Digital Communications

Chapter 11 Multichannel and Multicarrier Systems

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Motivation

In Chapter 9: *Communications Through Band-Limited Channels*, we have seen that

When channel is band-limited to $[-W, W]$, without extra care, the received signal at matched-filter output is

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

where

- I_k is the information symbol,
- x_k is the overall discrete impulse response,
- z_k is the additive noise

This gives an intersymbol interference (ISI) channel.

Motivation

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

With Nyquist pulse, it is possible to create an ISI-free channel

$$x_{k-n} = \delta_{k-n} \quad \text{and} \quad y_k = I_k + z_k.$$

However, due to

- mis-synchronization
- imperfect channel estimation, etc

ISI is sometimes inevitable.

Motivation

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

There is another simple solution to the ISI problem. The main idea is the following:

- Given the discrete channel impulse response x_k, we see

$$y_k = I_k \star x_k + z_k$$

- By Fourier duality, taking discrete Fourier transform (DFT) at both sides gives

$$Y_k = \mathfrak{I}_k X_k + Z_k$$

This transforms "convolution" to "multiplication."

- So, if we set, for example, $\{\Im_k\} \in \{-1, 1\}$ and transmit its IDFT $\{I_k\}$. Then, the ISI problem can be solved straightforwardly.

- Note that Y_k is only a function of $\Im_k$ and does not depend on $\ldots, \Im_{k-2}, \Im_{k-1}, \Im_{k+1}, \Im_{k+2}, \ldots$.

- This idea has been employed in many modern techniques such as **Orthogonal Frequency Division Multiplexing (OFDM)**.

11.2 Multicarrier communications:

11.2-3 Orthogonal frequency division multiplexing (OFDM)

11.2-4 Modulation and demodulation in an OFDM system

Let T be the symbol duration; then we know the set of waveforms

$$\left\{\kappa e^{\imath 2\pi \frac{k}{T} t} : t \in [0, T), k = 0, 1, \ldots, Q-1\right\}$$

is a set of orthonormal functions, where

$$\kappa = \sqrt{\frac{1}{T}}.$$

$$\begin{aligned}\left\langle \kappa e^{\imath 2\pi \frac{k}{T} t}, \kappa e^{\imath 2\pi \frac{j}{T} t}\right\rangle &= \int_0^T \kappa^2 e^{\imath 2\pi \frac{k}{T} t} e^{-\imath 2\pi \frac{j}{T} t} dt \\ &= \frac{1}{T}\int_0^T e^{\imath 2\pi \frac{(k-j)}{T} t} dt \\ &= \delta_{k-j}\end{aligned}$$

Let

$$X_{k,n} = I_{k,n} + \imath\, Q_{k,n}$$

be the QAM symbol at the kth subcarrier and at the nth symbol period; then the multicarrier waveform is given by

$$s_\ell(t) = \kappa \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} t} \right) g(t - nT)$$

where $g(t)$ is the pulse shaping function.

Hence,

$$s(t) = \mathbf{Re}\left\{ s_\ell(t) e^{\imath 2\pi f_c t} \right\}$$

At the first glance, it seems to be a single-carrier f_c system; but, it is actually a multi-carrier system with single-carrier implementation.

$$
\begin{aligned}
s(t) &= \mathbf{Re}\left\{s_\ell(t)e^{\imath 2\pi f_c t}\right\} \\
&= \mathbf{Re}\left\{\kappa \sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} t}\right) g(t-nT) e^{\imath 2\pi f_c t}\right\} \\
&= \sum_{k=0}^{Q-1} \mathbf{Re}\left\{\left(\kappa \sum_{n=-\infty}^{\infty} X_{k,n} g(t-nT)\right) e^{\imath 2\pi f_k t}\right\}
\end{aligned}
$$

where $f_k = f_c + \frac{k}{T}$ is the kth carrier.

11.2-6 Spectral characteristics of multicarrier signals

Clearly, $s_\ell(t)$ is a random process.

For simplicity, we may assume $I_{k,n}$ and $Q_{k,n}$ are i.i.d., zero mean, and variance $\frac{1}{2}\sigma^2$.

