9. Linearization of RIP Model

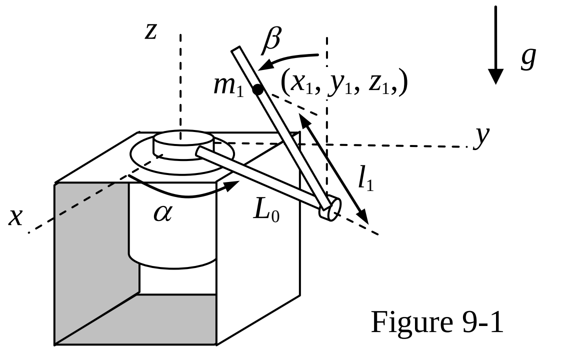

Figure 9-1

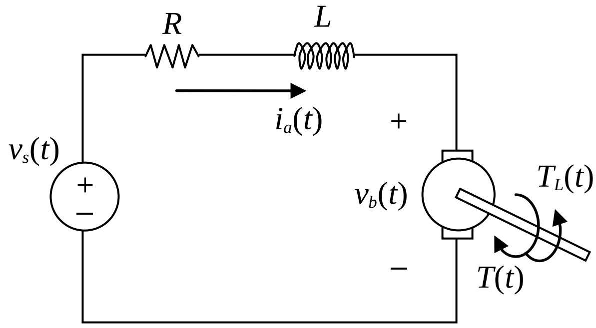

Figure 9-2

The system structure of a rotary inverted pendulum is shown in Figure 9-1, which is driven by a DC motor depicted in Figure 9-2. The dynamic equation has been derived by Lagrange's method and can be expressed as below:

$$\underbrace{\begin{bmatrix} J_0 + m_1 L_0^2 + m_1 l_1^2 \sin^2\beta & -m_1 L_0 l_1 \cos\beta \\ -m_1 L_0 l_1 \cos\beta & J_1 + m_1 l_1^2 \end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} \tag{9-1}$$

$$+ \underbrace{\begin{bmatrix} m_1 l_1^2 \dot{\beta} \sin\beta \cos\beta & m_1 l_1^2 \dot{\alpha} \sin\beta\cos\beta + m_1 L_0 l_1 \dot{\beta} \sin\beta \\ -m_1 l_1^2 \dot{\alpha} \sin\beta\cos\beta & 0 \end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -m_1 g l_1 \sin\beta \end{bmatrix} = \begin{bmatrix} \tau - C_0 \dot{\alpha} \\ -C_1 \dot{\beta} \end{bmatrix}$$

where $\boldsymbol{M}$ is the inertia matrix and $\boldsymbol{B}$ is the matrix related to the centrifugal forces and coriolis forces. There are two important properties concerning $\boldsymbol{M}$ and $\boldsymbol{B}$. First, the inertia matrix is symmetric and positive-definite, i.e., $\boldsymbol{M}=\boldsymbol{M}^{\mathrm{T}}$ and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x}>0$ for all $\boldsymbol{x}\neq\boldsymbol{0}$. Second, $\frac{1}{2}\dot{\boldsymbol{M}} - \boldsymbol{B}$ is skew-symmetic, i.e., $\boldsymbol{x}^T\left(\frac{1}{2}\dot{\boldsymbol{M}} - \boldsymbol{B}\right)\boldsymbol{x} = 0$ for all $\boldsymbol{x}$.

It is known that the torque to drive the motor can be approximately expressed as below:

$$\tau(t) = \frac{k}{R} v_s(t) + \left(B_M - \frac{k^2}{R} \right) \dot{\alpha}(t) - J_M \ddot{\alpha}(t) \tag{9-2}$$

such that (9-1) is rewritten as

$$\begin{bmatrix} J_M + J_0 + m_1 L_0^2 + m_1 l_1^2 \sin^2\beta & -m_1 L_0 l_1 \cos\beta \\ -m_1 L_0 l_1 \cos\beta & J_1 + m_1 l_1^2 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} \tag{9-3}$$

$$+ \begin{bmatrix} C_0 + \dfrac{k^2}{R} - B_M + m_1 l_1^2 \dot{\beta} \sin\beta \cos\beta & m_1 l_1^2 \dot{\alpha} \sin\beta \cos\beta + m_1 L_0 l_1 \dot{\beta} \sin\beta \\ -m_1 l_1^2 \dot{\alpha} \sin\beta \cos\beta & C_1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -m_1 g l_1 \sin\beta \end{bmatrix} = \begin{bmatrix} k/R \\ 0 \end{bmatrix} v_s$$

where $v_s(t)$ is the voltage source applied to drive the system.

