# 6. Controller Design of the DC Motor

This topic will focus on the controller design of the DC motor to drive its rotor moving in a desired speed $\omega(t)=\omega_d$ or to a specified angle $\theta(t)=\theta_d$.

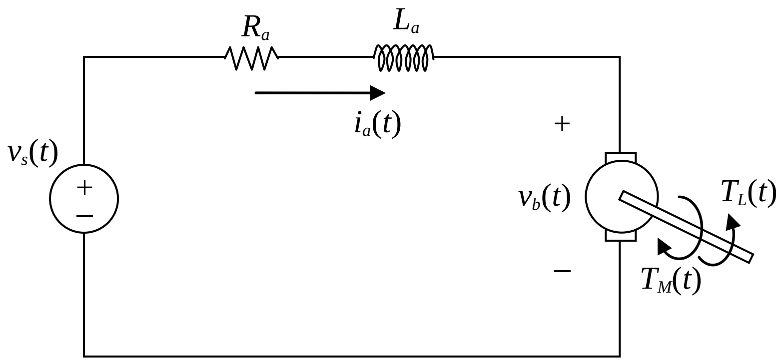


Figure 6-1

The system structure of a DC motor is shown in Figure 6-1, including the armature resistance $R_a$ and winding leakage inductance $L_a$. Its dynamic equation can be expressed as

$$L_a \frac{di_a(t)}{dt} + R_a i_a(t) + k\omega(t) = v_s(t) \tag{6-1}$$

$$J \frac{d\omega(t)}{dt} + B_M \omega(t) - k i_a(t) = 0 \tag{6-2}$$

where $J=J_M+J_L$ if the rotor is connected to a payload $T_L(t) = J_L \frac{d\omega(t)}{dt}$. In general, the electrical time constant $L_a/R_a$ is often neglected since it is usually at least one order in magnitude smaller than the mechanical time constant $J/B_M$. In other words, by neglecting the term $\frac{di_a(t)}{dt}$, (6-1) becomes

$$i_a(t) = \frac{1}{R_a} v_s(t) - \frac{k}{R_a}\omega(t) \tag{6-3}$$

Substituting it into (6-2), we have

$$\frac{d\omega(t)}{dt} + \left(\frac{B_M}{J} + \frac{1}{JR_a}k^2\right)\omega(t) = \frac{1}{JR_a} k v_s(t) \tag{6-4}$$

Now, based on (6-4), let's discuss the controller design of the rotor's angular velocity $\omega(t)$ and angular position $\theta(t)$.

To obtain the practical model of a DC motor described by (6-4), we can identify the modell via the experiment, not through the measurement of structure parameters $J$, $R_a$, $k$ and $B_M$. Assume the resulted dynamic model is

$$\frac{d\omega(t)}{dt}+\alpha\omega(t)=\beta v_s(t) \tag{6-5}$$

and then from $\omega(t)=\dfrac{d\theta(t)}{dt}$ we have

$$\frac{d^2\theta(t)}{dt^2}+\alpha\frac{d\theta(t)}{dt}=\beta v_s(t) \tag{6-6}$$

Hence, the transfer function of $\omega(t)$ with respect to $v_s(t)$ becomes

$$H_\omega(s)=\frac{\Omega(s)}{V_s(s)}=\frac{\beta}{s+\alpha} \tag{6-7}$$

and the transfer function of $\theta(t)$ with respect to $v_s(t)$ becomes

$$H_\theta(s)=\frac{\Theta(s)}{V_s(s)}=\frac{\beta}{s(s+\alpha)} \tag{6-8}$$

The block diagrams of (6-5) and (6-6) are shown in Figure 6-2 and Figure 6-3, respectively. Next, let's discuss the velocity controller design of the model (6-5).

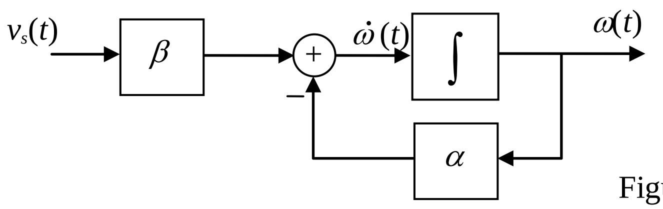


Figure 6-2

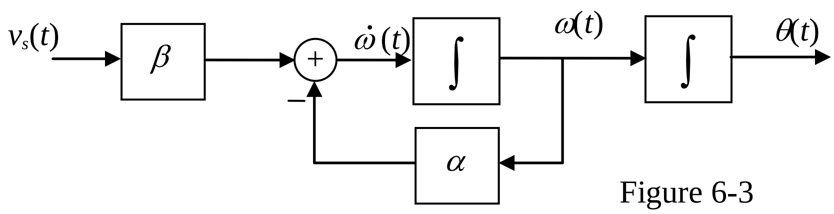


Figure 6-3

If the control goal is to drive the motor to a desired speed $\omega(t)=\omega_d$, the intuitive way is to give a constant input $v_s(t)=\dfrac{\alpha}{\beta}\omega_d$ and then (6-5) is changed into