With $\kappa = 1/\sqrt{T}$, the autocorrelation function of $s_\ell(t)$ is

$$\begin{aligned} R_{s_\ell}(t+\tau, t) &= \frac{1}{T}\mathbb{E}\left[\left(\sum_{n=-\infty}^{\infty}\sum_{k=0}^{Q-1} X_{k,n} g(t+\tau-nT) e^{\imath 2\pi \frac{k}{T}(t+\tau)}\right)\right. \\ &\qquad \left.\left(\sum_{m=-\infty}^{\infty}\sum_{j=0}^{Q-1} X^*_{j,m} g^*(t-mT) e^{-\imath 2\pi \frac{j}{T} t}\right)\right] \\ &= \frac{1}{T}\sigma^2 \sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty} g(t+\tau-nT) g^*(t-nT) \end{aligned}$$

- It is clear that

$$R_{s_\ell}(t+\tau, t) = R_{s_\ell}(t+\tau+mT, t+mT)$$

for any integer m; hence $s_\ell(t)$ is a cyclostationary random process with period T.

- The average autocorrelation function is thus given by

$$\begin{aligned}
\bar{R}_{s_\ell}(\tau) &= \frac{1}{T}\int_0^T R_{s_\ell}(t+\tau, t)\,dt \\
&= \frac{\sigma^2}{T^2}\sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty}\int_0^T g(t+\tau-nT)g^*(t-nT)\,dt \\
&= \frac{\sigma^2}{T^2}\sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \sum_{n=-\infty}^{\infty}\int_{-nT}^{-(n-1)T} g(u+\tau)g^*(u)\,du \\
&= \frac{\sigma^2}{T^2}\sum_{k=0}^{Q-1} e^{\imath 2\pi \frac{k}{T}\tau} \int_{-\infty}^{\infty} g(t+\tau)g^*(t)\,dt
\end{aligned}$$

Power spectral density

The time-average power spectral density of $s_\ell(t)$ is

$$
\begin{aligned}
\bar{S}_{s_\ell}(f) &= \int_{-\infty}^{\infty} \bar{R}_{s_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left(\int_{-\infty}^{\infty} g(t+\tau) e^{-\imath 2\pi (f - \frac{k}{T})\tau}\, d\tau \right) dt \\
&= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left(\int_{-\infty}^{\infty} g(u) e^{-\imath 2\pi (f - \frac{k}{T})u}\, du \right) e^{\imath 2\pi (f - \frac{k}{T})t} dt \\
&= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} G\left(f - \frac{k}{T}\right) \left(\int_{-\infty}^{\infty} g(t) e^{-\imath 2\pi (f - \frac{k}{T})t}\, dt \right)^* \\
&= \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T}\right) \right|^2 .
\end{aligned}
$$

Theorem 1

The time-average power spectral density of $s_\ell(t)$ is

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left| G\left(f - \frac{k}{T}\right) \right|^2$$

where Q is the number of subcarriers.

Example

Let $g(t)$ be the rectangular pulse shape of height 1 and duration T; then

$$G(f) = e^{-\imath\pi fT} T \operatorname{sinc}(fT).$$

Hence

$$\bar{S}_{s_\ell}(f) = \sigma^2 \sum_{k=0}^{Q-1} \left| \operatorname{sinc}\left(\left(f - \frac{k}{T} \right) T \right) \right|^2.$$

In particular,

$$\bar{S}_{s_\ell}\left(\frac{m}{T}\right) = \sigma^2 \sum_{k=0}^{Q-1} \left| \operatorname{sinc}(m-k) \right|^2 = \begin{cases} \sigma^2, & \text{if } 0 \le m < Q \\ 0, & \text{otherwise.} \end{cases}$$

