

1. Introduction

In control engineering, a controller is often designed and constructed based on four basic steps as shown in Figure 1-1, including mathematical modeling, controller design, numerical simulation and system implementation. In this course, we will learn how to practically design and construct controllers for a solar charger under maximum power point tracking and a rotary inverted pendulum balanced around the upright position. Before get into the first step, the mathematical modeling, in this chapter let's review some fundamental concepts of dynamic systems.

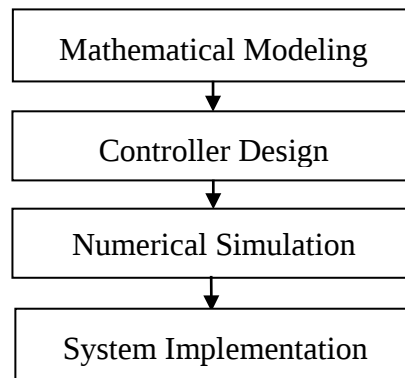


Figure 1-1

A practical system generally varies with time t and is often represented by a block diagram as shown in Figure 1-2, where the system H receives external inputs and generates outputs correspondingly. Assume that there are m inputs $u_i(t)$, $i=1,2,\dots,m$, and p outputs $y_j(t)$, $j=1,2,\dots,p$ and let them be denoted as the vector forms, i.e., $\mathbf{u}(t)=[u_1(t) \ u_2(t) \ \dots \ u_m(t)]^T \in \Re^m$ and $\mathbf{y}(t)=[y_1(t) \ y_2(t) \ \dots \ y_p(t)]^T \in \Re^p$.

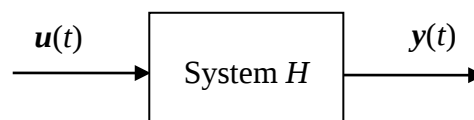


Figure1-2

According to the relationship between $\mathbf{u}(t)$ and $\mathbf{y}(t)$, systems can be classified into two categories: static and dynamic.

A system is said to be static if each output $y_j(t)$ only depends on the current inputs, mathematically described as

$$y_j(t) = h_j(\mathbf{u}(t)) = h_j(u_1(t), u_2(t), \dots, u_m(t)), \quad j=1, 2, \dots, p \quad (1-1)$$

where $h_j(\mathbf{u}(t))$ is an algebraic function of the current inputs $u_i(t)$, $i=1, 2, \dots, m$. An example of static systems is given in Figure 1-3, which is a resistive circuit with a voltage input $u(t)$.

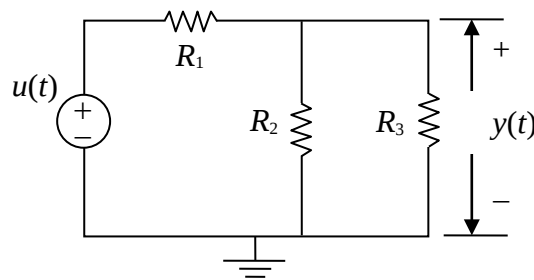


Figure 1-3

Let the output be the voltage across the resistance R_3 , then from the circuit theory we have

$$y(t) = h(u(t)) = \frac{R_2 R_3}{R_1(R_2 + R_3) + R_2 R_3} u(t) = k \cdot u(t) \quad (1-2)$$

where $h(u(t))$ is a linear function proportional to the current input $u(t)$ and the constant gain is $k = \frac{R_2 R_3}{R_1(R_2 + R_3) + R_2 R_3}$. Clearly, the output $y(t)$ can be simply determined by the current input $u(t)$, which means the resistive circuit is a static system. In fact, a static system is also referred to as a “*memoryless*” system since its outputs do not contain any information related to the past inputs.

Unlike static systems, a system is said to be dynamic if its outputs depend not only on the current inputs but also their past. In order to express the behavior of a

dynamic system, two kinds of description are often employed, called the state-space description and the input-output description. Next, let's adopt the linear time-invariant systems (LTI systems in brief) to discuss the description of dynamic systems. In system engineering, the LTI systems are known as the most useful ideal systems since almost every practical system can be approximated by an LTI system around the equivalent point in system state space.

