8. Modeling of Rotary Inverted Pendulum

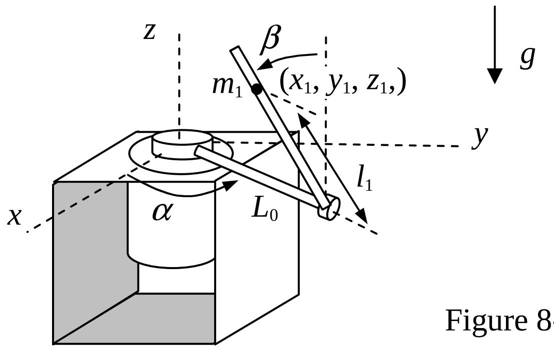

Figure 8-1

The dynamic model of a rotary inverted pendulum is shown in Figure 8-1, which is also called Furuta pendulum. The system structure is formed by a motor, an arm with length L_0 and a pendulum with effective mass m_1 and effective length l_1. The angular position α of the arm referring to x-axis is increasing when it rotates about the z-axis in right-hand rule. The angular position β of the pendulum referring to the upward axis is increasing when it is rotating about the axis along the arm, also in right-hand rule.

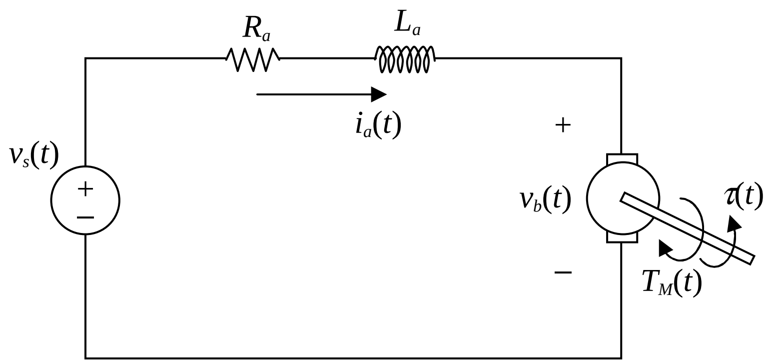

Figure 8-2

It is known that the motor is driven by the voltage $v_s(t)$ and its dynamic equation is given as

$$v_s(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) \approx R_a i_a(t) + v_b(t) \tag{8-1}$$

where the armature inductance L_a can be neglected and $v_b(t) = k\dot{\alpha}(t)$. Hence, the torque generated by the motor is

$$T_M(t) = ki_a(t) = \frac{k}{R_a}\left(v_s(t) - v_b(t)\right) = \frac{k}{R_a}v_s(t) - \frac{k^2}{R_a}\dot{\alpha}(t) \tag{8-2}$$

which is used to control the system via the following equation

$$J_M\ddot{\alpha}(t) + B_M\dot{\alpha}(t) = T_M(t) - \tau(t) \tag{8-3}$$

where J_M is the rotor moment of inertia, B_M is the frictional coefficient and the torque $\tau(t)$ is required to drive the rotary inverted pendulum. From (8-2) and (8-3), we have

$$\tau(t) = \frac{k}{R}v_s(t) + \left(B_M - \frac{k^2}{R}\right)\dot{\alpha}(t) - J_M\ddot{\alpha}(t) \tag{8-4}$$

For simplicity, we will omit the time variable t in the following equations.

Now, we will derive the dynamic model of the rotary inverted pendulum based on the Lagrange's equations, expressed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}}\right) - \frac{\partial L}{\partial\alpha} = \tau_\alpha \tag{8-5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\beta}}\right) - \frac{\partial L}{\partial\beta} = \tau_\beta \tag{8-6}$$

where $L=T-V$ is the Lagrangian, T is the total kinetic energy and V is the total potential energy. It is easy to find that the kinetic energy T_0 and the potential energy V_0 of the arm are

