2. Modeling of DC Motor

The most common device used as an actuator in mechanical control is the DC motor. For example, the control of a rotary inverted pendulum requires a DC motor to drive the arm and the pendulum as shown in Figure 2-1.

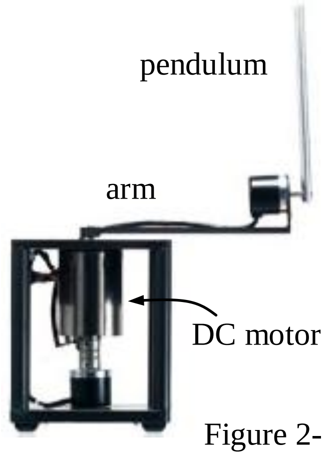

Figure 2-1

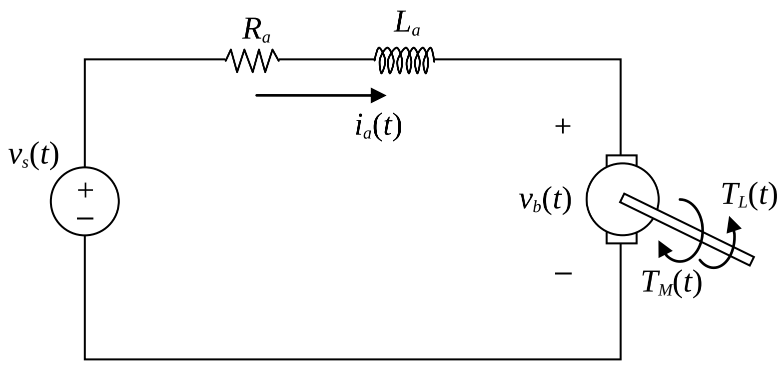

Figure2-2

The system structure of a DC motor is depicted in Figure2-2, including the armature resistance R_a and winding leakage inductance L_a. According to the Kirchhoff's voltage law, the electrical equation of the DC motor is described as

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) = v_s(t) \tag{2-1}$$

where $i_a(t)$ is the armature current, $v_b(t)$ is the back emf voltage and $v_s(t)$ is the voltage source. The back emf voltage $v_b(t)$ is proportional to the angular velocity $\omega(t)$ of the rotor in the motor, expressed as

$$v_b(t) = k_b \omega(t) \tag{2-2}$$

where k_b is the back emf constant. In addition, the motor generates a torque T_M proportional to the armature current, given as

$$T_M(t) = k_T i_a(t) \tag{2-3}$$

where k_T is the torque constant.

If the input voltage $v_s(t)=V_s$ is a constant, the resulted armature current $i_a(t)=I_a$, angular velocity $\omega(t)=\Omega$ and torque $T_M(t)=T$ are also constant in the steady state. From (2-1) to (2-3), we have

$$R_a I_a + k_b \Omega = V_s \tag{2-4}$$

$$T = k_T I_a \tag{2-5}$$

Under the conservation of power, we know that the input power I_aV_s is equal to the external power $T\Omega$ and the power $R_a I_a^2$ consumed in the resistance, i.e.,

$$V_s I_a = T\Omega + R_a I_a^2 \tag{2-6}$$

Substituting v_s in (2-4) into (2-6) yields

$$T = k_b I_a \tag{2-7}$$

From (2-5) and (2-7), we know that both k_T and k_b are the same. From (2-2), we can rewrite (2-1) and (2-3) as

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + k\omega(t) = v_s(t) \tag{2-8}$$

$$T_M(t) = k i_a(t) \tag{2-9}$$

where $k = k_T = k_b$. Besides, if the DC motor is used to drive an external torque T_L(t) of payload then its mechanical behavior is described as

$$J_M \frac{d\omega(t)}{dt}(t) + B_M \omega(t) = T_M(t) - T_L(t) \tag{2-10}$$

where J_M is the rotor moment of inertia and B_M is the frictional coefficient.