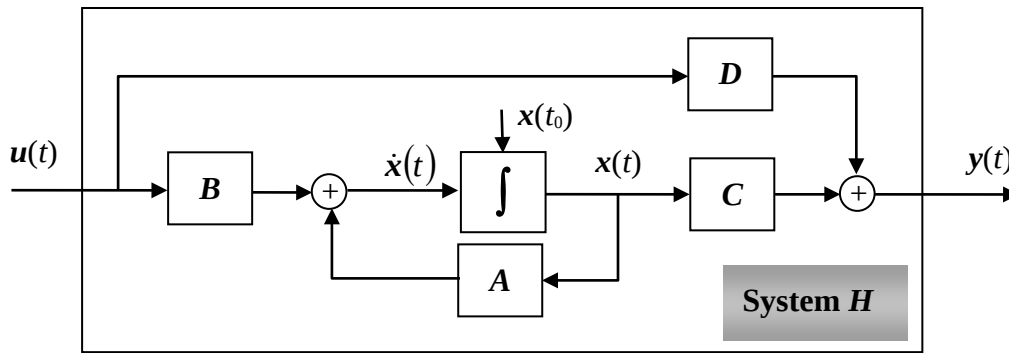


Figure 1-4

An LTI system can be modeled in time domain by the state-space description expressed as below:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1-3)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (1-4)$$

including the state equation (1-3) and the output equation (1-4) in vector form. Assume the system state $\mathbf{x}(t) = [x_1(t) \ x_2(t) \ \dots \ x_n(t)]^T \in \mathcal{R}^n$ is composed of n state variables $x_i(t)$, $i=1,2,\dots,n$ and all the system matrices $\mathbf{A} \in \mathcal{R}^{n \times n}$, $\mathbf{B} \in \mathcal{R}^{n \times m}$, $\mathbf{C} \in \mathcal{R}^{p \times n}$ and $\mathbf{D} \in \mathcal{R}^{p \times m}$ are constant. Besides, the system state at the initial time $t=t_0$ is represented by $\mathbf{x}(t_0) = \mathbf{x}_0$, called the initial state. Figure 1-4 shows the block diagram of an LTI system.

From the state equation (1-3), the system state has been solved and partitioned

into two vectors, $\mathbf{x}_h(t)$ for zero input and $\mathbf{x}_p(t)$ for zero initial state, expressed as

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_h(t) + \mathbf{x}_p(t) \\ &= e^{A(t-t_0)}\mathbf{x}_0 + \int_{t_0}^t e^{A(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\end{aligned}\quad (1-5)$$

where

$$\mathbf{x}_h(t) = e^{A(t-t_0)}\mathbf{x}_0 \quad (1-6)$$

$$\mathbf{x}_p(t) = \int_{t_0}^t e^{A(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (1-7)$$

For zero input $\mathbf{u}(t)=0$, the state equation (1-3) is rewritten as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1-8)$$

which is called the homogeneous state equation and $\mathbf{x}(t) = \mathbf{x}_h(t)$ is its solution called the zero-input state vector. As for $\mathbf{x}_p(t)$, it is called the zero-state state vector under the condition $\mathbf{x}(t_0) = \mathbf{x}_0 = 0$.

To verify that the system state (1-5) is really the solution of (1-3), it is required to employ three basic properties. The first one concerns the definition of e^{At} . Consider a square matrix $\mathbf{M}$ and define its exponential function as

$$e^{\mathbf{M}} = \mathbf{I} + \mathbf{M} + \frac{1}{2!}\mathbf{M}^2 + \frac{1}{3!}\mathbf{M}^3 + \dots \quad (1-9)$$

which is directly extended from the scalar exponential function

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots \quad (1-10)$$

If $\mathbf{M} = \mathbf{A}t$, then from (1-9) we have

$$e^{At} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{A}t)^k \quad (1-11)$$