$$T_0 = \frac{1}{2}J_0\dot{\alpha}^2 \tag{8-7}$$

$$V_0 = 0 \tag{8-8}$$

where J_0 is the moment of inertia related to the arm. As for the pendulum, its effective mass m_1 is concentrated at (x_1, y_1, z_1) as shown in Figure 8-1. The Cartesian coordinates can be obtained as

$$x_1 = L_0\cos\alpha + l_1\sin\alpha\sin\beta \tag{8-9}$$

$$y_1 = L_0\sin\alpha - l_1\cos\alpha\sin\beta \tag{8-10}$$

$$z_1 = l_1\cos\beta \tag{8-11}$$

and their derivatives are

$$\dot{x}_1 = -L_0\dot{\alpha}\sin\alpha + l_1\dot{\alpha}\cos\alpha\sin\beta + l_1\dot{\beta}\sin\alpha\cos\beta \tag{8-12}$$

$$\dot{y}_1 = L_0\dot{\alpha}\cos\alpha + l_1\dot{\alpha}\sin\alpha\sin\beta - l_1\dot{\beta}\cos\alpha\cos\beta \tag{8-13}$$

$$\dot{z}_1 = -l_1\dot{\beta}\sin\beta \tag{8-14}$$

Hence, we have

$$\begin{aligned}\dot{x}_1^2 &= L_0^2\dot{\alpha}^2\sin^2\alpha + l_1^2\left(\dot{\alpha}\cos\alpha\sin\beta + \dot{\beta}\sin\alpha\cos\beta\right)^2 \\ &\quad - 2L_0 l_1\dot{\alpha}\sin\alpha\left(\dot{\alpha}\cos\alpha\sin\beta + \dot{\beta}\sin\alpha\cos\beta\right)\end{aligned} \tag{8-15}$$

$$\begin{aligned}\dot{y}_1^2 &= L_0^2\dot{\alpha}^2\cos^2\alpha + l_1^2\left(\dot{\alpha}\sin\alpha\sin\beta - \dot{\beta}\cos\alpha\cos\beta\right)^2 \\ &\quad + 2L_0 l_1\dot{\alpha}\cos\alpha\left(\dot{\alpha}\sin\alpha\sin\beta - \dot{\beta}\cos\alpha\cos\beta\right)\end{aligned} \tag{8-16}$$

$$\dot{z}_1^2 = l_1^2\dot{\beta}^2\sin^2\beta \tag{8-17}$$

The kinetic energy and potential energy of the pendulum are then obtained as

$$\begin{aligned}T_1 &= \frac{1}{2}J_1\dot{\beta}^2 + \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right) \\ &= \frac{1}{2}J_1\dot{\beta}^2 + \frac{1}{2}m_1L_0^2\dot{\alpha}^2 + \frac{1}{2}m_1l_1^2\dot{\beta}^2 + \frac{1}{2}m_1l_1^2\dot{\alpha}^2\sin^2\beta - m_1L_0l_1\dot{\alpha}\dot{\beta}\cos\beta\end{aligned} \tag{8-18}$$

$$V_1 = m_1 g z_1 = m_1 g l_1\cos\beta \tag{8-19}$$

where J_1 is the moment related to the pendulum. The total kinetic energy and the total potential energy are $T = T_1 + T_0$ and $V = V_1 + V_0$ and the Lagrangian is expressed as

$$\begin{aligned}L &= T - V = T_0 + T_1 - V_0 - V_1 \\ &= \frac{1}{2}\left(J_0 + m_1L_0^2\right)\dot{\alpha}^2 + \frac{1}{2}\left(J_1 + m_1l_1^2\right)\dot{\beta}^2 + \frac{1}{2}m_1l_1^2\dot{\alpha}^2\sin^2\beta \\ &\quad - m_1L_0l_1\dot{\alpha}\dot{\beta}\cos\beta - m_1gl_1\cos\beta\end{aligned} \tag{8-20}$$

