7. Lagrange's Equations

To derive the dynamic equations of a mechanical system, we often apply the Lagrange's equations which is based on an energy-balance relation and expressed in terms of the kinetic energy *T*, the potential energy *U*, and a set of independent coordinates. Before introducing the Lagrange's equations, let's consider the simplest mechanical *MBK* system and adopt the *N*ewton's second law to obtain its dynamic systems.

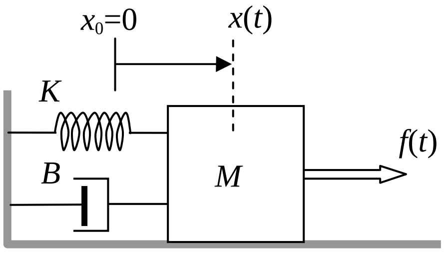

Figure7-1

In engineering, most of the systems are constructed by mechanical components such as damper and spring. Figure7-1 shows the simplest mechanical system, called the mass-damper-spring system or *MBK* system in brief, where *M* is the mass of the moving object, *B* is the damping coefficient of the damper and *K* is the stiffness of the spring. Let $f(t)$ be an extra force exerted on the object and assume $x(t)$ is the resulted deviation of the spring referred to its unforced status x_0=0. Then, there are two forces reacted to restrain the motion of the object, expressed as

$$f_B(t) = -B\dot{x}(t) \tag{7-1}$$

$$f_K(t) = -Kx(t) \tag{7-2}$$

where $f_B(t)$ is caused by the damper and $f_K(t)$ is the spring force. According to the Newton's second law of motion, we have

$$f(t) + f_B(t) + f_K(t) = M\ddot{x}(t) \tag{7-3}$$

i.e.,

$$M\ddot{x}(t)+B\dot{x}(t)+Kx(t)= f(t) \tag{7-4}$$

which is the dynamic equation of the MBK system.

For the mechanical system, the external force $f(t)$ is the input and the deviation $x(t)$ is commonly chosen as the output. Let the Laplace transforms be $F(s)$ and $X(s)$, then the system (7-4) can be described as

$$\left(Ms^2+Bs+K\right)X(s)=F(s) \tag{7-5}$$

The transfer function is then obtained as

$$H(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}=k_0\frac{a_0}{s^2+a_1s+a_0} \tag{7-6}$$

where $a_1=\dfrac{B}{M}$, $a_0=\dfrac{K}{M}$ and $k_0=\dfrac{1}{K}$. Clearly, it is a low-pass filter and can reject high-frequency inputs. For convenience, we define $a_0=\omega_n^2$ and $a_1=2\xi\omega_n$ and rewrite the transfer function in (7-6) as

$$H(s)=k_0\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \tag{7-7}$$

where ω_n is the natural frequency and ξ is the damping ratio. The analysis of (7-7) has been introduced in some fundamental courses.

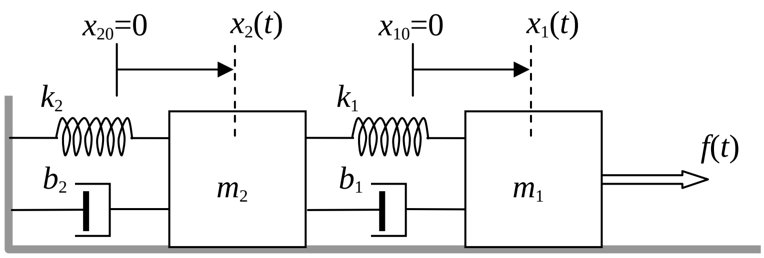

Figure7-2

Let's further consider a more complicated mechanical structure depicted in Figure7-2 where two masses m_1 and m_2 are linked to two dampers b_1 and b_2 and two springs k_1 and k_2. Let $f(t)$ be the extra force and assume $x_1(t)$ and $x_1(t)$ are the deviations of the springs referred to their unforced cases x_{10}=0 and x_{20}=0. According to the Newton's second law of motion, we have

$$f(t)+f_{b1}(t)+f_{k1}(t)=m_1\ddot{x}_1(t) \tag{7-8}$$

$$f_{b2}(t)+f_{k2}(t)-f_{b1}(t)-f_{k1}(t)=m_2\ddot{x}_2(t) \tag{7-9}$$

where

$$f_{b1}(t)=-b_1(\dot{x}_1(t)-\dot{x}_2(t)) \tag{7-10}$$

$$f_{k1}(t)=-k_1(x_1(t)-x_2(t)) \tag{7-11}$$

$$f_{b2}(t)=-b_2\dot{x}_2(t) \tag{7-12}$$

$$f_{k2}(t)=-k_2x_2(t) \tag{7-13}$$

Therefore, (7-8) and (7-9) can be rewritten as

$$m_1\ddot{x}_1(t)=-b_1\dot{x}_1(t)+b_1\dot{x}_2(t)-k_1x_1(t)+k_1x_2(t)+f(t) \tag{7-14}$$

$$m_2\ddot{x}_2(t)=b_1\dot{x}_1(t)-(b_1+b_2)\dot{x}_2(t)+k_1x_1(t)-(k_1+k_2)x_2(t) \tag{7-15}$$

which are the dynamic equations of two-mass system.

