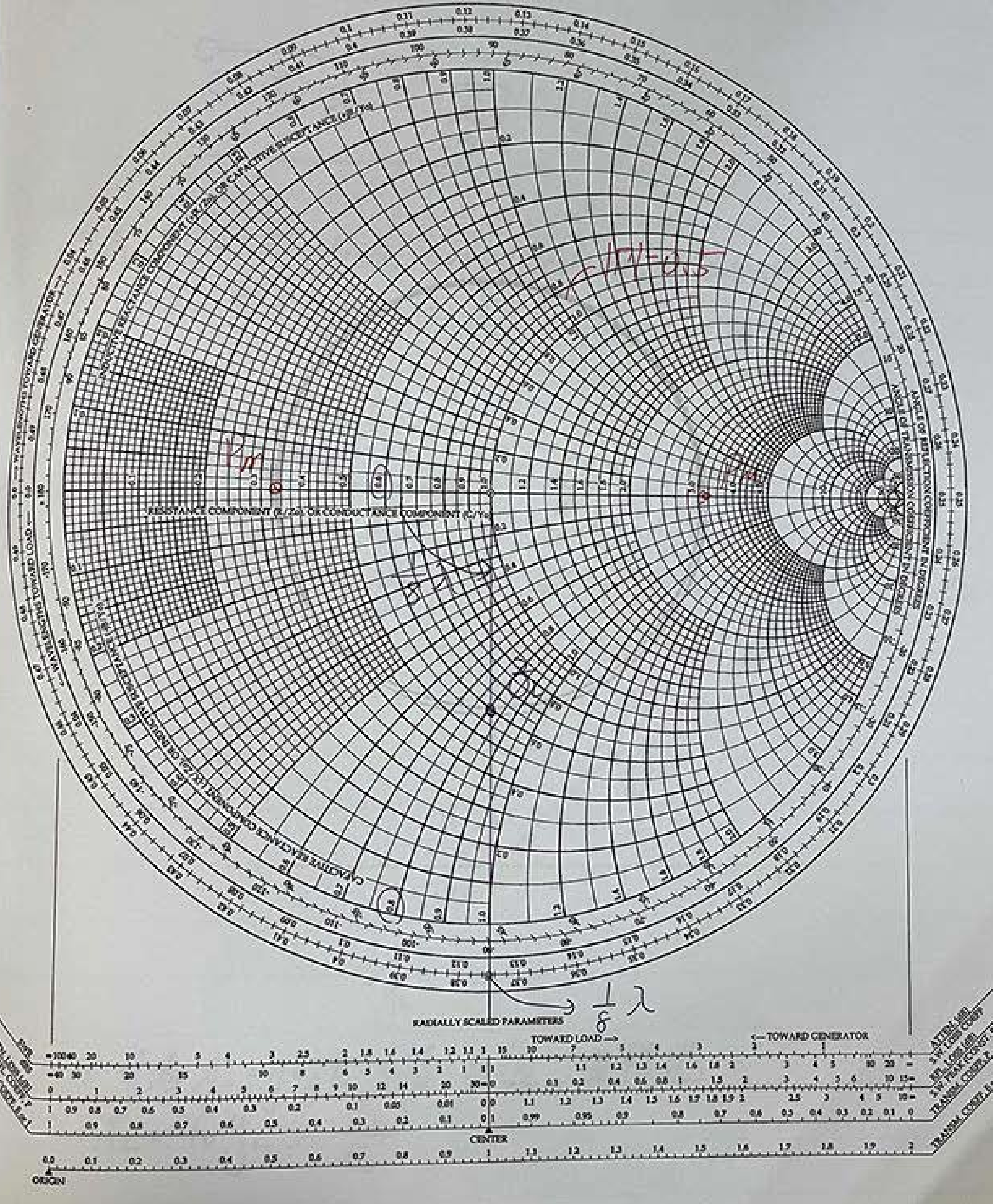
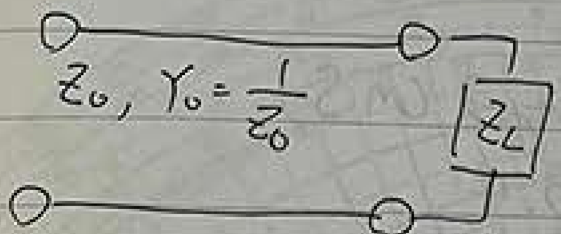