Concerning the generalized coordinate α, we have

$$\frac{\partial L}{\partial\dot{\alpha}} = \left(J_0 + m_1L_0^2\right)\dot{\alpha} + m_1l_1^2\dot{\alpha}\sin^2\beta - m_1L_0l_1\dot{\beta}\cos\beta \tag{8-21}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}}\right) &= \left(J_0 + m_1L_0^2 + m_1l_1^2\sin^2\beta\right)\ddot{\alpha} - m_1L_0l_1\ddot{\beta}\cos\beta \\ &\quad + 2m_1l_1^2\dot{\alpha}\dot{\beta}\sin\beta\cos\beta + m_1L_0l_1\dot{\beta}^2\sin\beta\end{aligned} \tag{8-22}$$

$$\frac{\partial L}{\partial \alpha} = 0 \tag{8-23}$$

From (8-5), it can be obtained that

$$\left(J_0 + m_1 L_0^2 + m_1 l_1^2 \sin^2 \beta\right)\ddot{\alpha} - m_1 L_0 l_1 \ddot{\beta} \cos\beta \tag{8-24}$$

$$+ 2m_1 l_1^2 \dot{\alpha}\dot{\beta} \sin\beta \cos\beta + m_1 L_0 l_1 \dot{\beta}^2 \sin\beta = \tau - C_0 \dot{\alpha}$$

where $\tau_\alpha = \tau - C_0\dot{\alpha}$, i.e., the generalized force τ_α includes the torque τ applied to the arm and the viscous friction $-C_0\dot{\alpha}$ of the arm. As to the generalized coordinate β, it can be found that

$$\frac{\partial L}{\partial \dot{\beta}} = \left(J_1 + m_1 l_1^2\right)\dot{\beta} - m_1 L_0 l_1 \dot{\alpha} \cos\beta \tag{8-25}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) = \left(J_1 + m_1 l_1^2\right)\ddot{\beta} - m_1 L_0 l_1 \ddot{\alpha} \cos\beta + m_1 L_0 l_1 \dot{\alpha}\dot{\beta} \sin\beta \tag{8-26}$$

$$\frac{\partial L}{\partial \beta} = m_1 l_1^2 \dot{\alpha}^2 \sin\beta \cos\beta + m_1 L_0 l_1 \dot{\alpha}\dot{\beta} \sin\beta + m_1 g l_1 \sin\beta \tag{8-27}$$

From (8-6), we obtain

$$- m_1 L_0 l_1 \ddot{\alpha} \cos\beta + \left(J_1 + m_1 l_1^2\right)\ddot{\beta} \tag{8-28}$$

$$- m_1 l_1^2 \dot{\alpha}^2 \sin\beta \cos\beta - m_1 g l_1 \sin\beta = -C_1 \dot{\beta}$$

where $\tau_\beta = -C_1\dot{\beta}$, i.e., τ_β is the viscous friction $-C_1\dot{\beta}$ of the pendulum.

From (8-24) and (8-28), the dynamic equation of the inverted pendulum is expressed in matrix form as below:

$$\underbrace{\begin{bmatrix} J_0 + m_1 L_0^2 + m_1 l_1^2 \sin^2\beta & -m_1 L_0 l_1 \cos\beta \\ -m_1 L_0 l_1 \cos\beta & J_1 + m_1 l_1^2 \end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} \tag{8-29}$$

$$+ \underbrace{\begin{bmatrix} m_1 l_1^2 \dot{\beta} \sin\beta \cos\beta & m_1 l_1^2 \dot{\alpha} \sin\beta \cos\beta + m_1 L_0 l_1 \dot{\beta} \sin\beta \\ -m_1 l_1^2 \dot{\alpha} \sin\beta \cos\beta & 0 \end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -m_1 g l_1 \sin\beta \end{bmatrix} = \begin{bmatrix} \tau - C_0\dot{\alpha} \\ -C_1\dot{\beta} \end{bmatrix}$$