Since the parameters in the model (9-3) may not easy to obtain, we have to estimate an appropriate model before control. For estimation, define the following constant parameters:

$$\phi_1 = J_M + J_0 + m_1 L_0^2, \quad \phi_2 = m_1 L_0 l_1, \quad \phi_3 = m_1 l_1^2,$$

$$\phi_4 = C_0 + \frac{k^2}{R} - B_M, \quad \phi_5 = C_1, \quad \phi_6 = \frac{k}{R}, \quad \phi_7 = m_1 g l_1$$

and thus, (9-3) is changed into

$$\phi_1 \ddot{\alpha} + \phi_3 \sin^2\beta \ddot{\alpha} - \phi_2 \cos\beta \ddot{\beta} + \phi_4 \dot{\alpha}$$

$$+ 2\phi_3 \dot{\alpha}\dot{\beta} \sin\beta \cos\beta + \phi_2 \dot{\beta}^2 \sin\beta = \phi_6 v_s \tag{9-4}$$

$$-\phi_2 \cos\beta \ddot{\alpha} + (J_1 + \phi_3)\ddot{\beta} - \phi_3 \dot{\alpha}^2 \sin\beta \cos\beta + \phi_5 \dot{\beta} - \phi_7 \sin\beta = 0 \tag{9-5}$$

or rewritten as

$$\rho_1 \ddot{\alpha} + \rho_3 \sin^2\beta \ddot{\alpha} - \rho_2 \cos\beta \ddot{\beta} + \rho_4 \dot{\alpha}$$

$$+ 2\rho_3 \dot{\alpha}\dot{\beta} \sin\beta \cos\beta + \rho_2 \dot{\beta}^2 \sin\beta = \rho_6 v_s \tag{9-6}$$

$$-\rho_2 \cos\beta \ddot{\alpha} + \ddot{\beta} - \rho_3 \dot{\alpha}^2 \sin\beta \cos\beta + \rho_5 \dot{\beta} - \rho_7 \sin\beta = 0 \tag{9-7}$$

where the related constants are

$$\rho_k = \frac{\phi_k}{J_1 + \phi_3}, \qquad k=1,2,\ldots,7$$

Further rearranging (9-7) will leads to

$$E(t)=\rho_2 g_2(t)+\rho_3 g_3(t)+\rho_5 g_5(t)+\rho_7 g_7(t) \tag{9-8}$$

where

$$E(t)=\ddot{\beta}\,,\quad g_2(t)=\cos\beta\ddot{\alpha}\,,\quad g_3(t)=\dot{\alpha}^2\sin\beta\cos\beta\,,$$

$$g_5(t)=-\dot{\beta}\,,\quad g_7(t)=\sin\beta\,.$$

and rearranging (9-6) will leads to

$$F(t)=\rho_1 h_1(t)+\rho_4 h_4(t)+\rho_6 h_6(t) \tag{9-9}$$

where

$$F(t)=\rho_3\sin^2\beta\ddot{\alpha}-\rho_2\cos\beta\ddot{\beta}+2\rho_3\dot{\alpha}\dot{\beta}\sin\beta\cos\beta+\rho_2\dot{\beta}^2\sin\beta$$

$$h_1(t)=-\ddot{\alpha}\,,\quad h_4(t)=-\dot{\alpha}\,,\quad h_6(t)=v_s$$

Note that the angular positions $\alpha(t)$ and $\beta(t)$ at time t are measurable and the derivatives $\dot{\alpha}(t)$, $\dot{\beta}(t)$, $\ddot{\alpha}(t)$ and $\ddot{\beta}(t)$ can be estimated from $\alpha(t)$ and $\beta(t)$ by the identification technology. As a results, all the values of $E(t)$, $F(t)$, $h_i(t)$ and $g_j(t)$, i=1,4,6 and j=2,3,5,7, at time t are known. To achieve the constant parameters ρ_j, j=1,2,…,7, we can employ the least squares method by the use of $E(kT)$, $F(kT)$, $h_i(kT)$ and $g_j(kT)$, i=1,4,6 and j=2,3,5,7, at time $t=kT$. Let $E[k]=E(kT)$, $F[k]=F(kT)$, $h_i[k]=h_i(kT)$ and $g_j[k]=g_j(kT)$, where k=1,2,…,n, then we have