Example: $T = 1$ and $Q = 5$

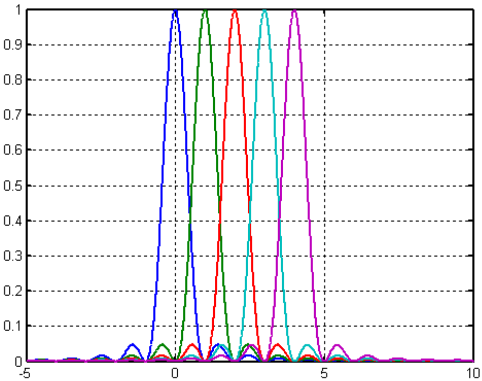


Figure: $\left|G\left(f - \frac{k}{T}\right)\right|^2$ for $k = 0, 1, 2, 3, 4$

Example: $T = 1$ and $Q = 5$

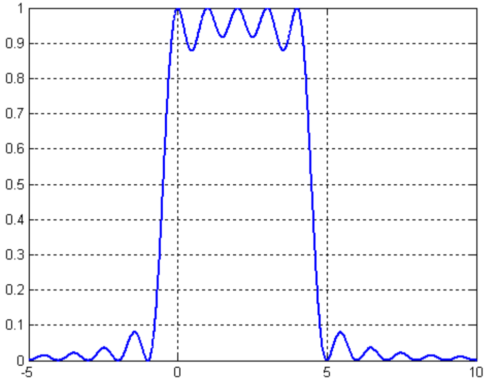


Figure: $S_{s_\ell}(f)$

$$\bar{S}_{s_\ell}(f) = \sigma^2 \sum_{k=0}^{Q-1} \left| \operatorname{sinc}\left(\left(f - \frac{k}{T}\right) T\right) \right|^2$$

- The PSD $\bar{S}_{s_\ell}(f)$ decays very slow at high frequencies at rate approximately

$$\bar{S}_{s_\ell}(f) \approx \frac{1}{f^2}.$$

- Out of band power leakage is severe and the resulting spectrum may not meet the FCC requirement.

- One can add a bandpass filter afterwards to remove the out-of-band signals, for example, using the root raised cosine filters.

11.2-5 An FFT algorithm implementation of an OFDM system

For simplicity, we again assume $g(t)$ is the rectangular pulse shape of height 1 and duration T such that for $0 \le t < T$,

$$s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t}$$

and zero, otherwise, where we drop the subscript n for symbol period for notational convenience.

Then, we will introduce an efficient way to generate the following waveform:

$$s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t}, \quad t \in [0, T)$$

Generating $s_\ell(t)$ using iFFT + DAC

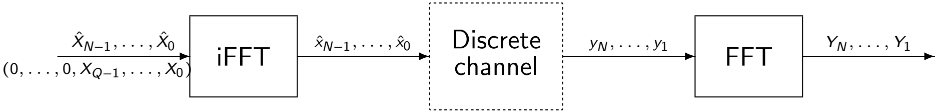
- Consider an N-point iFFT with $N \geq Q$. (Usually, N is equal to the power of two.)
- Set $\hat{X}_k = \begin{cases} X_k, & \text{if } 0 \leq k < Q \\ 0, & \text{if } Q \leq k < N \end{cases}$
- The iFFT of $\hat{X}_k$ is given by

$$\hat{x}_m = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{\imath 2\pi \frac{mk}{N}} = \frac{1}{N} \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{mk}{N}}$$

- Feeding $N\hat{x}_m$ to a digital-analog-converter (DAC) at rate $\frac{N}{T}$ gives

$$\hat{s}_\ell(t) = (\kappa N) \sum_{m=0}^{N-1} \hat{x}_m \, g_{\text{DAC}}\left(t - \frac{m}{N} T\right)$$

  where $g_{\text{DAC}}(t)$ is the rectangular pulse of height 1 and duration $\frac{T}{N}$.



Note that for $n = 0, 1, \ldots, N-1$,

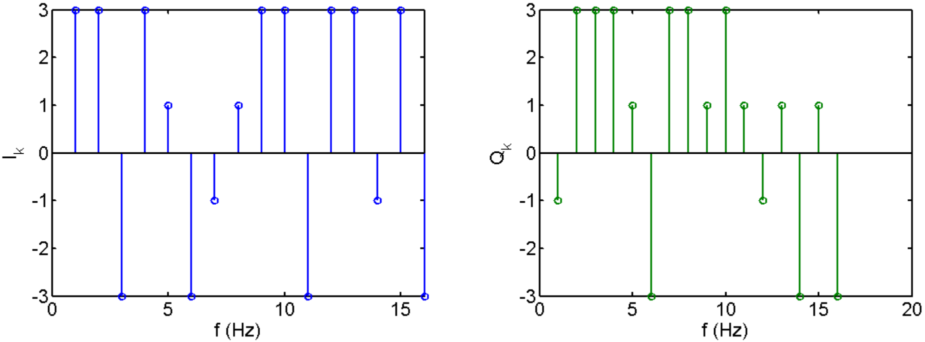
$$\begin{aligned} \hat{s}_\ell\left(\frac{n}{N}T\right) &= \kappa N \sum_{m=0}^{N-1} \hat{x}_m \, g_{\text{DAC}}\left(\frac{n}{N}T - \frac{m}{N}T\right) \\ &= \kappa N \hat{x}_n \\ &= \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{kn}{N}} = s_\ell\left(\frac{n}{N}T\right) \quad \text{(See Slide 11-20.)} \end{aligned}$$

We see

$$\hat{s}_\ell(t) = s_\ell(t) \text{ for } t = nT/N \text{ and } n = 0, 1, \ldots, N-1.$$

The technique we had used is called **Zero Padding** in DSP.