From (7-14) and (7-15), we may know that the dynamic equation of a mechanical system will become much more complicated if the number of masses are further increased. To deal with the modeling of a system with *n* masses, the famous Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=Q_i\,,\quad i=1,2,\ldots,n \tag{7-16}$$

are often employed to simplify the procedure, where q_i is the i-th generalized coordinate, $\dot{q}_i$ is its derivative, Q_i is the i-th generalized force and $L=T-U$ is a scalar function called Lagrangian. Clearly, the Lagrangian L is the difference between the total kinetic energy T and the total potential energy U. Note that the kinetic energy T depends on the generalized coordinates q_i and their derivatives $\dot{q}_i$, $i=1,2,\ldots,n$, while the potential energy only depends on the generalized coordinates q_i. Hence, substituting $L=T-U$ into (7-16) yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=Q_i\,,\quad i=1,2,\ldots,n \tag{7-17}$$

where the truth of $\frac{\partial U}{\partial \dot{q}_i}=0$ is used.

Now, instead of Newton's second law, let's derive (7-14) and (7-15) by the Lagrange's equations (7-17). First, the total kinetic energy is

$$T=\frac{1}{2}m_1\dot{x}_1^2(t)+\frac{1}{2}m_2\dot{x}_2^2(t) \tag{7-18}$$

and the total potential energy is

$$U=\frac{1}{2}k_1\left(x_1(t)-x_2(t)\right)^2+\frac{1}{2}k_2x_2^2(t) \tag{7-19}$$

From (7-17), we need to calculate the following terms:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)=m_1\ddot{x}_1(t) \tag{7-20}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right)=m_2\ddot{x}_2(t) \tag{7-21}$$

$$\frac{\partial T}{\partial x_1}=0 \tag{7-22}$$

$$\frac{\partial T}{\partial x_2}=0 \tag{7-23}$$

$$\frac{\partial U}{\partial x_1}=k_1\left(x_1(t)-x_2(t)\right) \tag{7-24}$$

$$\frac{\partial U}{\partial x_2}=-k_1\left(x_1(t)-x_2(t)\right)+k_2x_2(t) \tag{7-25}$$

Since the mass m_1 encounters the external force $f(t)$ and the force caused by the connected damper $-b_1\left(\dot{x}_1(t)-\dot{x}_2(t)\right)$ along the coordinate x_1, we have

$$Q_1=f(t)-b_1\left(\dot{x}_1(t)-\dot{x}_2(t)\right) \tag{7-26}$$

As for the mass m_2, it encounters the forces caused by the connected dampers $-b_1\left(\dot{x}_2(t)-\dot{x}_1(t)\right)$ and $-b_2\dot{x}_2(t)$. This results in

$$Q_2=-b_1\left(\dot{x}_2(t)-\dot{x}_1(t)\right)-b_2\dot{x}_2(t) \tag{7-27}$$

Hence, according to (7-17)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)-\frac{\partial T}{\partial x_i}+\frac{\partial U}{\partial x_i}=Q_i\,,\quad i{=}1{,}2 \tag{7-28}$$

the dynamic equations of the two-mass system are

$$m_1\ddot{x}_1(t)+k_1(x_1(t)-x_2(t))=f(t)-b_1(\dot{x}_1(t)-\dot{x}_2(t)) \tag{7-29}$$

$$m_2\ddot{x}_2(t)-k_1(x_1(t)-x_2(t))+k_2x_2(t)=-b_1(\dot{x}_2(t)-\dot{x}_1(t))-b_2\dot{x}_2(t) \tag{7-30}$$

i.e.,

$$m_1\ddot{x}_1(t)=-b_1\dot{x}_1(t)+b_1\dot{x}_2(t)-k_1x_1(t)+k_1x_2(t)+f(t) \tag{7-31}$$

$$m_2\ddot{x}_2(t)=b_1\dot{x}_1(t)-(b_1+b_2)\dot{x}_2(t)+k_1x_1(t)-(k_1+k_2)x_2(t) \tag{7-32}$$

both similar to (7-14) and (7-15).

Now, let's derive the Lagrange's equations for N masses, denoted as m_1, m_2, …, m_N, and their positions are described by the Cartesian coordinates. Since each mass is given by three coordinates, the total coordinates are x_i, i=1,2,…,3N. For simplicity, we assume the first mass m_1 is located at (x_1, x_2, x_3), the second mass m_2 is located at (x_4, x_5, x_6), and so on. As a result, the total kinetic energy is

$$T=\frac{1}{2}\sum_{i=1}^{3N}m_i'\dot{x}_i^2 \tag{7-33}$$

where $m_i'=m_{[(2+i)/3]}$. Further assume the system will be represented in terms of generalized coordinates q_j, j=1,2,…,n. This implies each coordinate x_i can be expressed as a function of q_j, j=1,2,…,n. Hence, we have

$$\dot{x}_i=\sum_{j=1}^{n}\frac{\partial x_i}{\partial q_j}\dot{q}_j+\frac{\partial x_i}{\partial t} \tag{7-34}$$

which leads to

$$T=\frac{1}{2}\sum_{i=1}^{3N}m_i'\left(\sum_{j=1}^{n}\frac{\partial x_i}{\partial q_j}\dot{q}_j+\frac{\partial x_i}{\partial t}\right)^2 \tag{7-35}$$

