

5. Identification of Dynamic Systems

Before processing or controlling a dynamic system, it is often required to identify its practical mathematical model by using parameter estimation techniques. There are two important estimation algorithms often used for system identification, including the Least squares method and the Lagrange's polynomial.

For example, let's consider a second order linear time invariant system which is described as below:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_1\dot{u}(t) + b_0u(t) \quad (5-1)$$

where $u(t)$ is the input, $y(t)$ is the output and all the coefficients a_1 , a_0 , b_1 and b_0 are unknown. In general, the input $u(t)$ is given by ourselves, so both $u(t)$ and $\dot{u}(t)$ are exactly known. As for the output, we assume that only $y(t)$ is measurable, but $\dot{y}(t)$ and $\ddot{y}(t)$ are not obtainable. Based on the above statement, now let's find a way to determine the unknown coefficients a_1 , a_0 , b_1 and b_0 by applying the data related to $u(t)$, $\dot{u}(t)$ and $y(t)$.

If the sampling time to measure the output $y(t)$ is T , then the data available for the determination of a_1 , a_0 , b_1 and b_0 are $u(kT)$, $\dot{u}(kT)$ and $y(kT)$ where $k=0,1,2,\dots,n$. Since there are four unknown coefficients, we need to adopt at least four equations formed by (7-1) at $t=k_iT$, $i=1,2,\dots,n$ and $n>4$. For simplicity, we arbitrarily choose five data, i.e., $n=5$, at $k_1=100$, $k_2=140$, $k_3=200$, $k_4=250$ and $k_5=310$. Then, we can write five equations from (7-1) as below:

$$\ddot{y}(100T) + a_1\dot{y}(100T) + a_0y(100T) = b_1\dot{u}(100T) + b_0u(100T) \quad (5-2)$$

$$\ddot{y}(140T) + a_1\dot{y}(140T) + a_0y(140T) = b_1\dot{u}(140T) + b_0u(140T) \quad (5-3)$$

$$\ddot{y}(200T) + a_1\dot{y}(200T) + a_0y(200T) = b_1\dot{u}(200T) + b_0u(200T) \quad (5-4)$$

$$\ddot{y}(250T) + a_1\dot{y}(250T) + a_0y(250T) = b_1\dot{u}(250T) + b_0u(250T) \quad (5-5)$$

$$\ddot{y}(310T) + a_1 \dot{y}(310T) + a_0 y(310T) = b_1 \dot{u}(310T) + b_0 u(310T) \quad (5-6)$$

which can be rearranged as the following matrix form:

$$\underbrace{\begin{bmatrix} \dot{u}(100T) & u(100T) & -\dot{y}(100T) & -y(100T) \\ \dot{u}(140T) & u(140T) & -\dot{y}(140T) & -y(140T) \\ \dot{u}(200T) & u(200T) & -\dot{y}(200T) & -y(200T) \\ \dot{u}(250T) & u(250T) & -\dot{y}(250T) & -y(250T) \\ \dot{u}(310T) & u(310T) & -\dot{y}(310T) & -y(310T) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} b_1 \\ b_0 \\ a_1 \\ a_0 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \ddot{y}(100T) \\ \ddot{y}(140T) \\ \ddot{y}(200T) \\ \ddot{y}(250T) \\ \ddot{y}(310T) \end{bmatrix}}_{\mathbf{p}} \quad (5-7)$$

Without loss of generality, $\mathbf{A}$ is often of full rank and then $\mathbf{A}^T \mathbf{A}$ is invertible. As a result, the vector $\mathbf{x}$ concerning the unknown parameters is solved as below:

$$\mathbf{v} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{p} \quad (5-8)$$

Actually, (7-8) is referred to the least squares method which is introduced in the end of this topic as an appendix.

Unfortunately, (7-8) is not available yet because we still need to get the data of $\dot{y}(t)$ and $\ddot{y}(t)$ at $t=100T, 140T, 200T, 250T$ and $310T$. Here, let's adopt the data of $\dot{y}(t)$ and $\ddot{y}(t)$ at $t=100T$ to explain how to estimate them from the data $y(t)$ based on the lagrange's polynomial.