where $\boldsymbol{M}$ is the inertia matrix and $\boldsymbol{B}$ is the matrix related to the centrifugal forces and coriolis forces. There are two important properties concerning $\boldsymbol{M}$ and $\boldsymbol{B}$. First, the inertia matrix is symmetric and positive-definite, i.e., $\boldsymbol{M}=\boldsymbol{M}^T$ and $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}>0$ for all $\boldsymbol{x}\neq\boldsymbol{0}$. Second, $\frac{1}{2}\dot{\boldsymbol{M}}-\boldsymbol{B}$ is skew-symmetic, i.e., $\boldsymbol{x}^T\left(\frac{1}{2}\dot{\boldsymbol{M}}-\boldsymbol{B}\right)\boldsymbol{x}=0$ for all $\boldsymbol{x}$.

For the first property, it can be seen from the elements of $\boldsymbol{M}$ that the symmetricity $\boldsymbol{M}=\boldsymbol{M}^T$ is true. Furthermore, by direct calculation we have

$$\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}=\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} J_0+m_1L_0^2+m_1l_1^2\sin^2\beta & -m_1L_0l_1\cos\beta \\ -m_1L_0l_1\cos\beta & J_1+m_1l_1^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (8\text{-}30)$$

$$\begin{aligned}
&=\left(J_0+m_1L_0^2+m_1l_1^2\sin^2\beta\right)x_1^2-2m_1L_0l_1\cos\beta x_1x_2+\left(J_1+m_1l_1^2\right)x_2^2 \\
&=\left(J_0+m_1l_1^2\sin^2\beta\right)x_1^2+m_1\left(L_0^2x_1^2-2L_0l_1\cos\beta x_1x_2+l_1^2x_2^2\right)+J_1x_2^2 \\
&=\left(J_0+m_1l_1^2\sin^2\beta\right)x_1^2+m_1\left(\left(L_0x_1-l_1\cos\beta x_2\right)^2+\left(l_1\sin\beta x_2\right)^2\right)+J_1x_2^2>0
\end{aligned}$$

which shows $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}>0$ for all $\boldsymbol{x}\neq\boldsymbol{0}$. This proves the first property. For the second property, let's calculate the matrix $\frac{1}{2}\dot{\boldsymbol{M}}-\boldsymbol{B}$, which can be obtained as

$$\frac{1}{2}\dot{\boldsymbol{M}}-\boldsymbol{B}=\begin{bmatrix} 0 & -\phi \\ \phi & 0 \end{bmatrix} \quad (8\text{-}31)$$

where $\phi=m_1l_1^2\dot{\alpha}\sin\beta\cos\beta+\frac{1}{2}m_1L_0l_1\dot{\beta}\sin\beta$. For all $\boldsymbol{x}$, it is easy to check that $\boldsymbol{x}^T\left(\frac{1}{2}\dot{\boldsymbol{M}}-\boldsymbol{B}\right)\boldsymbol{x}=0$, i.e., the second property is true.

From (8-2), we substitute the torque τ applied to the arm into (8-29) and then the dynamic equation is rewritten as

$$\begin{bmatrix} J_M+J_0+m_1L_0^2+m_1l_1^2\sin^2\beta & -m_1L_0l_1\cos\beta \\ -m_1L_0l_1\cos\beta & J_1+m_1l_1^2 \end{bmatrix}\begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} \quad (8\text{-}32)$$

$$+\begin{bmatrix} C_0+\dfrac{k^2}{R}-B_M+m_1l_1^2\dot{\beta}\sin\beta\cos\beta & m_1l_1^2\dot{\alpha}\sin\beta\cos\beta+m_1L_0l_1\dot{\beta}\sin\beta \\ -m_1l_1^2\dot{\alpha}\sin\beta\cos\beta & C_1 \end{bmatrix}\begin{bmatrix}\dot{\alpha}\\ \dot{\beta}\end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ -m_1gl_1\sin\beta \end{bmatrix}=\begin{bmatrix} k/R \\ 0 \end{bmatrix}v_s$$

where $v_s(t)$ is the voltage source to drive the system.