Based on (2-8), (2-9) and (2-10), the dynamic equation of the DC motor can be expressed as

$$L_a \frac{di_a(t)}{dt} + R_a i_a(t) + k\omega(t) = v_s(t) \tag{2-11}$$

$$J_M \frac{d\omega(t)}{dt}(t) + B_M\omega(t) - ki_a(t) = -T_L(t) \qquad (2\text{-}12)$$

Note that the electrical time constant L_a/R_a is often neglected since it is at least one order in magnitude smaller than the mechanical time constant J_M/B_M. In other words, by neglecting the term $\frac{di_a(t)}{dt}$, (2-11) becomes

$$i_a(t) = \frac{1}{R_a}v_s(t) - \frac{k}{R_a}\omega(t) \qquad (2\text{-}13)$$

Substituting it into (2-12), we have

$$\frac{d\omega(t)}{dt}(t) + \left(\frac{B_M}{J_M} + \frac{k^2}{J_M R_a}\right)\omega(t) = -\frac{1}{J_M}T_L(t) + \frac{k}{J_M R_a}v_s(t) \qquad (2\text{-}14)$$

Clearly, the motor will encounter two external sources, the input voltage $v_s(t)$ to drive the motor and the torque $T_L(t)$ reacted from the payload.

Now, based on the above analysis, let's discuss the model of a DC motor in state-space description and input-output description.

First, let's consider the case which requires the DC motor to move in a constant speed. Then, the angular velocity is selected as the output, expressed as

$$y(t)=\omega(t) \qquad (2\text{-}15)$$

From (2-11) and (2-12) and choosing the state variables as $x_1(t)=i_a(t)$ and $x_2(t)=\omega(t)$, we have

$$R_a x_1(t) + L_a\dot{x}_1(t) + kx_2(t) = v_s(t) \qquad (2\text{-}16)$$

$$J_M\dot{x}_2(t) + B_M x_2(t) - kx_1(t) = -T_L(t) \qquad (2\text{-}17)$$

Further rearranging (2-15) to (2-17) yields the state equations

$$\dot{x}_1(t) = -\frac{R_a}{L_a}x_1(t) - \frac{k}{L_a}x_2(t) + \frac{1}{L_a}v_s(t) \qquad (2\text{-}18)$$

$$\dot{x}_2(t) = \frac{k}{J_M}x_1(t) - \frac{B_M}{J_M}x_2(t) - \frac{1}{J_M}T_L(t) \qquad (2\text{-}19)$$

and the output equation

$$y(t)= x_2(t) \tag{2-20}$$

Hence, the state-space description is given as

$$\text{State equation: } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{2-21}$$

$$\text{Output equation: } y(t) = \boldsymbol{c}\boldsymbol{x}(t) \tag{2-22}$$

where the state vector is $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, the input vector is $\boldsymbol{u}(t) = \begin{bmatrix} v_s(t) \\ T_L(t) \end{bmatrix}$, and the system matrices are $\boldsymbol{A} = \begin{bmatrix} -\dfrac{R_a}{L_a} & -\dfrac{k}{L_a} \\ \dfrac{k}{J_M} & -\dfrac{B_M}{J_M} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} \dfrac{1}{L_a} & 0 \\ 0 & -\dfrac{1}{J_M} \end{bmatrix}$ and $\boldsymbol{c} = [0 \quad 1]$. Note that the state equation (2-21) can be rearranged as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}_1 u_1(t) + \boldsymbol{b}_2 u_2(t) \tag{2-23}$$

where $u_1(t)=v_s(t)$, $u_2(t)=T_L(t)$, $\boldsymbol{b}_1 = \begin{bmatrix} 1/L_a \\ 0 \end{bmatrix}$ and $\boldsymbol{b}_2 = \begin{bmatrix} 0 \\ -1/J_M \end{bmatrix}$. If the motor is operated without any payload $T_L(t)$, i.e., $u_2(t)=T_L(t)=0$, then the state equation (2-23) can be rewritten as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \tag{2-24}$$

where the input is $u(t)= v_s(t)$ and the input matrix is $\boldsymbol{b} = \begin{bmatrix} 1/L_a \\ 0 \end{bmatrix}$.