6. Normalized admittance



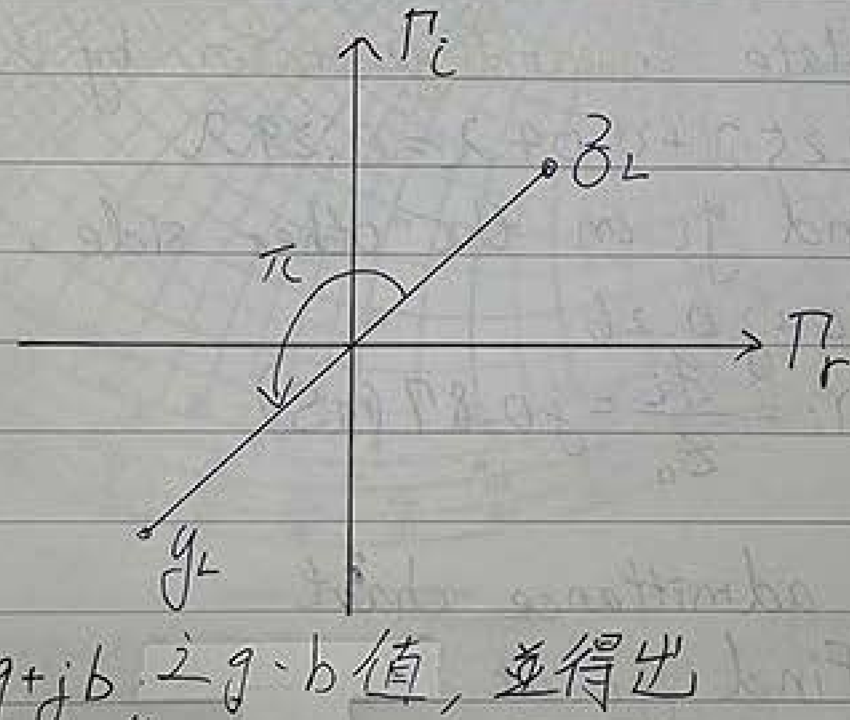
定義 $Y_L = \frac{1}{z_L}$，以及 normalized admittance

$$y_L = \frac{Y_L}{Y_0} = \frac{z_0}{z_L} = \frac{1}{\mathfrak{z}_L}$$

⇒ 在 Smith Chart 上，

$$\Gamma = \frac{\mathfrak{z}_L - 1}{\mathfrak{z}_L + 1} = \frac{\frac{1}{y_L} - 1}{\frac{1}{y_L} + 1} = -\frac{y_L - 1}{y_L + 1}$$

定義 $\Gamma' \triangleq -\Gamma = \frac{y_L - 1}{y_L + 1}$，故 Γ' 與 y_L 之關係 = Γ 與 $\mathfrak{z}_L$ 之關係。



⇒ 可看 $y_L = g + jb$ 之 g、b 值，並得出

$$Y_L = Y_0 y_L = \frac{y_L}{z_0}$$

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ex9-18. 假設 $Z_L = 95 + j20\,\Omega$, $Y_L = ?$ (設 $R_0 = 50\,\Omega$)

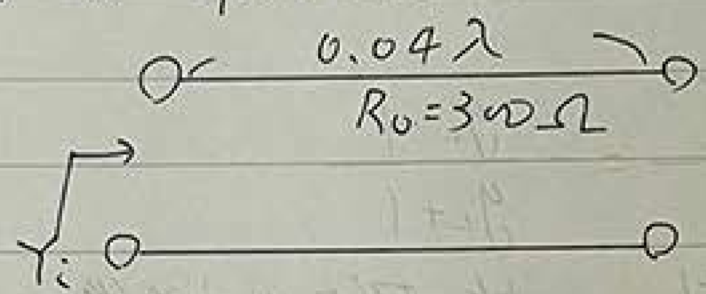
-sol- 〈法I〉 $Y_L = \frac{1}{Z_L} = 10 - j2$ (mS)