If we define the generalized momentum with respect to q_k, k=1,2,…,n, as

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{3N} m_i' \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \sum_{i=1}^{3N} m_i' \dot{x}_i \frac{\partial x_i}{\partial q_k} \tag{7-36}$$

where the truth of $\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}$ can be obtained from (7-34). Differentiating p_k with

$$\dot{p}_k = \sum_{i=1}^{3N} m_i' \ddot{x}_i \frac{\partial x_i}{\partial q_k} + \sum_{i=1}^{3N} m_i' \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) \tag{7-37}$$

where $\frac{\partial x_i}{\partial q_k}$ is also a function of t and q_j j1,2,…,n. Thus,

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 x_i}{\partial q_j \partial q_k} \dot{q}_j + \frac{\partial^2 x_i}{\partial t \partial q_k} \tag{7-38}$$

From (7-34), we have

$$\frac{\partial \dot{x}_i}{\partial q_k} = \sum_{j=1}^{n} \frac{\partial^2 x_i}{\partial q_j \partial q_k} \dot{q}_j + \frac{\partial^2 x_i}{\partial t \partial q_k} \tag{7-39}$$

Both (7-38) and (7-39) are the same, i.e.,

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) = \frac{\partial \dot{x}_i}{\partial q_k} \tag{7-40}$$

which will change (7-37) into

$$\dot{p}_k = \sum_{i=1}^{3N} m_i' \ddot{x}_i \frac{\partial x_i}{\partial q_k} + \sum_{i=1}^{3N} m_i' \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} \tag{7-41}$$

Its last term $\sum_{i=1}^{3N} m_i' \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k}$ can be also shown related to the kinetic energy (7-33) as below:

$$\frac{\partial T}{\partial q_k} = \sum_{i=1}^{3N} m_i' \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} \tag{7-42}$$

Therefore, we have

$$\dot{p}_k = \sum_{i=1}^{3N} m_i' \ddot{x}_i \frac{\partial x_i}{\partial q_k} + \frac{\partial T}{\partial q_k} \tag{7-43}$$

According to the Newton's second law, the mass m_i should obey the following equation

$$m_i' \ddot{x}_i = F_i + R_i \tag{7-44}$$

where F_i represents the sum of the applied forces and R_i is the sum of the workless constraint forces. Now, (7-43) is written as

$$\dot{p}_k = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_k} + \sum_{i=1}^{3N} R_i \frac{\partial x_i}{\partial q_k} + \frac{\partial T}{\partial q_k} \tag{7-45}$$

For the sum of the workless constraint forces R_i, we assume it has done a virtual work δW by moving a virtual displacement, expressed as

$$\delta W = \sum_{j=1}^{n} \left(\sum_{i=1}^{3N} R_i \frac{\partial x_i}{\partial q_k} \right) \delta q_j = 0 \tag{7-46}$$

Note that the virtual work δW done by constraint forces should be zero for any independent virtual displacement δq_j. This implies the term $\sum_{i=1}^{3N} R_i \frac{\partial x_i}{\partial q_k}$ in (7-46) should be zero, i.e.,

$$\sum_{i=1}^{3N} R_i \frac{\partial x_i}{\partial q_k} = 0 \tag{7-47}$$

Consequently, (7-45) becomes

$$\dot{p}_k = Q_k + \frac{\partial T}{\partial q_k} \tag{7-48}$$

where

$$Q_k = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_k} \tag{7-49}$$

is defined as the generalized force Q_k with respect to q_k. Further from the definition of $p_k = \frac{\partial T}{\partial \dot{q}_k}$ in (7-36), we rewrite (7-48) as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k, \quad k=1,2,\ldots,n \tag{7-50}$$

whish are known as the fundamental form of Lagrange's equations.

Under the condition that the system only encounters the conservative forces, all the generalized forces Q_k can be obtained by the gradient of the potential energy V with respect to q_k. The generalized forces Q_k, k=1,2,…,n, are then expressed as

$$Q_k = -\frac{\partial V}{\partial q_k}, \quad k=1,2,\ldots,n \tag{7-51}$$

As a result, the Lagrange's equations under conservative forces are governed by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0, \quad k=1,2,\ldots,n \tag{7-52}$$

In case that there are non-conservative forces applying to the system, the Lagrange's equations will be described as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k, \quad k=1,2,\ldots,n \tag{7-53}$$

where Q_k, $k=1,2,\ldots,n$, are the non-conservative forces. The standard form of Lagrange's equations are then shown as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k, \quad k=1,2,\ldots,n \tag{7-54}$$

where $L=T-V$ is called the Lagrangian. Note that here uses the truth that the potential energy V is independent to $\dot{q}_k$, the derivative of generalized coordinates q_k.