If the goal of control is to drive the DC motor to a desired angle, not a speed, then the output should be set as the angular position $y(t) = \theta(t) = \int_0^t \omega(\tau)d\tau$. To include the angulr possition, we often change the integral form $\theta(t) = \int_0^t \omega(\tau)d\tau$ into the differential form as below:

$$\dot{\theta}(t) = \omega(t) \tag{2-25}$$

and choose the new state variable $x_3(t)=\theta(t)$. As a result, the total system is changed into the state equations

$$\dot{x}_1(t) = -\frac{R_a}{L_a}x_1(t) - \frac{k}{L_a}x_2(t) + \frac{1}{L_a}v_s(t) \tag{2-26}$$

$$\dot{x}_2(t) = \frac{k}{J_M}x_1(t) - \frac{B_M}{J_M}x_2(t) - \frac{1}{J_M}T_L(t) \tag{2-27}$$

$$\dot{x}_3(t) = x_2(t) \tag{2-28}$$

and the output equation

$$y(t)= x_3(t) \tag{2-29}$$

In matrix form, we have

$$\text{State equation:} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{2-30}$$

$$\text{Output equation:} \quad y(t) = \boldsymbol{c}\boldsymbol{x}(t) \tag{2-31}$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \boldsymbol{u}(t) = \begin{bmatrix} v_s(t) \\ T_L(t) \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} -R_a/L_a & -k/L_a & 0 \\ k/J_M & -B_M/J_M & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1/L_a & 0 \\ 0 & -1/J_M \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Similarly, without any payload $T_L(t)$, the state equation (2-30) can be expressed as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \tag{2-32}$$

where the input is $u(t)= v_s(t)$ and the input matrix is $\boldsymbol{b} = \begin{bmatrix} \frac{1}{L_a} & 0 & 0 \end{bmatrix}^T$.

Now let’s consider the following numerical example of a DC motor with parameters R_a=0.5Ω, L_a=1.5×10^{-3} H, J_M=2.5×10^{-4} N-m/(rad/s^2), k_T=0.05 N-m/A, k_b=0.05 V/(rad/s) and B_M=1.0×10^{-4} N-m/(rad/s). Without any payload $T_L(t)$, the motor is described by (2-32) and listed as below:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} -333.33 & -33.33 & 0 \\ 200 & -0.40 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 666.67 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{b}} \cdot u(t) \tag{2-33}$$

where $x_1(t)=i_a(t)$, $x_2(t)=\omega(t)$, $x_3(t)=\theta(t)$ and $u(t)=v_s(t)$. If the output is $x_3(t)=\theta(t)$, then

$$y(t) = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{c}} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{2-34}$$

Based on (2-33) and (2-34), let's find the input-output description of the DC motor.

First, let's determine the characteristic polynomial of the DC motor from (2-33), which is obtained as

$$|s\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} s+333.33 & 33.33 & 0 \\ -200 & s+0.40 & 0 \\ 0 & -1 & s \end{vmatrix} = s^3 + 333.73s^2 + 6799.33s \tag{2-35}$$

According to the Cayley-Hamilton theory, we know that

$$\boldsymbol{A}^3 + 333.73\boldsymbol{A}^2 + 6799.33\boldsymbol{A} = \boldsymbol{0} \tag{2-36}$$

After calculating the following terms

$$\begin{aligned} &\dddot{y}(t) + 333.73\ddot{y}(t) + 6799.33\dot{y}(t) \\ &= \boldsymbol{c}\left(\boldsymbol{A}^3 + 333.73\boldsymbol{A}^2 + 6799.33\boldsymbol{A}\right)\boldsymbol{x} + \boldsymbol{c}\boldsymbol{A}^2\boldsymbol{b}u(t) \end{aligned} \tag{2-37}$$

we have the input-output description as below:

$$\dddot{y}(t) + 333.73\ddot{y}(t) + 6799.33\dot{y}(t) = 133334u(t) \tag{2-38}$$

where $\boldsymbol{c}\boldsymbol{A}^2\boldsymbol{b} = 133334$. It is obvious that the transfer function is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{133334}{s^3 + 333.73s^2 + 6799.33s} \tag{2-39}$$