〈法II〉 ∵ $z_L = 1.9 + j0.4$

∴ $y_L = g + jb = 0.5 - j0.1$

⇒ $Y_L = y_L / Z_0 = 10 - j2$ (mS) #

ex9-19 有一 open circuit 如下圖，求 $Y_i = ?$



〈法I〉 impedance chart：自 P_{oc} 看到 z_L

⇒ rotate toward generator by 0.04λ to P_3

⇒ $0.25\lambda + 0.04\lambda = 0.29\lambda$

⇒ Find y_i on the other side, $y_i = g + jb$

$= 0 + j0.26$

⇒ $Y_i = \frac{y_i}{Z_0} = j0.87$ (mS)

〈法II〉 admittance chart

⇒ Find y_L at P_{sc}

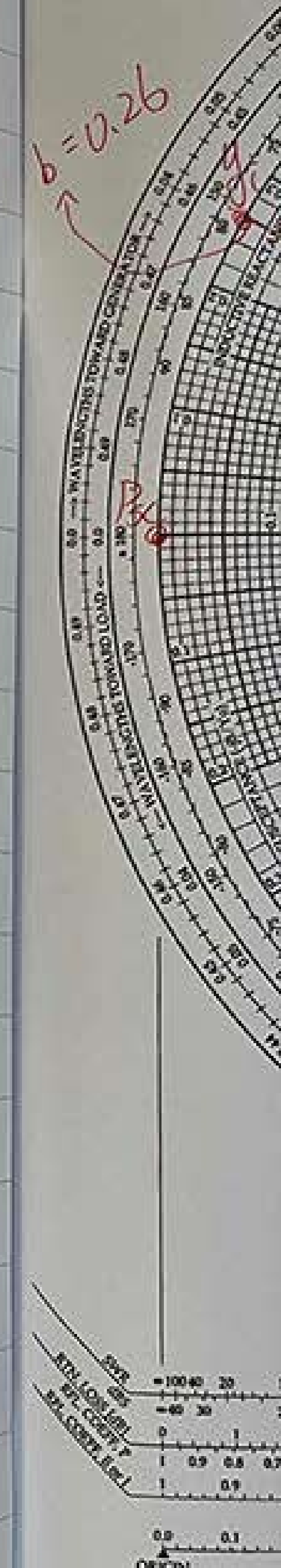
⇒ rotate toward generator by 0.04λ

⇒ $y_i = 0 + j0.26$

⇒ $Y_i = y_i / Z_0 = j0.87$ (mS)