$$\frac{d\omega(t)}{dt}+\alpha\omega(t)=\alpha\omega_d \tag{6-9}$$

whose response becomes

$$\omega(t)=(\omega(0)-\omega_d)e^{-\alpha t}+\omega_d \tag{6-10}$$

Since $\alpha$>0, if $t\to\infty$ we have $\omega(t)\to\omega_d$ and reach the control goal. This is a kind of direct feed-forward control and it is only suitable for a system precisely modeled.

In case that the values $\alpha$ and $\beta$ are possessed of uncertainties, the direct feed-forward control is not able to appropriately drive the motor to operate at a desired speed. To deal with such problem, the first step of controller design is often to assign an error function as below:

$$e(t)=\omega(t)-\omega_d \tag{6-11}$$

which shows the difference for the current speed to reach its desired value. It is clear that the control purpose is fulfilled when the error $e(t)$ vanishes, i.e., $e(t)=0$. Based on (6-11), we change (6-5) into

$$\frac{de(t)}{dt}+\alpha e(t)=\beta v_s(t)-\alpha\omega_d \tag{6-12}$$

and design the controller by feedback technology.

First, let's employ the proportional feedback control by setting the input as below:

$$v_s(t)=k_p e(t) \tag{6-13}$$

and then (6-12) becomes

$$\frac{de(t)}{dt}+(\alpha-\beta k_p)e(t)=-\alpha\omega_d \tag{6-14}$$

Suitably choosing $k_p$ such that $\alpha_p=\alpha-\beta k_p>0$ yields

$$e(t)=\left(e(0)+\frac{\alpha}{\alpha_p}\omega_d\right)e^{-\alpha_p t}-\frac{\alpha}{\alpha_p}\omega_d \tag{6-15}$$

which leads to

$$e(\infty)=-\frac{\alpha}{\alpha_p}\omega_d\neq 0 \tag{6-16}$$

In other words, there exists a steady-state error and the error can be only reduced by choosing $\alpha_p$. large enough.

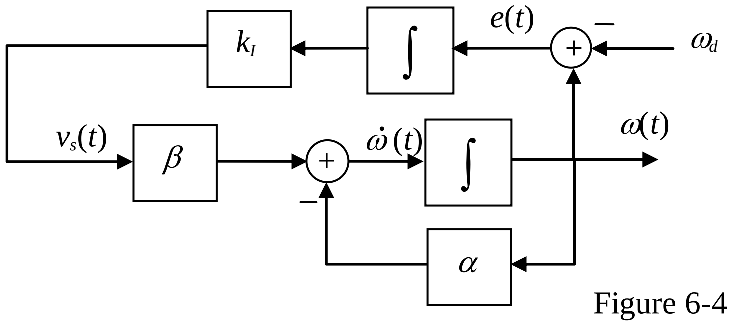

Figure 6-4

In order to eliminate the steady-state error, we often employ the integral control which is set to be

$$v_s(t)=k_I\int_0^t e(\tau)d\tau \tag{6-17}$$

and then (6-12) becomes

$$\frac{de(t)}{dt}(t)+\alpha e(t)=\beta k_I\int_0^t e(\tau)d\tau-\alpha\omega_d \tag{6-18}$$

Further taking the derivative of (6-18) leads to

$$\frac{d^2e(t)}{dt^2}+\alpha\frac{de(t)}{dt}-\beta k_I e(t)=0 \tag{6-19}$$

By choosing $k_I$ such that $-\beta k_I>0$, we can conclude that the eigenvalues of (6-19) are stable and thus the error $e(t)\to 0$ as $t\to\infty$. This makes the control successful. Figure 6-4 shows the block diagram with the integral control.