In fact, the most simplest way to estimate $\dot{y}(t)$ at $t=100T$ can be implemented by the definition:

$$\dot{y}(100T) = \lim_{h \rightarrow 0} \frac{y(100T) - y(100T - h)}{h} \approx \frac{y(100T) - y(100T - mT)}{mT} \quad (5-9)$$

where m is an integer but not necessary to be 1. As for the estimation of $\ddot{y}(t)$ at $t=100T$, we further calculate

$$\begin{aligned} \ddot{y}(100T - mT) &= \lim_{h \rightarrow 0} \frac{y(100T - mT) - y(100T - mT - h)}{h} \\ &\approx \frac{y(100T - mT) - y(100T - 2mT)}{mT} \end{aligned} \quad (5-10)$$

then

$$\begin{aligned}\ddot{y}(100T) &= \lim_{h \rightarrow 0} \frac{\dot{y}(100T) - \dot{y}(100T - h)}{h} \approx \frac{\dot{y}(100T) - \dot{y}(100T - mT)}{mT} \\ &\approx \frac{y(100T) - 2y(100T - mT) + y(100T - 2mT)}{m^2 T^2}\end{aligned}\quad (5-11)$$

Although (5-9) and (5-11) is simple, they often lead to undesirable estimation errors. In instead, we add two other data $y(100T+mT)$ and $y(100T+2mT)$, i.e., we use five data of $y(t)$ to estimate $\dot{y}(t)$ and $\ddot{y}(t)$ at $t=100T$. For convenient, let's define

$$t_1=100T-2mT, t_2=100T-mT, t_3=100T, t_4=100T+mT, t_5=100T+2mT$$

$$y_1=y(t_1), y_2=y(t_2), y_3=y(t_3), y_4=y(t_4), y_5=y(t_5)$$

then a possible curve of $y(t)$ passing y_i at $t=t_i, i=1,2,3,4,5$, can be formed as

$$y(t) = \sum_{i=1}^5 \left(y_i \prod_{j=1, j \neq i}^5 \frac{(t-t_j)}{(t_i-t_j)} \right) \quad (5-12)$$

or

$$\begin{aligned}y(t) &= y_1 \frac{(t-t_2)(t-t_3)(t-t_4)(t-t_5)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)(t_1-t_5)} \\ &+ y_2 \frac{(t-t_1)(t-t_3)(t-t_4)(t-t_5)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)(t_2-t_5)} + y_3 \frac{(t-t_1)(t-t_2)(t-t_4)(t-t_5)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)(t_3-t_5)} \\ &+ y_4 \frac{(t-t_1)(t-t_2)(t-t_3)(t-t_5)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)(t_4-t_5)} + y_5 \frac{(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{(t_5-t_1)(t_5-t_2)(t_5-t_3)(t_5-t_4)}\end{aligned}\quad (5-13)$$

which is called the Lagrange's polynomial. Hence, after directly taking derivative with respect to t and then setting $t=t_3=100T$, we obtain

$$\dot{y}(100T) = \dot{y}(t_3) = \frac{y_1 - 8y_2 + 8y_3 - y_5}{12mT} \quad (5-14)$$

Similarly, taking derivative twice with respect to t yields

$$\ddot{y}(100T) = \ddot{y}(t_3) = \frac{-y_1 + 16y_2 - 42y_3 + 16y_4 - y_5}{12m^2 T^2} \quad (5-15)$$

The above procedure can be applied to estimate $\dot{y}(t)$ and $\ddot{y}(t)$ at $t=140T, 200T, 250T$ and $310T$. Then, with all these estimated data, the vector $\mathbf{v}$ in (5-8) can be implemented to get the unknown parameters a_1, a_0, b_1 and b_0 in (5-1) which describes the dynamic system under processing.

Appendix: Least Square Method

Consider an engineering problem which have to determine the unknown coefficients $a_i, i=1,2,\dots,n$, in the following linear equation

$$y = a_1x_1 + a_2x_2 + \dots + a_nx_n \quad (\text{A-1})$$

where the values of $y, x_1, x_2, \dots, x_n$ are measured up to m times, where $m>n$. Assume $y_k, x_{k1}, x_{k2}, \dots$ and x_{kn} are the values obtained at the k -th measurement. Unfortunately, measurement often exists inaccuracy and results in errors such that

$$y_k \neq a_1x_{k1} + a_2x_{k2} + \dots + a_nx_{kn} \quad (\text{A-2})$$

Hence, it is impossible to find the n coefficients $a_i, i=1,2,\dots,n$, just by n times of measurement. Instead, a large amount of measurements are needed to obtain a set of coefficients $a_i, i=1,2,\dots,n$, to fit (A-1) best. Now, let's introduce how to apply the least squares method to determine the best coefficients for (A-1). First, define the k -th error function as