ex9-18 —

ex9-19 —



EX9-18 —

EX9-19 —

The Complete Smith Chart

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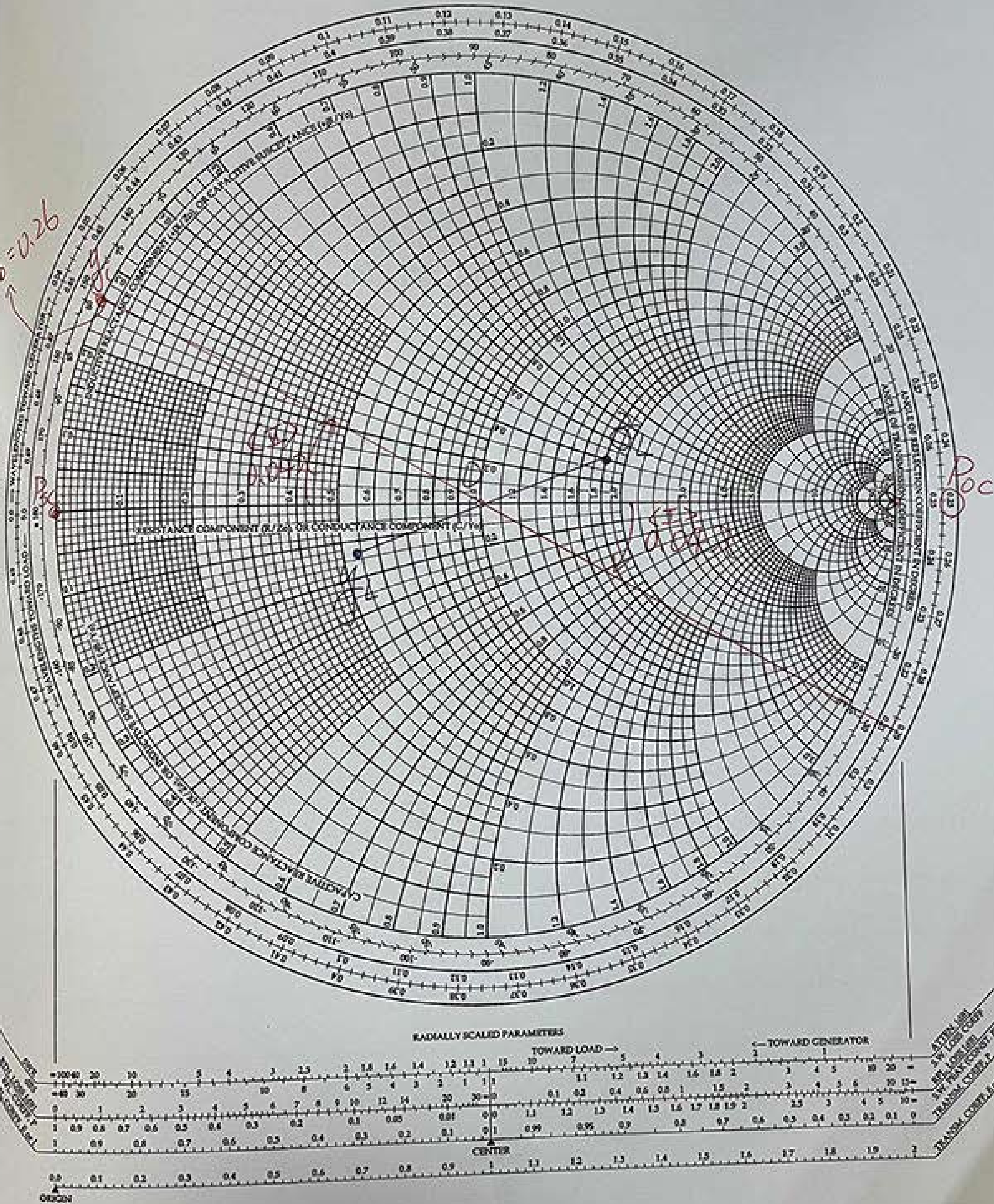


§9.7 Impedence Matching

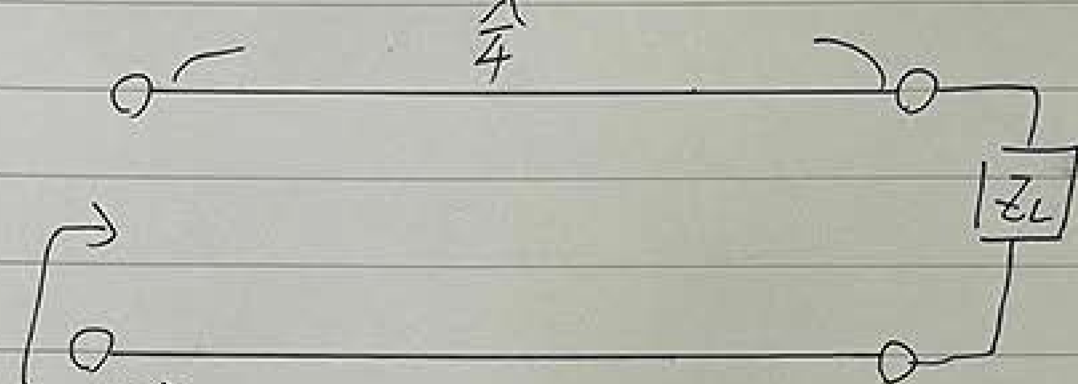
Q: Why impedance matching?

