10. State-feedback Control of LTI Systems

From this section, we will focus on the state-feedback control applied to an LTI system in state-space description, which is expressed as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(t_0)=\boldsymbol{x}_0 \tag{10-1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \tag{10-2}$$

where $\boldsymbol{u}(t)=[u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^T \in \Re^m$ is the input, $\boldsymbol{y}(t)=[y_1(t)\ y_2(t)\ \cdots\ y_p(t)]^T \in \Re^p$ is the output and $\boldsymbol{x}(t)=[x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T \in \Re^n$ is the system state with initial condition $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$ at the initial time $t=t_0$.

First, let's consider the stability problem. It is well-known that if the system satisfies the following condition

$$rank\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n \tag{10-3}$$

then the system is controllable, i.e., with the use of state feedback

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}(t) \tag{10-4}$$

the system state $\boldsymbol{x}(t)$ will approach the origin $\boldsymbol{0}$ from any initial state $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$.

Next, let's show how to determine the matrix $\boldsymbol{K}$. Commonly, the so-called pole-placement method is adopted. With the use of (10-4), the state equation is changed into

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}(t) \tag{10-5}$$

which implies that the system can be stabilized when the eigenvalues of $\boldsymbol{A}-\boldsymbol{BK}$ are located in the left-half complex plane, i.e., each eigenvalue possesses a negative real part. To apply the pole-placement method, first we have to choose n desired eigenvalues, such as λ_i, $i=1,2,\ldots,n$. Then, determine $\boldsymbol{K}$ from the following characteristic polynomial

$$\left|\lambda\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\right| = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \tag{10-6}$$

Note that the matrix $\boldsymbol{K}$ is not unique unless m=1. In other words, $\boldsymbol{K}$ is unique only for single input systems, not for multiple input systems. By adding other limitation on $\boldsymbol{K}$, the pole-placement method can be used to uniquely determine the matrix $\boldsymbol{K}$. Now, let's consider a system with four state variables and two inputs, described as below:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} \tag{10-7}$$

with initial condition $\boldsymbol{x}(0) = \begin{bmatrix} 5 & 4 & -2 & 1 \end{bmatrix}^T$. First, let's check the system stability without any input, i.e., check the eigenvalues of $\boldsymbol{A}$. By the use of MATLAB, we have

```
==========================================

>>% Create A and check eigenvalues of A
>> A=[1 0 -1 0; 0 -1 0 1; -1 0 1 1; 0 1 -1 0];
>> eig(A)

ans =

    1.6903
    0.4074 + 0.4766i
    0.4074 - 0.4766i
   -1.5051
==========================================
```

Clearly, the eigenvalues are not all located in the left-half complex plane, which means the system is unstable without input. To stabilize the system, before the use of state feedback control (10-4), we have to check whether the system is controllable or not by the condition (10-3). From MATLAB, we have

```
==========================================

>>% Create B and check rank[B AB A^2B A^3B]
>> B=[0 0; 0 0; 1 0; 0 1];
>> rank([B A*B A^2*B A^3*B])

ans =

    4

==========================================
```

Evidently, the system satisfies the condition (10-3) and thus, it is controllable. Then, based on the pole-placement method, we assign four eigenvalues −1, −2±j2, −4 for the matrix ***A***−***BK*** whose character polynomial is

$$\left|\lambda \boldsymbol{I}-(\boldsymbol{A}-\boldsymbol{BK})\right|=(\lambda+1)(\lambda+2+j2)(\lambda+2-j2)(\lambda+4) \tag{10-8}$$

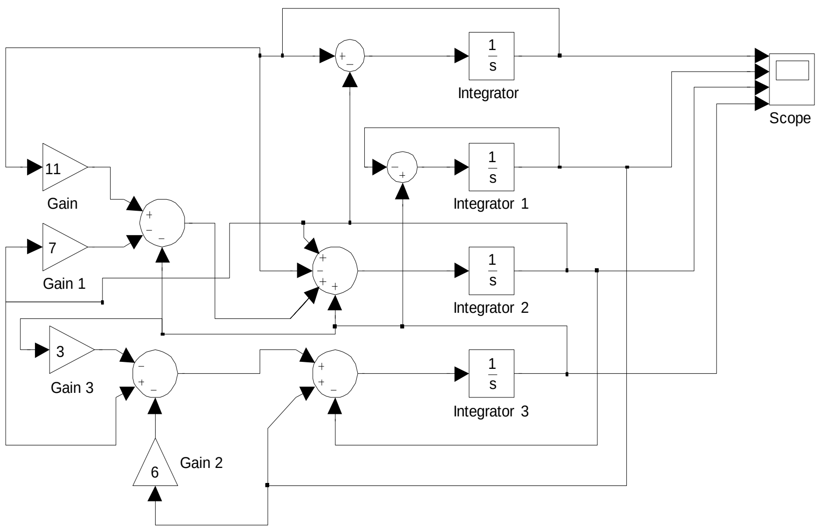
and solve ***K*** by MATLAB as below:

```
==========================================
>>% Solve K by pole-placement method
>> p=[-1 -2+2j -2-2j -4];
>> K=place(A, B, p)

K =

   -11.0000         0    7.0000    1.0000
          0    6.0000   -1.0000    3.0000
==========================================
```