Example: $Q = 16$ and $T = 1$



$$X_k = I_k + \imath\, Q_k \text{ for } 0 \le k < Q = 16$$

Example: $Q = 16$ and $T = 1$

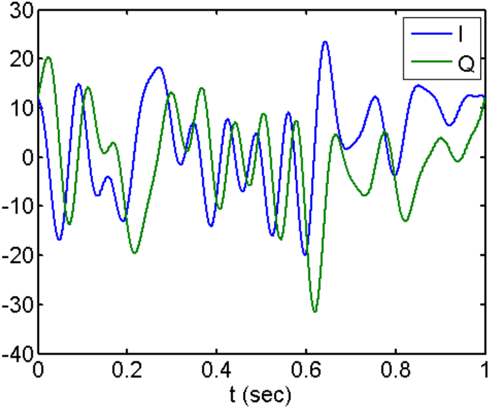
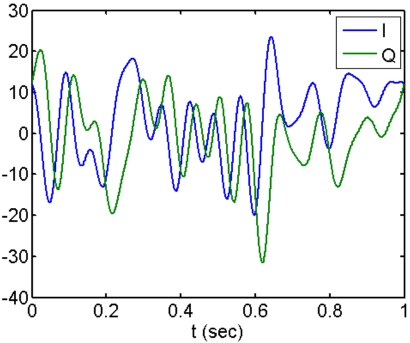


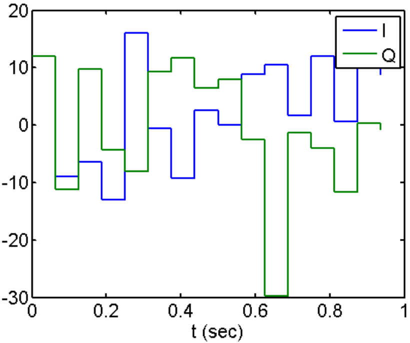
Figure: $s_\ell(t) = I(t) + \imath\, Q(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t}, \quad t \in [0, T)$

Example: $Q = 16$ and $T = 1$ and $N = 16$

$s_\ell(t)$



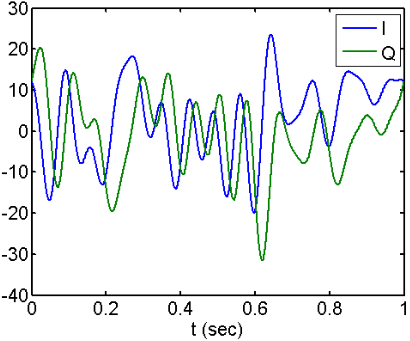
$\hat{s}_\ell(t)$



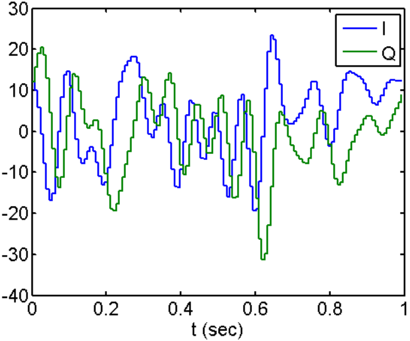
$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = 0, \frac{1}{16}, \frac{2}{16}, \ldots, \frac{15}{16} \text{ (sec)}$$

Example: $Q = 16$ and $T = 1$ and $N = 128$

$s_\ell(t)$



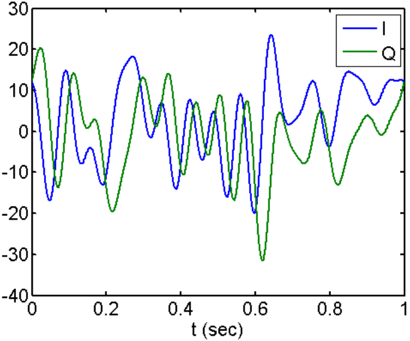
$\hat{s}_\ell(t)$



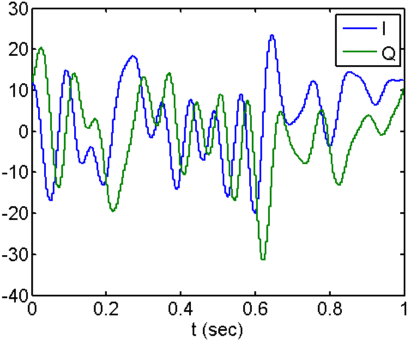
$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = 0, \frac{1}{128}, \frac{2}{128}, \ldots, \frac{127}{128} \text{ (sec)}$$