Moreover, it can be further decomposed as

$$H(s) = \frac{133334}{s(s+21.80)(s+311.93)} \tag{2-40}$$

which implies the pole at $s=-311.93$ can be omitted since it is much faster than the pole at $s=-21.80$. In other words, the transfer function $H(s)$ can be approximated by a second order transfer function $H_a(s)$ expressed as

$$H_a(s) = \frac{\beta}{s(s+\alpha)} = \frac{\beta}{s^2 + \alpha s} \tag{2-41}$$

where α is near to 21.80. There are several methods to obtain the approximate transfer function $H_a(s)$. One of the simplest one is to determine $H_a(s)$ under the condition that

$$\frac{|H(j\omega)|}{|H_a(j\omega)|} = 1, \quad \text{for } 0 \le \omega < \infty \tag{2-42}$$

which leads to

$$\frac{|H(j\omega)|^2}{|H_a(j\omega)|^2} = \left.\frac{H(s)H(-s)}{H_a(s)H_a(-s)}\right|_{s=j\omega} = 1, \quad \text{for } 0 \le \omega < \infty \tag{2-43}$$

To adopt the condition, let's rewrite the transfer function as

$$H(s) = \frac{19.610}{s\left(1 + 0.0491s + 0.0001471s^2\right)} \tag{2-44}$$

and choose the approximate transfer function as

$$H_a(s) = \frac{19.610}{s(1+\rho s)} \tag{2-45}$$

Hecnce, we have

$$\begin{aligned}
&\frac{H(s)H(-s)}{H_a(s)H_a(-s)} \\
&= \frac{(1+\rho s)}{\left(1 + 0.0491s + 0.0001471s^2\right)} \frac{(1-\rho s)}{\left(1 - 0.0491s + 0.0001471s^2\right)} \\
&= \frac{1-\rho^2 s^2}{1 - 0.0021s^2 + 2.1638 \times 10^{-8} s^4} \\
&= 1 + \frac{-\left(\rho^2 - 0.0021\right)s^2 - 2.1638 \times 10^{-8} s^4}{1 - 0.0021s^2 + 2.1638 \times 10^{-8} s^4}
\end{aligned} \tag{2-46}$$

Clearly, between $\frac{H(s)H(-s)}{H_a(s)H_a(-s)}$ and 1, there exists an error

$$E(s) = \frac{-\left(\rho^2 - 0.0021\right)s^2 - 2.1638 \times 10^{-8} s^4}{1 - 0.0021s^2 + 2.1638 \times 10^{-8} s^4} \tag{2-47}$$

which is minimized when $\rho^2 - 0.0021 = 0$ or $\rho = 0.0458$. From (2-45), the approximate transfer function is then obtained as

$$H_a(s)=\frac{19.610}{s(1+0.0458s)}=\frac{428.17}{s(s+21.83)} \tag{2-48}$$

which implies α=21.83 near to 21.80 and β=428.17 as shown in (2-41).

The other one is just neglect the state variable $x_1(t)=i_a(t)$ since the convergence rate of $i_a(t)$ is much faster than that of angular velocity $x_2(t)=\omega(t)$. From (2-33), we assume that $\dot{x}_1(t)\approx 0$ and then

$$\begin{bmatrix} 0 \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}=\begin{bmatrix} -333.33 & -33.33 & 0 \\ 200 & -0.40 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}+\begin{bmatrix} 666.67 \\ 0 \\ 0 \end{bmatrix}\cdot u(t) \tag{2-49}$$

or

$$-333.33x_1(t)-33.33x_2(t)+666.67u(t)=0 \tag{2-50}$$

$$\dot{x}_2(t)=200x_1(t)-0.40x_2(t) \tag{2-51}$$

$$\dot{x}_3(t)=x_2(t) \tag{2-52}$$

From (2-50), we have

$$x_1(t)=-0.1x_2(t)+2u(t) \tag{2-53}$$

and substituting it into (2-51) becomes

$$\dot{x}_2(t)=-20.4x_2(t)+400u(t) \tag{2-54}$$

Hence, differentiating (2-52) yields

$$\ddot{x}_3(t)=\dot{x}_2(t)=-20.4x_2(t)+400u(t)=-20.4\dot{x}_3(t)+400u(t) \tag{2-55}$$

Since $y(t)=x_3(t)$, we obtain

$$\ddot{y}(t)+20.4\dot{y}(t)=400u(t) \tag{2-56}$$

and the approximate transfer function is

$$H_a(s)=\frac{Y(s)}{U(s)}=\frac{400}{s(s+20.4)} \tag{2-57}$$

which is approximate to the one derived in (2-48).