$$e_k = y_k - (a_1x_{k1} + a_2x_{k2} + \dots + a_nx_{kn}) \quad (\text{A-3})$$

where $k=1,2,\dots,m$, and a performance index as

$$E = e_1^2 + e_2^2 + \dots + e_m^2 = \sum_{k=1}^m e_k^2 \quad (\text{A-4})$$

It is required to determine the coefficients $a_i, i=1,2,\dots,n$, such that the performance index is minimal, i.e., reaches the smallest value. Since the index is the sum of error squares, this method is usually called "least squares method."

Further rewrite the set of m error functions defined in (A-3) as the following matrix form

$$\underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}}_{\mathbf{e}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{\mathbf{y}} - \underbrace{\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \dots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}}_{\mathbf{X}} \cdot \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\mathbf{a}} \quad (\text{A-5})$$

where $\mathbf{y} \in \mathcal{R}^m$ and $\mathbf{X} \in \mathcal{R}^{m \times n}$ are fixed and known since both of them contain all the measured values. The vector $\mathbf{a} \in \mathcal{R}^n$ consists of the unknown n coefficients.

Let's define

$$\underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}}_{\boldsymbol{\alpha}} = \underbrace{\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}}_{\mathbf{X}} \cdot \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\mathbf{a}} = a_1 \underbrace{\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \end{bmatrix}}_{\mathbf{x}_1} + a_2 \underbrace{\begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \end{bmatrix}}_{\mathbf{x}_2} + \cdots + a_n \underbrace{\begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix}}_{\mathbf{x}_n} \quad (\text{A-6})$$

which implies the vector $\boldsymbol{\alpha} \in \mathcal{R}^n$ is in the space formed by the column vectors of $\mathbf{X}$, denoted as $\mathbf{x}_i$, $i=1,2,\dots,n$, i.e., $\boldsymbol{\alpha}$ is in the column space of $\mathbf{X}$. Now, (A-5) can be expressed as

$$\mathbf{e} = \mathbf{y} - \boldsymbol{\alpha} \quad (\text{A-7})$$

depicted in Figure A-1. Without loss of generality, the column vectors $\mathbf{x}_i$, $i=1,2,\dots,n$, are all assumed to be independent, i.e., $\mathbf{X}$ is of full rank or $\text{rank}(\mathbf{X})=n$ for $m>n$. Under the condition $\text{rank}(\mathbf{X})=n$, it has been proved that the square matrix $\mathbf{X}^T \mathbf{X} \in \mathcal{R}^{n \times n}$ is invertible and its inverse $(\mathbf{X}^T \mathbf{X})^{-1}$ exists.

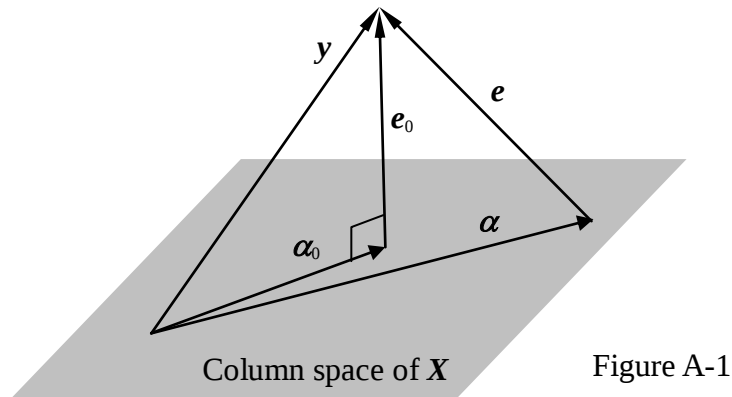


Figure A-1

It is required to find a suitable coefficient vector $\mathbf{a} \in \mathcal{R}^n$ such that the error $\mathbf{e}$ is reduced as small as possible based on the performance index (A-4), given as below:

$$E = \sum_{k=1}^m e_k^2 = \mathbf{e}^T \mathbf{e} = \|\mathbf{e}\|^2 = \|\mathbf{y} - \mathbf{X}\mathbf{a}\|^2 = \|\mathbf{y} - \boldsymbol{\alpha}\|^2 \quad (\text{A-8})$$