Example: $Q = 16$ and $T = 1$ and $N = 256$

$s_\ell(t)$



$\hat{s}_\ell(t)$

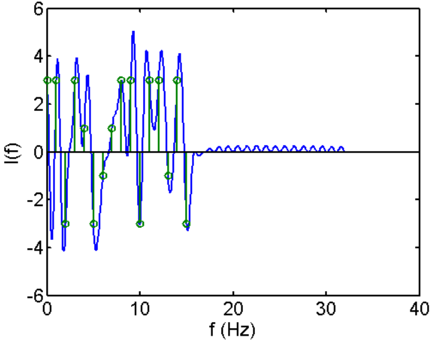


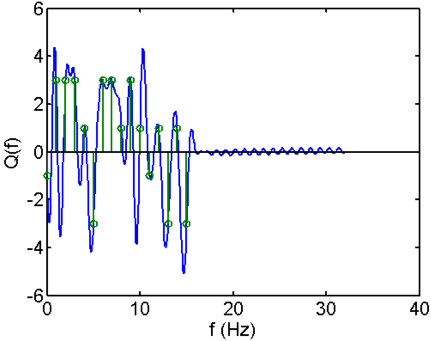
$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = \frac{1}{256}, \frac{2}{256}, \ldots, \frac{255}{256} \text{ (sec)}$$

Example: $Q = 16$ and $T = 1$

$$s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t} g(t), \quad t \in [0, T)$$

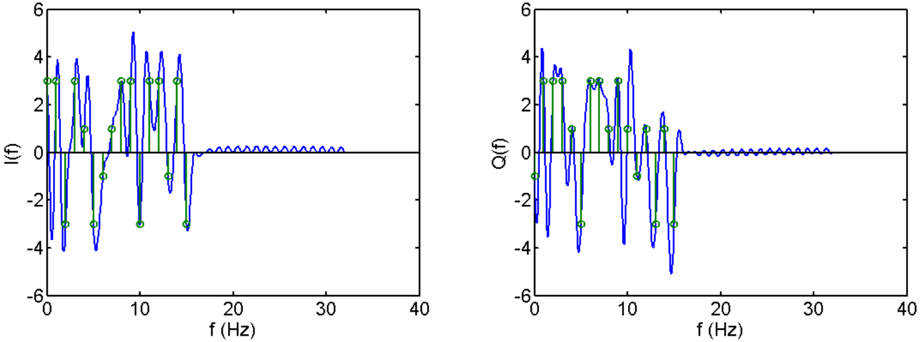
$$S_\ell(f) = \mathcal{F}\{s_\ell(t)\} = I(f) + \imath\, Q(f) = \kappa \sum_{k=0}^{Q-1} X_k G\left(f - \frac{k}{T}\right)$$





Out-of-band leakage due to rectangular pulse shape $g(t)$

Example: $Q = 16$ and $T = 1$



$$S_\ell\left(f = \frac{k}{T}\right) = X_k = I_k + \imath\, Q_k \text{ for } k = 0, 1, \ldots, Q = 15 \text{ and } \kappa = \frac{1}{T}$$

Transmission of multicarrier signal

$$
\begin{aligned}
\hat{s}_\ell(t) &= (\kappa N)\sum_{n=-\infty}^{\infty}\left(\sum_{m=0}^{N-1}\hat{x}_{m,n}\; g_{\text{DAC}}\left(t-\frac{m}{N}T\right)\right)g(t-nT)\\
&= \kappa\sum_{n=-\infty}^{\infty}\left(\sum_{m=0}^{N-1}\left(\sum_{k=0}^{Q-1}X_{k,n}e^{\imath 2\pi\frac{mk}{N}}\right)\; g_{\text{DAC}}\left(t-\frac{m}{N}T\right)\right)g(t-nT)\\
&= \kappa\sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{Q-1}X_{k,n}\left(\sum_{m=0}^{N-1}e^{\imath 2\pi\frac{mk}{N}}\; g_{\text{DAC}}\left(t-\frac{m}{N}T\right)\right)\right)g(t-nT)\\
&= \kappa\sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{Q-1}X_{k,n}e^{\imath 2\pi\frac{k}{T}\frac{\lfloor t(N/T)\rfloor}{(N/T)}}\right)g(t-nT)\\
s_\ell(t) &= \kappa\sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{Q-1}X_{k,n}e^{\imath 2\pi\frac{k}{T}t}\right)g(t-nT)
\end{aligned}
$$