A: 1. 為了減少反射訊号跟反射訊号所帶有的功率

2. 希望source送出來的功率能100%送到負載

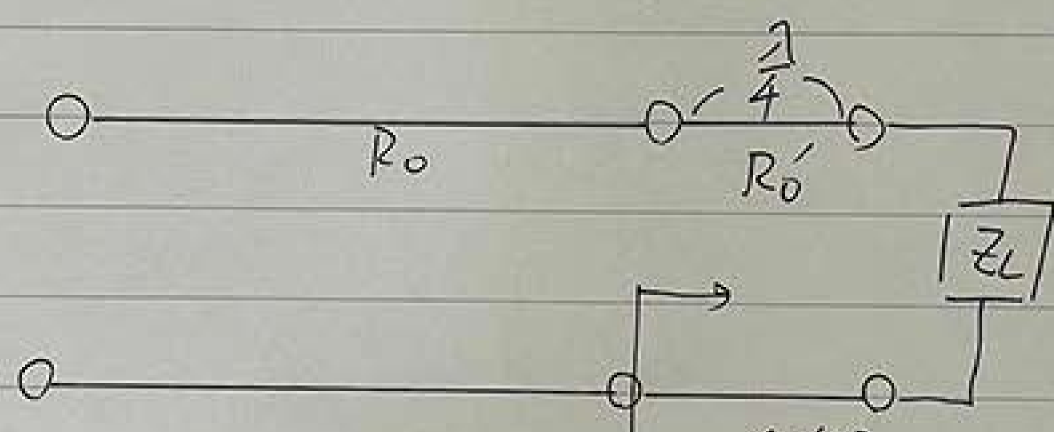
3. 減少訊号間干擾(∵会不斷反射)