Then, its minimum is determined by the following condition:

$$\frac{\partial E}{\partial a_i} = \sum_{k=1}^m \frac{\partial e_k^2}{\partial a_i} = 2 \sum_{k=1}^m e_k \frac{\partial e_k}{\partial a_i} = -2 \sum_{k=1}^m e_k x_{ki} = 0, \quad i=1,2,\dots,n \quad (\text{A-9})$$

which can be rearranged as

$$\sum_{k=1}^m e_k x_{ki} = \underbrace{[e_1 \quad e_2 \quad \dots \quad e_m]}_{\mathbf{e}^T} \cdot \underbrace{\begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{mi} \end{bmatrix}}_{\mathbf{x}_i} = 0, \quad i=1,2,\dots,n \quad (\text{A-10})$$

Clearly, if the index E in (A-8) is minimal, then the error vector $\mathbf{e}$ must be perpendicular to all the column vectors of $\mathbf{X}$, which implies the error vector $\mathbf{e}$ must be orthogonal to the column space of $\mathbf{X}$. It can be seen from Figure A-1 that the error vector $\mathbf{e}_0$ is the one satisfying (A-10); therefore, we have

$$\mathbf{e}_0^T \cdot \underbrace{\begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{mi} \end{bmatrix}}_{\mathbf{x}_i} = 0, \quad i=1,2,\dots,n \quad (\text{A-11})$$

which implies

$$\mathbf{e}_0^T \cdot \underbrace{\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \dots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}}_{\mathbf{X}} = \underbrace{[0 \quad 0 \quad \dots \quad 0]}_{\mathbf{0}} \quad \text{or} \quad \mathbf{X}^T \mathbf{e}_0 = \mathbf{0} \quad (\text{A-12})$$

From (A-5), let $\mathbf{a}_0$ be the coefficient vector related to $\mathbf{e}_0$, i.e.,

$$\mathbf{e}_0 = \mathbf{y} - \mathbf{X}\mathbf{a}_0 \quad (\text{A-13})$$

then

$$\mathbf{X}^T \mathbf{e}_0 = \mathbf{X}^T (\mathbf{y} - \mathbf{X}\mathbf{a}_0) = \mathbf{0} \quad (\text{A-14})$$

i.e.,

$$\mathbf{X}^T \mathbf{X}\mathbf{a}_0 = \mathbf{X}^T \mathbf{y} \quad (\text{A-15})$$

Due to the fact that $(\mathbf{X}^T \mathbf{X})^{-1}$ exists, the coefficient vector is attained as

$$\mathbf{a}_0 = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (\text{A-16})$$

which is the solution of the least squares problem. In addition, it can be found that the smallest error vector is

$$\mathbf{e}_0 = \mathbf{y} - \mathbf{X}\mathbf{a}_0 = \mathbf{y} - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} = (\mathbf{I} - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T) \mathbf{y} \quad (\text{A-17})$$

and the minimal performance index is

$$\begin{aligned} E = \mathbf{e}_0^T \mathbf{e}_0 &= \mathbf{y}^T (\mathbf{I} - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T)^T (\mathbf{I} - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T) \mathbf{y} \\ &= \mathbf{y}^T (\mathbf{I} - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T) \mathbf{y} = \mathbf{y}^T \mathbf{e}_0 \end{aligned} \quad (\text{A-18})$$

i.e.,

$$(\mathbf{y} - \mathbf{e}_0)^T \mathbf{e}_0 = 0 \quad (\text{A-19})$$

Further from (A-6), we define

$$\boldsymbol{\alpha}_0 = \mathbf{X}\mathbf{a}_0 \quad (\text{A-20})$$

then (A-13) leads to

$$\mathbf{y} - \mathbf{e}_0 = \boldsymbol{\alpha}_0 \quad (\text{A-21})$$

Hence, from (A-19) and (A-21) we obtain

$$(\mathbf{y} - \mathbf{e}_0)^T \mathbf{e}_0 = \boldsymbol{\alpha}_0^T \mathbf{e}_0 = 0 \quad (\text{A-22})$$

which demonstrates the truth shown in Figure 3-1 that $\boldsymbol{\alpha}_0$ is perpendicular to $\mathbf{e}_0$.