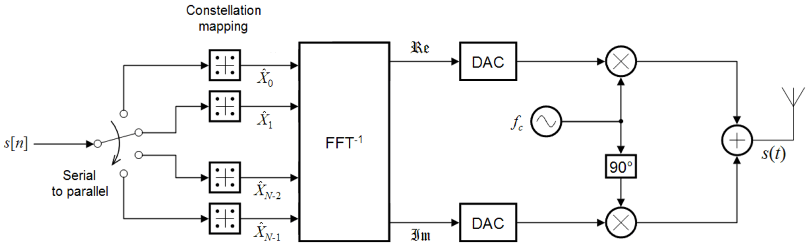
The difference between ideal $s_\ell(t)$ and physically realizable $\hat{s}_\ell(t)$ is that the latter uses a "digitized" time scale.

Denote $a = \frac{\lfloor t(N/T) \rfloor}{t(N/T)}$, which is approximately 1 when N large.

Then the transmitted signal is given by

$$\begin{aligned}
\hat{s}(t) &= \mathbf{Re}\left\{\hat{s}_\ell(t) e^{\imath 2\pi f_c t}\right\} \\
&= \kappa \sum_{n=-\infty}^{\infty} \mathbf{Re}\left\{\left(\sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} \frac{\lfloor t(N/T) \rfloor}{(N/T)}}\right) e^{\imath 2\pi f_c t}\right\} g(t - nT) \\
&= \kappa \sum_{n=-\infty}^{\infty} \sum_{k=0}^{Q-1} \left\{ I_{k,n} \cos\left[2\pi\left(f_c + a\frac{k}{T}\right) t\right]\right. \\
&\qquad \left. - Q_{k,n} \sin\left[2\pi\left(f_c + a\frac{k}{T}\right) t\right]\right\} g(t - nT)
\end{aligned}$$

Transmission of multicarrier signal



OFDM = Multicarrier + Cyclic prefix

- Why adding cyclic prefix?
  To combat the channel effect due to $c_\ell(t)$.

- We can virtually think that

$$s_\ell(t) = \begin{cases} \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

  or more physically

$$\hat{s}_\ell(t) = \begin{cases} \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} a t}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

- Virtually extend $s_\ell(t)$ to make it periodic

$$\tilde{s}_\ell(t) = \sum_{n=-\infty}^{\infty} s_\ell(t - nT) = \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t} \text{ for } t \in \mathbb{R}$$

We will transmit $\tilde{s}_\ell(t)$ (of duration $P+T$) instead of $s_\ell(t)$ (of duration T) for OFDM, where P is the length of $c_\ell(t)$.

In other words, we essentially assume that

$$c_\ell(t) = 0 \text{ for } t < 0 \text{ and } t \geq P.$$

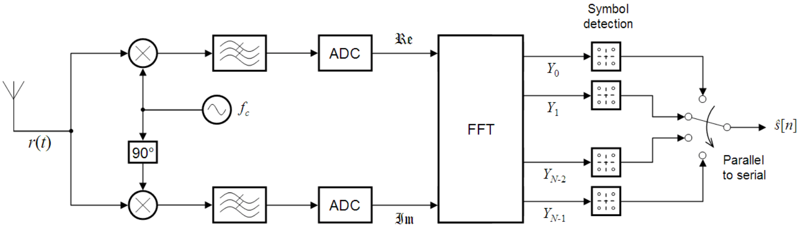
The extra periodic part P is called **cyclic prefix** in OFDM.

Usually, T should be made much larger than P in order to reduce the loss in transmission time and to save extra transmission power. For example, $T = 3.2\mu s$ and $P = 0.8\mu s$ for IEEE 802.11.

The necessity of adding CP will be clear in the analysis of Rx.

Receiver for multicarrier signal

Receiver for multicarrier signal



Oversampling

While there are only Q tones transmitted, oversampling is required to avoid aliasing caused by out-of-band signals from other users.