For the second property, it refers to the derivative of e^{At} , which can be obtained as

$$\frac{d}{dt}e^{At} = \mathbf{A} + \mathbf{A}(\mathbf{A}t) + \mathbf{A}\frac{1}{2!}(\mathbf{A}t)^2 + \dots = \mathbf{A} \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{A}t)^k = \mathbf{A}e^{At} \quad (1-12)$$

In fact, it is also true that

$$\frac{d}{dt}e^{At} = \mathbf{A} + (\mathbf{A}t)\mathbf{A} + \frac{1}{2!}(\mathbf{A}t)^2\mathbf{A} + \cdots = \left(\sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{A}t)^k \right) \mathbf{A} = e^{At} \mathbf{A} \quad (1-13)$$

Finally, the third property is given as

$$\frac{d}{dt} \int_{t_0}^t f(\tau, t) d\tau = f(t, t) + \int_{t_0}^t \frac{\partial}{\partial t} f(\tau, t) d\tau \quad (1-14)$$

which has been learned in Calculus.

Based on the above three properties, taking derivative of the system state (1-5) with respect to time yields

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \frac{d}{dt} e^{A(t-t_0)} \mathbf{x}_0 + \frac{d}{dt} \int_{t_0}^t e^{A(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau \\ &= \mathbf{A}e^{A(t-t_0)} \mathbf{x}_0 + e^{A(t-t)} \mathbf{B}\mathbf{u}(t) + \int_{t_0}^t \frac{\partial}{\partial t} e^{A(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau \\ &= \mathbf{A}e^{A(t-t_0)} \mathbf{x}_0 + \mathbf{B}\mathbf{u}(t) + \int_{t_0}^t \mathbf{A}e^{A(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau \\ &= \mathbf{A} \left(e^{A(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t e^{A(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau \right) + \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \end{aligned} \quad (1-15)$$

which proves the system state (1-5) is indeed the solution of state equation (1-3).

From the system state (1-5) and the output equation (1-4), the system output can be expressed as

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \\ &= \mathbf{C}e^{A(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t \mathbf{C}e^{A(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau + \mathbf{D}\mathbf{u}(t) \end{aligned} \quad (1-16)$$

Clearly, $\mathbf{y}(t)$ is related to the current and past input vector $\mathbf{u}(\tau)$ for $t_0 \leq \tau \leq t$, which shows that the LTI system is really a dynamic system. In addition, the output $\mathbf{y}(t)$ only depends on the initial state $\mathbf{x}_0$ and the input vector $\mathbf{u}(\tau)$ for $t_0 \leq \tau \leq t$; in other words, no information of $\mathbf{x}(t)$ for $t > 0$ is required to represent the output $\mathbf{y}(t)$.

In addition to the state-space description, an LTI system can be also described by the input-out description, i.e., without the system state $\mathbf{x}(t)$.

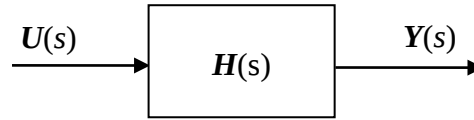


Figure1-5

By taking the Laplace transform of (1-3) and (1-4), the system can be changed into the following form:

$$s\mathbf{X}(s) - \mathbf{x}(t_0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \quad (1-7)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s) \quad (1-8)$$

where $\mathbf{X}(s)$, $U(s)$ and $\mathbf{Y}(s)$ are the Laplace transforms of $\mathbf{x}(t)$, $u(t)$ and $\mathbf{y}(t)$, respectively.

Without loss of generality, assume $s\mathbf{I} - \mathbf{A}$ is invertible and thus (1-7) is rearranged as

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(t_0) \quad (1-9)$$

and (1-8) becomes

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}] U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(t_0) \quad (1-10)$$

Further neglecting the initial condition yields

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) \quad (1-11)$$

which is the input-output description of the LTI system in frequency domain and

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D} \quad (1-12)$$

is called the transfer function. The block diagram of (1-11) is depicted in Figure 1-5 correspondingly.