B. An Example: Estimation of a Circular Boundary

Let's take an example concerning an object with circular boundary where all the points (x, y) form a circle given as

$$(x - c_x)^2 + (y - c_y)^2 = r^2 \quad (\text{A-23})$$

The center (c_x, c_y) and radius r are unknown and will be determined by measuring 12 boundary points with values listed as below:

i -th points	x_i	y_i
1	4.0	-0.95
2	3.0	1.26
3	2.0	1.81
4	1.0	2.01
5	0.0	-3.80
6	-1.0	-3.23
7	-2.0	-0.97
8	-1.0	1.21
9	0.0	1.83
10	1.0	-4.00
11	2.0	-3.78
12	3.0	-3.26

As expected, due to the measurement error, the boundary points (x_i, y_i) , $i=1,2,\dots,12$, are a little deviated from the circle, i.e., $(x_i - c_x)^2 + (y_i - c_y)^2 \neq r^2$. Let the error be defined as

$$\begin{aligned} e_i &= (x_i - c_x)^2 + (y_i - c_y)^2 - r^2 \\ &= (x_i^2 + y_i^2) - 2c_x x_i - 2c_y y_i - r^2 + (c_x^2 + c_y^2) \\ &= (x_i^2 + y_i^2) - 2c_x x_i - 2c_y y_i - c_r \end{aligned} \quad (\text{A-24})$$

where

$$c_r = r^2 - (c_x^2 + c_y^2) \quad (\text{A-25})$$

It is required to determine c_x , c_y and c_r in (A-24). Once c_x , c_y and c_r are solved, the circle is then obtained with the center (c_x, c_y) and the radius

$$r = \sqrt{c_r + c_x^2 + c_y^2} \quad (\text{A-26})$$

Follow the process of least squares method, first let's rewrite (A-24) as

$$e_i = (x_i^2 + y_i^2) - [2x_i \quad 2y_i \quad 1] \cdot \begin{bmatrix} c_x \\ c_y \\ c_r \end{bmatrix} \quad (\text{A-27})$$

for $i=1,2,\dots,12$. Hence,

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{12} \end{bmatrix} = \underbrace{\begin{bmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ \vdots \\ x_{12}^2 + y_{12}^2 \end{bmatrix}}_{\mathbf{z}} - \underbrace{\begin{bmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ \vdots & \vdots & \vdots \\ 2x_{12} & 2y_{12} & 1 \end{bmatrix}}_{\mathbf{X}} \cdot \underbrace{\begin{bmatrix} c_x \\ c_y \\ c_r \end{bmatrix}}_{\mathbf{a}} = \mathbf{z} - \mathbf{Xa} \quad (\text{A-28})$$

which is the same as (A-5). From (A-16), the coefficient vector is solved as

$$\mathbf{a}_0 = \begin{bmatrix} c_x \\ c_y \\ c_r \end{bmatrix} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{z} \quad (\text{A-29})$$

The calculation in MATLAB is given as below:

```
>> % Calculation in MATLAB
>> %Input (xi, yi)
>> x=[4 3 2 1 0 -1 -2 -1 0 1 2 3];
>> y=[-0.95 1.26 1.81 2.01 -3.80 -3.23 -0.97 1.21 1.83 -4.00 -3.78 -3.26];
>> %Calculate X and z
>> for i=1:12;
    X(i,:)=2*x(i) 2*y(i) 1;
    z(i,:)=x(i)^2+y(i)^2;
end
>> X
```

X =

8.0000	-1.9000	1.0000
6.0000	2.5200	1.0000
4.0000	3.6200	1.0000
2.0000	4.0200	1.0000
0	-7.6000	1.0000
-2.0000	-6.4600	1.0000
-4.0000	-1.9400	1.0000
-2.0000	2.4200	1.0000
0	3.6600	1.0000
2.0000	-8.0000	1.0000
4.0000	-7.5600	1.0000
6.0000	-6.5200	1.0000

```
>> z=z(:,1)
```

z =

16.9025
10.5876
7.2761
5.0401
14.4400
11.4329
4.9409
2.4641
3.3489
17.0000
18.2884
19.6276

```
>> % Calculate the rank of X  
>> rank(X)
```

```
ans=  
    3
```

```
>> a=inv(X'*X)*X'*z
```

```
a =
```

```
    1.0065  
   -0.9934  
    6.9675
```

```
>> % Calculate the radius r  
>> r=sqrt(a(3)+a(1)^2+a(2)^2)
```

```
r =  
    2.9946
```

From the numeric results, we obtain the circle with the center (1.0065,-0.9934) and the radius $r=2.9946$.