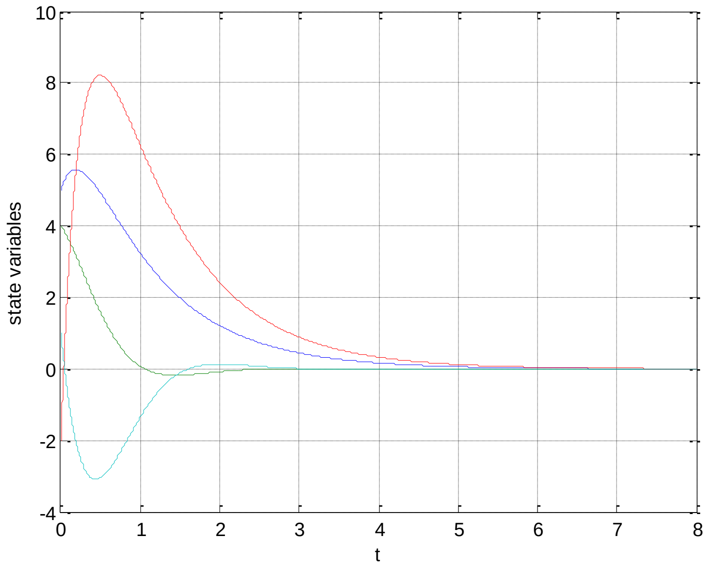
The control inputs are expressed as below:

$$\begin{aligned} u_1(t) &= 11x_1(t)-7x_3(t)-x_4(t) \\ u_2(t) &= -6x_2(t)+x_3(t)-3x_4(t) \end{aligned} \tag{10-9}$$

By the use of SIMULINK, we have the block diagram as below:



The numerical result of the four state variables can be obtaine from the Scope block and shown as below:



Clearly, the system is stabilized since all the state variables are driven to approach the origin under state-feedback control.

Next, let's consider the so-called regulation problem: Drive the output $\boldsymbol{y}(t)$ to the fixed destination $\boldsymbol{y}_d$, i.e., $\boldsymbol{y}(t)=\boldsymbol{y}_d$. To deal with such problem, define the error vector as

$$\boldsymbol{e}(t)=\boldsymbol{y}(t)-\boldsymbol{y}_d=\boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{y}_d \tag{10-10}$$

and genetate a new state by the use of integrator as below:

$$\boldsymbol{z}(t)=\int_0^t \boldsymbol{e}(\tau)d\tau=\int_0^t(\boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{y}_d)d\tau \tag{10-11}$$

which implies

$$\dot{\boldsymbol{z}}(t)=\boldsymbol{e}(t)=\boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{y}_d\,, \qquad \boldsymbol{z}(t_0)=\boldsymbol{0} \tag{10-12}$$

Combined with (1), it leads to the following augmented system

$$\underbrace{\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{z}}(t)\end{bmatrix}}_{\dot{\hat{\boldsymbol{x}}}(t)}=\underbrace{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{C} & \boldsymbol{0}\end{bmatrix}}_{\hat{\boldsymbol{A}}}\cdot\underbrace{\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{z}(t)\end{bmatrix}}_{\hat{\boldsymbol{x}}(t)}+\underbrace{\begin{bmatrix}\boldsymbol{B}\\ \boldsymbol{0}\end{bmatrix}}_{\hat{\boldsymbol{B}}}u(t)+\begin{bmatrix}0\\ \boldsymbol{y}_d\end{bmatrix},\qquad \underbrace{\begin{bmatrix}\boldsymbol{x}(t_0)\\ \boldsymbol{z}(t_0)\end{bmatrix}}_{\hat{\boldsymbol{x}}(t_0)}=\begin{bmatrix}\boldsymbol{x}_0\\ \boldsymbol{0}\end{bmatrix} \tag{10-13}$$