$$\underbrace{\begin{bmatrix} E[1] \\ E[2] \\ \vdots \\ E[n] \end{bmatrix}}_{\boldsymbol{E}}=\underbrace{\begin{bmatrix} g_2[1] & g_3[1] & g_5[1] & g_7[1] \\ g_2[2] & g_3[2] & g_5[2] & g_7[2] \\ \vdots & \vdots & \vdots & \vdots \\ g_2[n] & g_3[n] & g_5[n] & g_7[n] \end{bmatrix}}_{\boldsymbol{G}}\cdot\underbrace{\begin{bmatrix} \rho_2 \\ \rho_3 \\ \rho_5 \\ \rho_7 \end{bmatrix}}_{\boldsymbol{a}} \tag{9-10}$$

$$\underbrace{\begin{bmatrix} F[1] \\ F[2] \\ \vdots \\ F[n] \end{bmatrix}}_{F} = \underbrace{\begin{bmatrix} h_1[1] & h_4[1] & h_6[1] \\ h_1[2] & h_4[2] & h_6[2] \\ \vdots & \vdots & \vdots \\ h_1[n] & h_4[n] & h_6[n] \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} \rho_1 \\ \rho_4 \\ \rho_6 \end{bmatrix}}_{b} \tag{9-11}$$

Based on the least squares method, first we obtain

$$\boldsymbol{a} = \left(\boldsymbol{G}^T\boldsymbol{G}\right)^{-1}\boldsymbol{G}^T\boldsymbol{E} \tag{9-12}$$

After $\boldsymbol{a}$ is achieved, we can have $\boldsymbol{F}$ from (9-9) and then

$$\boldsymbol{b} = \left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T\boldsymbol{F} \tag{9-13}$$

Therefore, the identification of the dynamic model (9-6) and (9-7) are completed.

Now, we can design the controller to drive the RIP to the upright angular position β=0. In general, to simplify the controller design, we often assume the RIP is initially loctated around $\alpha(0)\approx 0$, $\beta(0)\approx 0$, $\dot{\alpha}(0)\approx 0$ and $\dot{\beta}(0)\approx 0$. Therefore, the model (9-6) and (9-7) are linearized as

$$\ddot{\alpha} - \sigma_2\ddot{\beta} + \sigma_3\dot{\alpha} = \sigma_4 v_s \tag{9-14}$$

$$-\rho_1\ddot{\alpha} + \ddot{\beta} + \rho_2\dot{\beta} - \rho_3\beta = 0 \tag{9-15}$$

where $cos\beta\approx 1$, $sin\beta\approx\beta$ and all the nonlinear terms are neglected. This linearized model can be rewritten as

$$\begin{bmatrix} 1 & -\sigma_2 \\ -\rho_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & -\rho_2 \end{bmatrix} \cdot \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \rho_3 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \sigma_4 \\ 0 \end{bmatrix} \cdot v_s \tag{9-16}$$

or

$$\begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \cdot \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} \cdot v_s \tag{9-17}$$

where

$$\begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & -\sigma_2 \\ -\rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\sigma_3 & 0 \\ 0 & -\rho_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a_{32} \\ 0 & a_{42} \end{bmatrix} = \begin{bmatrix} 1 & -\sigma_2 \\ -\rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \rho_3 \end{bmatrix}$$

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 & -\sigma_2 \\ -\rho_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_4 \\ 0 \end{bmatrix}$$

Hence, the state equation of the RIP is expressed as

$$\underbrace{\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{bmatrix}}_{\boldsymbol{b}} \cdot u \tag{9-18}$$

where $\boldsymbol{x} = \begin{bmatrix} \alpha & \beta & \dot{\alpha} & \dot{\beta} \end{bmatrix}^T$ and $u=v_s$. From (9-18), we can control the RIP in upright angular position by the use of state-feedback control.