Assuming the channel has a lowpass equivalent impulse response $c_\ell(t)$, the received noise-free received signal is

$$r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t) = \int_0^P c_\ell(\tau)\tilde{s}_\ell(t-\tau)d\tau,$$

where $\tilde{s}_\ell(t)$ periodic with period T.

Since all we need is $r_\ell(t)$ for $t \in [0, T)$, it is clear from the above formula that we only need $\tilde{s}_\ell(t)$ for $t \in [-P, T)$.

By this CP technique, the received signal is simplified to:

$$\begin{aligned} r_\ell(t) &= \tilde{s}_\ell(t) \star c_\ell(t) \\ &= \kappa \left(\sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t} \right) \star c_\ell(t) \\ &= \kappa \sum_{k=0}^{Q-1} X_k \int_{-\infty}^{\infty} c_\ell(\tau) e^{\imath 2\pi \frac{k}{T}(t-\tau)} d\tau \end{aligned}$$

$$
\begin{aligned}
r_\ell(t) &= \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t} \int_{-\infty}^{\infty} c_\ell(\tau) e^{-\imath 2\pi \frac{k}{T}\tau}\, d\tau \\
&= \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t} C_\ell\left(\frac{k}{T}\right).
\end{aligned}
$$

Note $r_\ell(t)$ is actually periodic with period T.

Sample $r_\ell(t)$ at rate $\frac{\tilde{N}}{T}$, where $\tilde{N}$ is not necessarily equal to N.

$$
r_m = r_\ell\left(\frac{m}{\tilde{N}} T\right) = \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi \frac{km}{\tilde{N}}}
$$

* The extra receptions for $m = -1, -2, \ldots, -\frac{P}{T}\tilde{N}$ due to CP are unused.

When using the physical $\hat{s}_\ell(t)$ instead of ideal $s_\ell(t)$,

$$\begin{aligned}\hat{r}_m &= \hat{r}_\ell\left(\frac{m}{\tilde{N}}T\right) = \kappa\sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi \frac{k}{T}\left(\frac{\lfloor m(N/\tilde{N})\rfloor}{m(N/\tilde{N})}\right)\frac{m}{\tilde{N}}T} \\ &= \kappa\sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi k \frac{\lfloor m(N/\tilde{N})\rfloor}{N}} \text{ for } 0 \le m \le \tilde{N}-1\end{aligned}$$

So, if $N = \tilde{N}$ or N is a multiple of $\tilde{N}$ (i.e., the sampling rate at Tx is higher), then $\hat{r}_m = r_m$.
However, if $\tilde{N}$ is a multiple of N, say, $\tilde{N} = uN$, then

$$\hat{r}_m = \kappa\sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi k \frac{\lfloor m/u\rfloor}{N}} = r_{u\lfloor m/u\rfloor}.$$

In other words, we only have N different samples at Rx since Tx only transmits N samples.

FFT/iFFT duality

The FFT/iFFT duality we adopt here is:

$$\begin{cases} \text{FFT} \quad \hat{X}_k = \sum_{m=0}^{N-1} \hat{x}_m e^{-\imath 2\pi \frac{mk}{N}} \\ \text{iFFT} \quad \hat{x}_m = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{\imath 2\pi \frac{mk}{N}} \end{cases}$$

Channel equalization

Given the received signal vector $\boldsymbol{r} = [r_0, \cdots, r_{\tilde{N}-1}]$, the receiver applies FFT to $\boldsymbol{r}$ (Implicitly, N is a multiple of $\tilde{N}$ with $\tilde{N} > Q$.)

$$
\begin{aligned}
R_n &= \sum_{m=0}^{\tilde{N}-1} r_m e^{-\imath 2\pi \frac{mn}{\tilde{N}}} \\
&= \sum_{m=0}^{\tilde{N}-1} \left(\kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi \frac{km}{\tilde{N}}} \right) e^{-\imath 2\pi \frac{mn}{\tilde{N}}} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k \sum_{m=0}^{\tilde{N}-1} e^{-\imath 2\pi \frac{m(n-k)}{\tilde{N}}} \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k \cdot \tilde{N}\delta_{n-k} \\
&= \begin{cases} \kappa \tilde{N} C_\ell\left(\frac{n}{T}\right) X_n, & 0 \le n < Q \\ 0, & Q \le n < \tilde{N} \end{cases}
\end{aligned}
$$