In order to show the input-output description in time domain of an LTI system, we take a single-input single-output (SISO) system as an example, which is given as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1-13)$$

$$y(t) = \mathbf{c}\mathbf{x}(t) + du(t) \quad (1-14)$$

From (1-10), this SISO system is expressed as

$$Y(s) = \left[c(sI - A)^{-1}b + d \right] U(s) + c(sI - A)^{-1}x_0 \quad (1-15)$$

where

$$H(s) = c(sI - A)^{-1}b + d = \frac{c \cdot adj(sI - A)b + d|sI - A|}{|sI - A|} \quad (1-16)$$

Let the characteristic polynomial be

$$|sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \quad (1-17)$$

and the numerator of $H(s)$ be

$$c \cdot adj(sI - A)b + d|sI - A| = b_ms^m + b_{n-1}s^{n-1} + \cdots + b_1s + b_0 \quad (1-18)$$

where $m \leq n$, then (1-15) becomes

$$\begin{aligned} Y(s) &= H(s)U(s) + c(sI - A)^{-1}x_0 \\ &= \frac{b_ms^m + b_{n-1}s^{n-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} U(s) \\ &\quad + \frac{c \cdot adj(sI - A)x_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} \end{aligned} \quad (1-19)$$

Further, neglecting the initial condition x_0 leads to

$$\begin{aligned} &(s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0) Y(s) \\ &= (b_ms^m + b_{n-1}s^{n-1} + \cdots + b_1s + b_0) U(s) \end{aligned} \quad (1-20)$$

Taking the inverse Laplace transform yields

$$\begin{aligned} &y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + y(t) \\ &= b_mu^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots + b_1\dot{u}(t) + b_0u(t) \end{aligned} \quad (1-21)$$

where $y^{(k)}(t) \equiv \frac{d^k}{dt^k} y(t)$. This is the input-output description of the LTI system in time domain.

Actually, (1-21) can be also derived from the Cayley-Hamilton theory which states that if the characteristic polynomial of a square matrix $A \in \Re^{n \times n}$ is given as

$$|sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \quad (1-22)$$

then the following equation must be true:

$$\sum_{k=0}^n a_k \mathbf{A}^k = \mathbf{A}^n + a_{n-1} \mathbf{A}^{n-1} + \cdots + a_1 \mathbf{A} + a_0 \mathbf{I} = \mathbf{0} \quad (1-23)$$

where $a_n=1$. Now, from (1-13) and (1-14) we have

$$\begin{cases} y(t) = \mathbf{c}\mathbf{x}(t) + du(t) \\ \dot{y}(t) = \mathbf{c}\dot{\mathbf{x}}(t) + d\dot{u}(t) = \mathbf{c}\mathbf{A}\mathbf{x}(t) + \mathbf{c}\mathbf{b}u(t) + d\dot{u}(t) \\ \ddot{y}(t) = \mathbf{c}\mathbf{A}\dot{\mathbf{x}}(t) + \mathbf{c}\mathbf{b}\dot{u}(t) + d\ddot{u}(t) = \mathbf{c}\mathbf{A}^2\mathbf{x}(t) + \mathbf{c}\mathbf{A}\mathbf{b}u(t) + \mathbf{c}\mathbf{b}\dot{u}(t) + d\ddot{u}(t) \\ \vdots \end{cases} \quad (1-24)$$

or

$$\begin{aligned} y^{(k)}(t) = & \mathbf{c}\mathbf{A}^k \mathbf{x}(t) + \mathbf{c}\mathbf{A}^{k-1} \mathbf{b}u(t) + \mathbf{c}\mathbf{A}^{k-2} \mathbf{b}\dot{u}(t) + \cdots \\ & + \mathbf{c}\mathbf{A}\mathbf{b}u^{(k-2)}(t) + \mathbf{c}\mathbf{b}u^{(k-1)}(t) + du^{(k)}(t) \end{aligned} \quad (1-25)$$