To drive the motor to a specified angle, we can use the model (6-6) and choose the error function as below:

$$e(t) = \theta(t) - \theta_d \tag{6-20}$$

which shows the difference for the current angle to reach its desired value. Similarly, the control purpose is reached if the error $e(t)$ vanishes, i.e., $e(t)$=0. Based on (6-20), we change (6-6) into

$$\frac{d^2 e(t)}{dt^2} + \alpha \frac{de(t)}{dt} = \beta v_s(t) \tag{6-21}$$

and design the controller by proportional feedback technology.

$$v_s(t) = k_p e(t) \tag{6-22}$$

The resulted system becomes

$$\frac{d^2 e(t)}{dt^2} + \alpha \frac{de(t)}{dt} - \beta k_p e(t) = 0 \tag{6-23}$$

Obviously, if $-\beta k_p > 0$, the error $e(t)$ will approach zero as $t \to \infty$. The angular position control is thus completed. Figure 6-5 shows the block diagram with the proportional control.

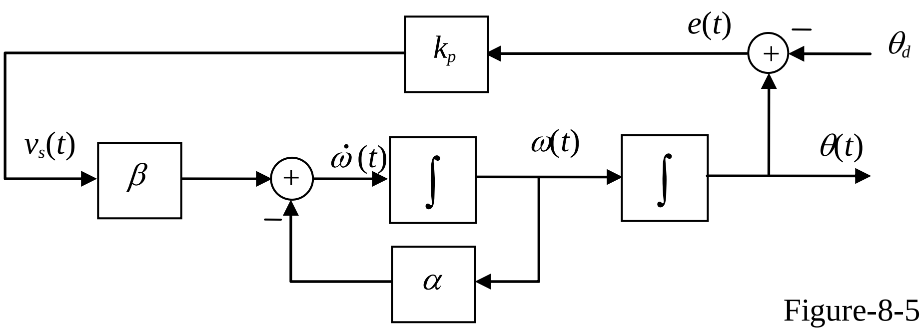


Figure-8-5

Although the error can be reduced to zero, its approaching rate may not be as desired, too fast or too slow. To avoid such drawback, we could employ the prroportional-derivative control as below:

$$v_s(t) = k_p e(t) + k_d \dot{e}(t) \tag{6-24}$$

which leads to

$$\frac{d^2e(t)}{dt^2}+(\alpha-k_d)\frac{de(t)}{dt}-\beta k_p e(t)=0 \tag{6-25}$$

Now, the approaching rate of the error $e(t)$ to reach zero can be assigned by choosing appropriate roots of the characteristic equation $\lambda^2+(\alpha-k_d)\lambda-\beta k_p=0$.

The other way is to trace a desired trajectory by setting a feedforward filter given as

$$H(s)=\frac{\sigma}{s+\sigma} \tag{6-26}$$

such that the desired angular position is changed into $\theta_D(t)=(\theta_D(0)-\theta_d)e^{-\sigma t}+\theta_d$ where $\theta_D(0)\approx\theta(0)$. Note that $\theta_D(t)\to\theta_d$ as $t\to\infty$. Figure 6-6 shows the block diagram with a feedforward filter.

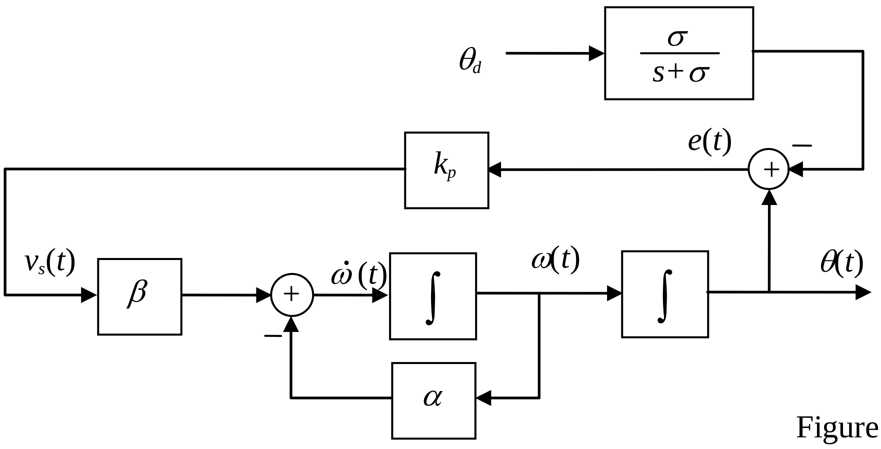


Figure 6-6