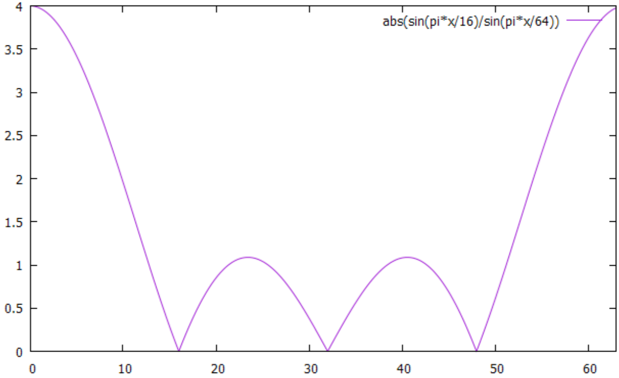
When oversampling occurs

When $\tilde{N} = uN$,

$$
\begin{aligned}
R_n &= \sum_{m=0}^{\tilde{N}-1} \hat{r}_m e^{-\imath 2\pi \frac{mn}{\tilde{N}}} \\
&= \sum_{m=0}^{\tilde{N}-1} \left(\kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k e^{\imath 2\pi \frac{km}{\tilde{N}} \frac{\lfloor m/u \rfloor}{m/u}} \right) e^{-\imath 2\pi \frac{mn}{\tilde{N}}} \quad (m = ui + j) \\
&= \kappa \sum_{k=0}^{Q-1} C_\ell\left(\frac{k}{T}\right) X_k \left(\sum_{j=0}^{u-1} e^{-\imath 2\pi \frac{nj}{\tilde{N}}} \sum_{i=0}^{N-1} e^{-\imath 2\pi \frac{i(n-k)}{N}} \right) \\
&= \begin{cases} \kappa \left(\sum_{j=0}^{u-1} e^{-\imath 2\pi \frac{nj}{uN}} \right) N C_\ell\left(\frac{n \bmod N}{T}\right) X_{n \bmod N}, & 0 \le n \bmod N < Q \\ 0, & Q \le n \bmod N < N \end{cases} \\
&= \begin{cases} \kappa \dfrac{e^{-\imath \pi \frac{(u-1)n}{uN}} \sin\left(\frac{\pi n}{N}\right)}{\sin\left(\frac{\pi n}{uN}\right)} N C_\ell\left(\frac{n \bmod N}{T}\right) X_{n \bmod N}, & 0 \le n \bmod N < Q \\ 0, & Q \le n \bmod N < N \end{cases}
\end{aligned}
$$

Example. $N = 16$ and $\tilde{N} = 64$

$$\left|\frac{\sin\left(\frac{\pi n}{N}\right)}{\sin\left(\frac{\pi n}{uN}\right)}\right|$$



n

Channel equalization

With noise present, we have

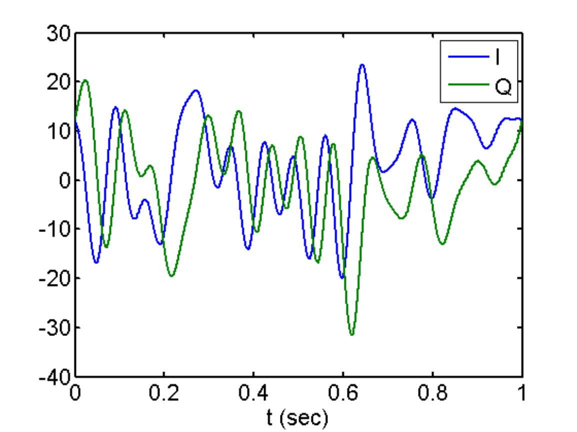
$$R_k = \kappa \tilde{N} C_\ell\left(\frac{k}{T}\right) X_k + Z_k$$

- Only one-tap equalization (i.e., $\kappa \tilde{N} C_\ell\left(\frac{k}{T}\right)$) is needed.

Disadvantages of OFDM

While OFDM allows for simple equalization, it also introduces other problems such as:

High peak-to-average power ratio (PAPR) at $s_\ell(t)$



What you learn from Chapter 11

- Spectral characteristics of multicarrier signals
- An FFT implementation of an OFDM system with DAC consideration
- Physical transmission of multicarrier signal over digitized time scale
- Multicarrier + Cyclic prefix
- Oversampling and undersampling at RX