4. Source 可能承受不了反射之功率 (ex. 放大器)



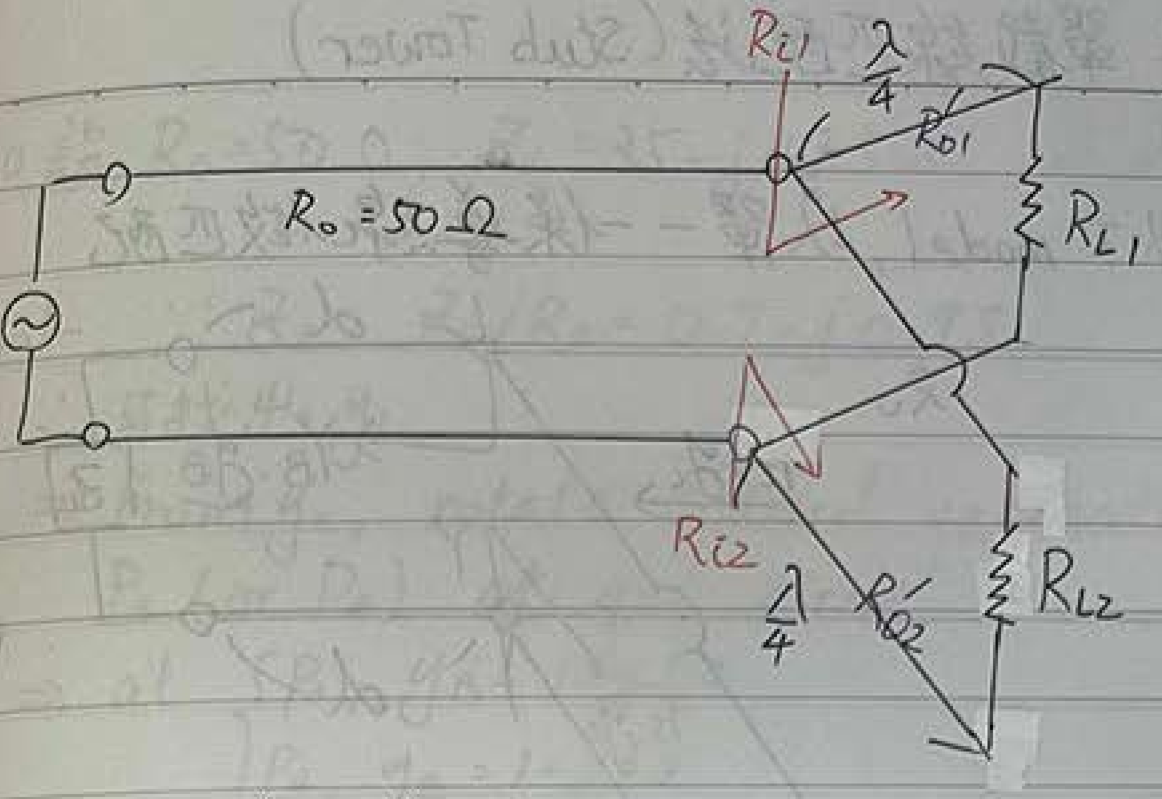
$$Z_i = \frac{R_0^2}{Z_L}$$

⇓ 选一適当之R_0'使$Z_i' = R_0$



$$Z_i' = \frac{(R_0')^2}{Z_L} = R_0$$

ex9-17 有一$R_0 = 50\Omega$之傳輸線，若要供应相同的功率至兩了並联之負載 $R_{L1} = 64\Omega$, $R_{L2} = 25\Omega$, 設這兩條相同之導線長$\lambda/4$，且特性阻抗為R_{01}', R_{02}'，如下图



求：(a) $R'_{01}, R'_{02} = ?$

(b) $\hat{R_0},$ R'_{01}, R'_{02} 上之駐波比 $S = ?$

-sol-. (a) ∵ power 要相同

∴ $R_{i1} = R_{i2}$

$\Rightarrow R_{i1} = R_{i2} = 2R_0 = 100\,\Omega$

$$\Rightarrow \begin{cases} R'_{01} = \sqrt{2R_0R_{L1}} = 80\,(\Omega) \\ R'_{02} = \sqrt{2R_0R_{L2}} = 50\,(\Omega) \end{cases}$$

(b) ∵ matching

∴ $S|_{R_0} = 1$

在 R'_{01} 上，$\Gamma_1 = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = -0.1$