Calculating $\sum_{k=0}^n a_k y^{(k)}(t)$ yields

$$\begin{aligned} \sum_{k=0}^n a_k y^{(k)}(t) = & \mathbf{c} \left(\sum_{k=0}^n a_k \mathbf{A}^k \right) \mathbf{x}(t) + \left(\sum_{k=1}^n a_k \mathbf{c}\mathbf{A}^{k-1} \mathbf{b} + a_0 d \right) u(t) \\ & + \left(\sum_{k=2}^n a_k \mathbf{c}\mathbf{A}^{k-2} \mathbf{b} + a_1 d \right) \dot{u}(t) + \cdots + (a_n \mathbf{c}\mathbf{A}\mathbf{b} + a_{n-1} \mathbf{c}\mathbf{b} + a_{n-2} d) u^{(n-2)}(t) \\ & + (a_n \mathbf{c}\mathbf{b} + a_{n-1} d) u^{(n-1)}(t) + a_n du^{(n)}(t) \end{aligned} \quad (1-26)$$

From (1-23), we have

$$\begin{aligned} \sum_{k=0}^n a_k y^{(k)}(t) = & \left(\sum_{k=1}^n a_k \mathbf{c}\mathbf{A}^{k-1} \mathbf{b} + a_0 d \right) u(t) + \left(\sum_{k=2}^n a_k \mathbf{c}\mathbf{A}^{k-2} \mathbf{b} + a_1 d \right) \dot{u}(t) + \cdots \\ & + (a_n \mathbf{c}\mathbf{A}\mathbf{b} + a_{n-1} \mathbf{c}\mathbf{b} + a_{n-2} d) u^{(n-2)}(t) \\ & + (a_n \mathbf{c}\mathbf{b} + a_{n-1} d) u^{(n-1)}(t) + a_n du^{(n)}(t) \end{aligned} \quad (1-27)$$

or

$$\begin{aligned} & y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + y(t) \\ & = b_n u^{(n)}(t) + b_{n-1} u^{(n-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t) \end{aligned} \quad (1-28)$$

with coefficients

$$\begin{aligned} b_n = & a_n d, \quad b_{n-1} = a_n \mathbf{c}\mathbf{b} + a_{n-1} d, \quad b_{n-2} = a_n \mathbf{c}\mathbf{A}\mathbf{b} + a_{n-1} \mathbf{c}\mathbf{b} + a_{n-2} d, \\ & \cdots, \quad b_1 = \sum_{k=2}^n a_k \mathbf{c}\mathbf{A}^{k-2} \mathbf{b} + a_1 d, \quad b_0 = \sum_{k=1}^n a_k \mathbf{c}\mathbf{A}^{k-1} \mathbf{b} + a_0 d \end{aligned} \quad (1-29)$$

To sum up, the input-output description of a SISO LTI system is often described as the following form

$$\begin{aligned} & y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + y(t) \\ & = b_mu^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots + b_1\dot{u}(t) + b_0u(t) \end{aligned} \quad (1-30)$$

where $n \geq m$. In general, a system satisfying $n \geq m$ is called a proper system, otherwise it is an improper system. Viewing from (1-29), if $d=0$ then $b_n=0$ which leads to $n > m$. In control engineering, we often deal with the SISO LTI system described in (1-30), which satisfies $n > m$. A system represented by (1-30) is called an n -th order system with relative degree $p=n-m$. From (1-29), it can be concluded that if a system with relative degree $p=n-m$ must satisfy

$$d = 0, \quad \mathbf{c}\mathbf{b} = 0, \quad \mathbf{c}\mathbf{A}\mathbf{b} = 0, \quad \cdots, \quad \mathbf{c}\mathbf{A}^{p-1}\mathbf{b} = 0, \quad \mathbf{c}\mathbf{A}^p\mathbf{b} \neq 0 \quad (1-31)$$

The relative degree $p=n-m$ is an important property in controller design based on output feedback technology.

What is output feedback technology? Also, what is state feedback? Besides, several important properties concerning systems are required to discuss in this course, such as the impulse response, poles, zeros, pole-zero cancellation, and so on. Let's leave them as the discussion points in the class.