It has been proved that if $m \geq p$ and (10-3) is satisfied then

$$rank\begin{bmatrix}\hat{\boldsymbol{B}} & \hat{\boldsymbol{A}}\hat{\boldsymbol{B}} & \hat{\boldsymbol{A}}^2\hat{\boldsymbol{B}} & \cdots & \hat{\boldsymbol{A}}^{n+p-1}\hat{\boldsymbol{B}}\end{bmatrix}=n+p \tag{10-14}$$

i.e., the augmented system (10-13) is also controllable. Hence, with the use of state feedback

$$\boldsymbol{u} = -\hat{\boldsymbol{K}}\hat{\boldsymbol{x}}(t) \tag{10-15}$$

the system state $\hat{\boldsymbol{x}}(t)$ will approach the origin $\boldsymbol{0}$ from the initial state $\hat{\boldsymbol{x}}(t_0) = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix}$.

Based on the pole- placement method, the augmented state equation is changed into

$$\dot{\hat{\boldsymbol{x}}}(t) = \left(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\hat{\boldsymbol{x}}(t) \tag{10-16}$$

where the eigenvalues of $\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}$ are assigned as $\hat{\lambda}_i$, i=1,2,…,n+p, all located in the left-half complex plane. Then, determine $\hat{\boldsymbol{K}}$ by the pole-placement method from the following characteristic polynomial

$$\left|\lambda \boldsymbol{I} - \left(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\right| = \left(\lambda - \hat{\lambda}_1\right)\left(\lambda - \hat{\lambda}_2\right)\cdots\left(\lambda - \hat{\lambda}_{n+p}\right) \tag{10-17}$$

It will be demonstrated by the following example:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} \tag{10-18}$$

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\boldsymbol{y}(t)} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{C}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} \tag{10-19}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 = [5 \quad 4 \quad -2 \quad 1]^T$. Let the destination be $\boldsymbol{y}_d = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then we can choose the error vector as

$$\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{y}_d = \begin{bmatrix} y_1(t) - 1 \\ y_2(t) + 1 \end{bmatrix} = \begin{bmatrix} x_1(t) - x_2(t) + x_4(t) - 1 \\ -x_1(t) + x_4(t) + 1 \end{bmatrix} \tag{10-20}$$

Further genetate a new state as (10-11) to construct the augmented system as below:

$$\underbrace{\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{z}}(t) \end{bmatrix}}_{\dot{\hat{\boldsymbol{x}}}(t)} = \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}}_{\hat{\boldsymbol{A}}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{z}(t) \end{bmatrix}}_{\hat{\boldsymbol{x}}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}}_{\hat{\boldsymbol{B}}} \boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{y}_d \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{z}(0) \end{bmatrix}}_{\hat{\boldsymbol{x}}(0)} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix} \quad (10\text{-}21)$$

First, let's check the condition (10-14) to make sure whether the system is controllable or not. From MATLAB, we have

```
============================================
>>% Create Ah and Bh; Check rank[Bh AhBh … Ah^5Bh ]
>> Ah=[1 0 -1 0 0 0; 0 -1 0 1 0 0; -1 0 1 1 0 0; 0 1 -1 0 0 0;
        1 -1 0 1 0 0; -1 0 0 1 0 0];
>> Bh=[0 0; 0 0; 1 0; 0 1; 0 0; 0 0];
>> rank([Bh Ah*Bh Ah^2*Bh Ah^3*Bh Ah^4*Bh Ah^5*Bh])

ans =

    6
============================================
```

As expected, the augmented system is also controllable. Then, based on the pole-placement method, we assign six eigenvalues −1, −2±j2, −3±j3, −4 for the matrix $\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}$ whose character polynomial is

$$\left|\lambda \boldsymbol{I} - \left(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\right| = (\lambda+1)(\lambda+2+j2)(\lambda+2-j2)(\lambda+3+j3)(\lambda+3-j3)(\lambda+4) \quad (10\text{-}22)$$

and solve $\hat{\boldsymbol{K}}$ by MATLAB as below:

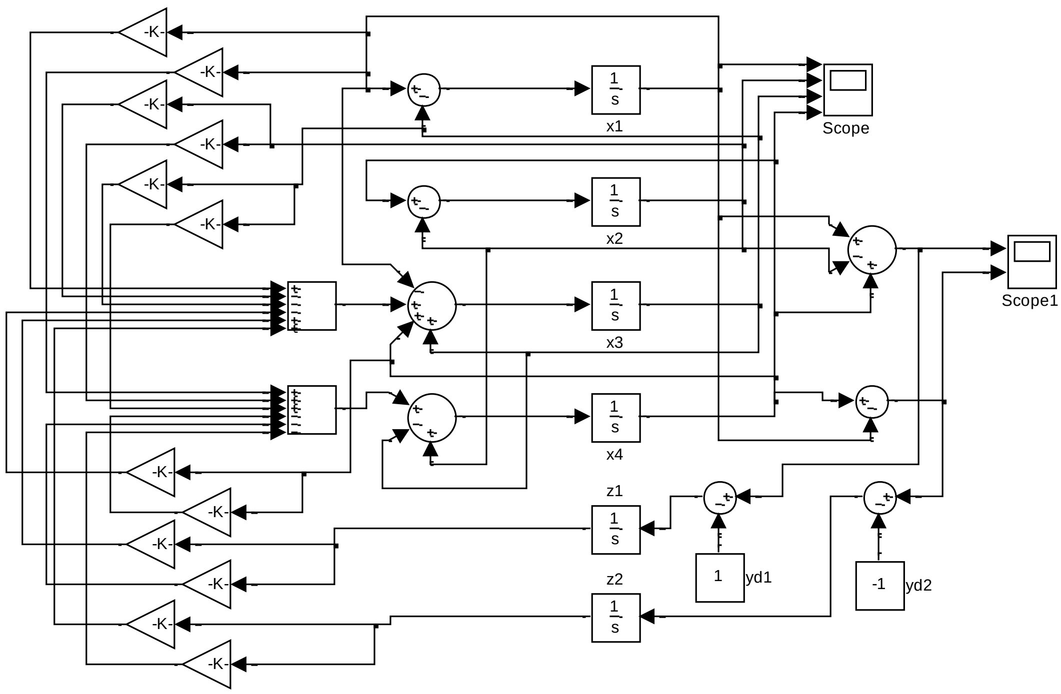
```
============================================
>>% Solve K by pole-placement method
>> p=[-1 -2+2j -2-2j -3+3j -3-3j -4];
>> Kh=place(Ah, Bh, p)

Kh =

     -35.5688    36.7645    10.0393     0.7304   -34.1787    -1.6543
      -1.3863   -24.6179    -1.1292     5.9607    24.1804    18.0229
============================================
```

The control inputs are expressed as below:

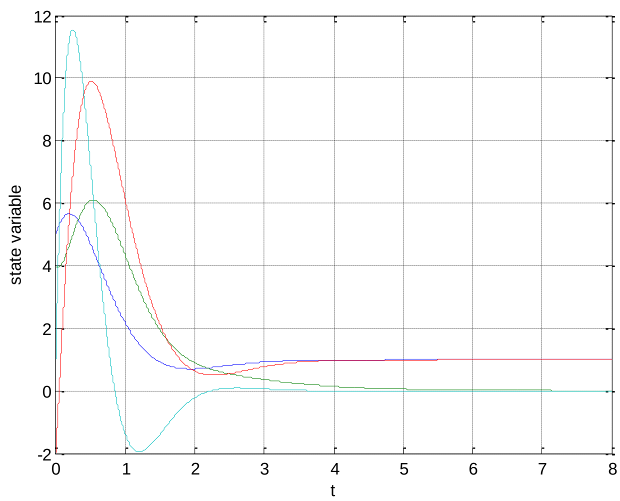
$$u_1(t)=35.57x_1(t)-36.76x_2(t)-10.04x_3(t)-0.73x_4(t)+34.18z_1(t)+1.65z_2(t) \tag{10-23}$$

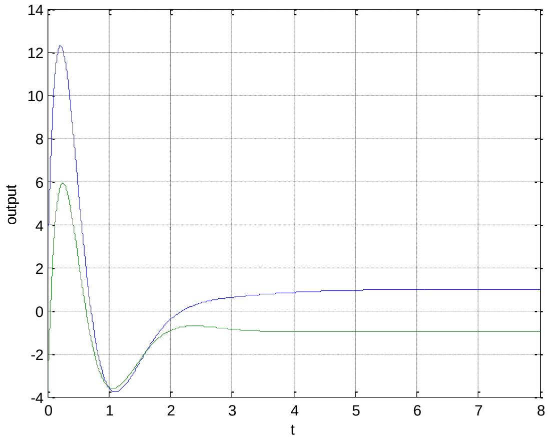
$$u_2(t)=1.39x_1(t)+24.62x_2(t)+1.13x_3(t)-5.96x_4(t)-24.18z_1(t)-18.02z_2(t) \tag{10-24}$$

By the use of SIMULINK, we have the block diagram as below:



The numerical result of the four state variables can be obtaine from the Scope block and shown as below:





Clearly, the system outputs $y_1(t)$ and $y_2(t)$ are successfully regulated to y_{1d}=1 and y_{2d}=−1.