$\Rightarrow S|_{R'_{01}} = \frac{1+|\Gamma_1|}{1-|\Gamma_1|} = 1.25$

同理，$S|_{R'_{02}} = 1.99$ #

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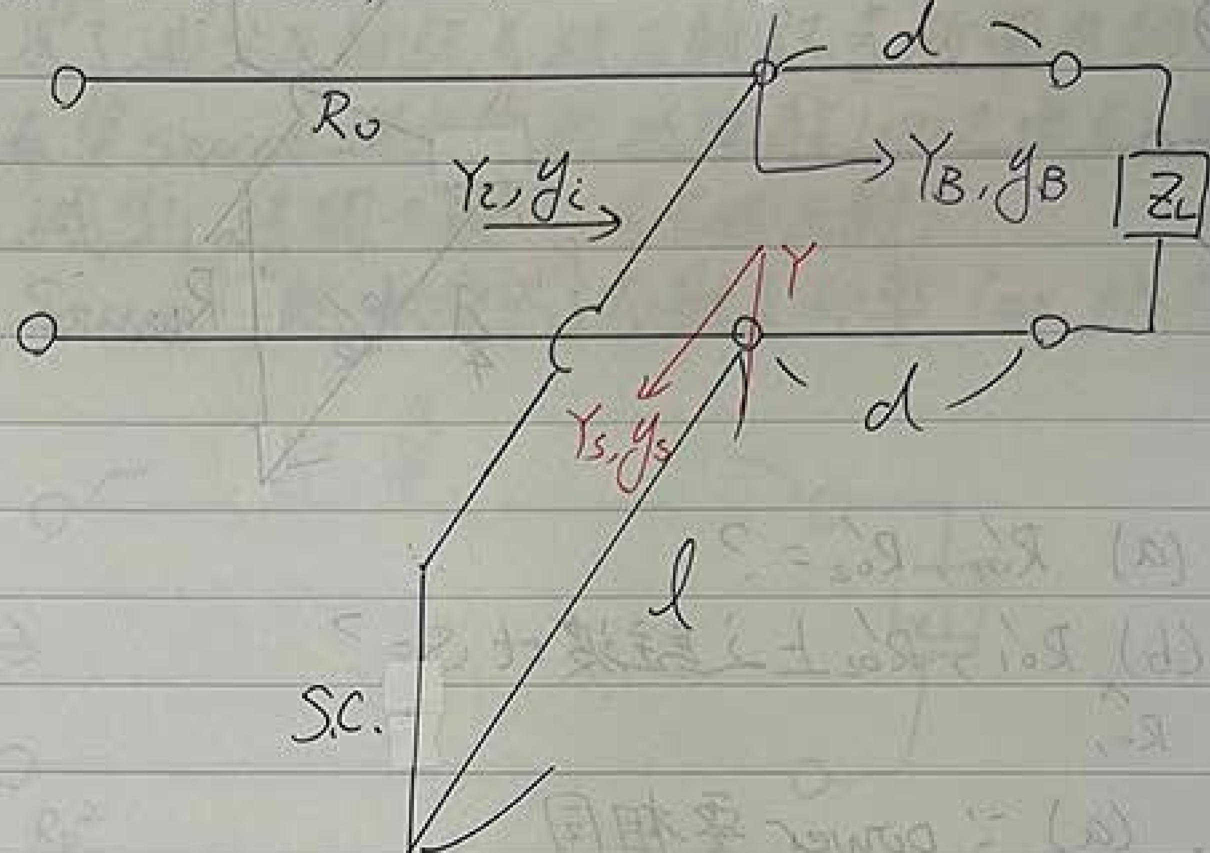
§9.7-2 單截線匹配法(Stub Tower)

考慮以下 model, 以單一條導線做匹配



則: 要取 l 為何? 要在 R_0 之導線自 Z_L 往回退之距離 d 為多少?

$\Rightarrow \because Y_i = Y_B + Y_S = Y_0 = \frac{1}{R_0}$ (matching)

$\therefore$ normalized admittance $y_B + y_S = 1$

$\Rightarrow y_B = 1 - y_S$

$\Rightarrow$ 可令 $\begin{cases} y_B = g + jb = jb_B + 1 \\ y_S = -jb_B \end{cases}$

$\Rightarrow$ ① Find d s.t. y_B has unity real part $= g + jb$ on $g=1$ circle

② Find l s.t. y cancels the y_B imaginary part of y_B

ex9-20. 設 $R_0 = 50\,\Omega$, $Z_L = 35 - j47.5\,\Omega$, 求 $d, l = ?$

-sol-. $\because z_L = Z_L / R_0 = 0.7 - j0.95$

$\therefore$ 可找出 y_L

$\Rightarrow$ From y_L, rotate toward generator to P_3 (or P_4) at $g=1$ circle

$\Rightarrow$ at $\begin{cases} P_3, & y_B = 1 + j1.2 \\ P_4, & y_B = 1 - j1.2 \end{cases}$

$\Rightarrow \begin{cases} d_3 = (0.168 - 0.109)\lambda = 0.059\lambda \\ d_4 = (0.332 - 0.109)\lambda = 0.223\lambda \end{cases}$ 都可取為 d

$\Rightarrow$ ① For P_3', $y_{s1} = -jb_{B3} = -j1.2$,
From y_{sc} rotate $l_3 = (0.36 - 0.25)\lambda$
$= 0.11\lambda$ toward load

② For P_4', $y_{s2} = -jb_{B4} = j1.2$,
From y_{sc} 〃 $l_4 = (0.332 - 0.109)\lambda$
$= 0.39\lambda$ 〃

$\Rightarrow l_3, l_4$ 都可當成 $l_\#$

ex9-20

The Complete Smith Chart

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