Digital Communications

Chapter 9 Digital Communications Through Band-Limited Channels

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9.2 Signal design for band-limited channels

Motivation

For the baseband waveform

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$$

- Channel is band-limited to $[-W, W]$.
- Transmitted signals outside $[-W, W]$ will be truncated.

How to design $g(t)$ to yield optimal performance?

[1] Here we use $s_\ell(t)$ instead of $v(t)$ as being used in text to clearly indicate that $s_\ell(t)$ is the lowpass equivalent signal of $s(t)$.

Recall that

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$$

is a cyclostationary random process with i.i.d. discrete random process $\{I_n\}$.

- Autocorrelation function of $s_\ell(t)$ is

$$R_{s_\ell}(t+\tau, t) = \mathbb{E}[s_\ell(t+\tau)s_\ell^*(t)]$$

  is periodic with period T.

- The time-average autocorrelation function

$$\begin{aligned}\bar{R}_{s_\ell}(\tau) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_{s_\ell}(t+\tau, t)\, dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_I(n) \int_{-\infty}^{\infty} g(t+\tau-nT)g^*(t)\, dt\end{aligned}$$

$$\bar{R}_{s_\ell}(\tau) = \frac{1}{T}\sum_n R_I(n)\int_{-\infty}^{\infty} g(t+\tau-nT)g^*(t)\,dt$$

- The time-average power spectral density of $s_\ell(t)$ is

$$\begin{aligned}\bar{S}_{s_\ell}(f) &= \int_{-\infty}^{\infty} \bar{R}_{s_\ell}(\tau)e^{-\imath 2\pi f\tau}\,d\tau \\ &= \frac{1}{T}\left[\sum_{n=-\infty}^{\infty} R_I(n)e^{-\imath 2\pi f nT}\right]|G(f)|^2\end{aligned}$$

- Assuming I_n is zero mean and i.i.d., $R_I(n) = \sigma^2\delta_n$, hence

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T}|G(f)|^2$$

For static channel with impulse response $c_\ell(t)$, band-limited to $[-W, W]$, i.e.

$$C_\ell(f) = 0 \quad \text{for } |f| > W$$

The received signal is

$$\begin{aligned} r_\ell(t) &= c_\ell(t) \star s_\ell(t) + n_\ell(t) \\ &= \int_{-\infty}^{\infty} c_\ell(\tau) \left(\sum_{n=-\infty}^{\infty} I_n g(t - \tau - nT) \right) d\tau + n_\ell(t) \\ &= \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} g(t - nT - \tau) c_\ell(\tau)\, d\tau + n_\ell(t) \end{aligned}$$

whose time-average power spectral density is

$$\bar{S}_{r_\ell}(f) = \frac{\sigma^2}{T} |G(f)|^2 |C_\ell(f)|^2 + 2N_0 \, \mathrm{rect}\left(\frac{f}{2W} \right)$$

Note that $n_\ell(t)$ is a band-limited white noise process.

$$r_\ell(t) = \sum_{n=-\infty}^{\infty} I_n h_\ell(t-nT) + n_\ell(t) \text{ with } h_\ell(t) = g(t) \star c_\ell(t)$$

- Assume channel Impulse response $c_\ell(t)$ known to Rx.

- The **match-filtered** received signal is

$$y_\ell(t) = r_\ell(t) \star h_\ell^*(T-t) = \sum_n I_n x_\ell(t-nT) + z_\ell(t)$$

where $x_\ell(t) = h_\ell(t) \star h_\ell^*(T-t)$ and $z_\ell(t) = n_\ell(t) \star h_\ell^*(T-t)$.

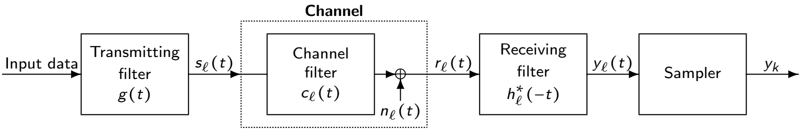
For simplicity, one may use $h_\ell^*(-t)$ instead of $h_\ell^*(T-t)$.

- Sampling at $t = kT$, $k \in \mathbb{Z}$, we get

$$y_k = y_\ell(kT) = \sum_{n=-\infty}^{\infty} I_n x_\ell(kT-nT) + z_\ell(kT) = \sum_n I_n x_{k-n} + z_k$$

where $x_{k-n} = x_\ell(kT-nT)$.

Quick summary



- Transmitted signal $s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$
- Received signal $r_\ell(t) = \sum_{n=-\infty}^{\infty} I_n h_\ell(t - nT) + n_\ell(t)$ with $h_\ell(t) = g(t) \star c_\ell(t)$.
- Matched filter output $y_\ell(t) = \sum_{n=-\infty}^{\infty} I_n x_\ell(t - nT) + z_\ell(t)$ with $x_\ell(t) = h_\ell(t) \star h_\ell^*(-t)$ or equivalently

$$X_\ell(f) = |H_\ell(f)|^2 = |G(f)|^2 \, |C_\ell(f)|^2 .$$

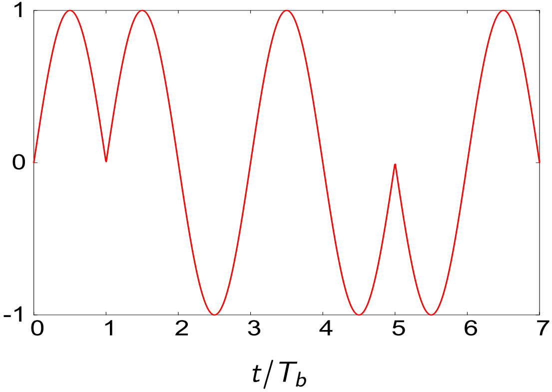
- Sampling at $t = kT$ and $y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$.

The transmission power is not $\|g(t)\star c_\ell(t)\|^2$, which the receiver filter is to match! The additive interference to the transmitted information I_k is not just z_k but includes $\sum_{n=-\infty, n\neq k}^{\infty} I_n x_{k-n}$. So match filter is good for single transmission but may not be proper for consecutive transmissions!

Eye pattern

A good tool to examine inter-symbol interference is the eye pattern.

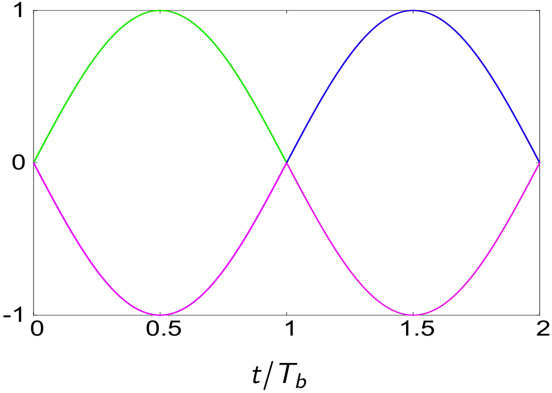
- Eye pattern: The synchronized superposition of all possible realizations of the signal of interest viewed within a particular signaling interval.



BPSK signals

Perfect eye pattern (at Tx)

Eye pattern for $r_\ell(t)$ for $g(t)$ being half-cycle sine wave with duration T_b, $c_\ell(t) = \delta(t)$ and error-free BPSK transmission.



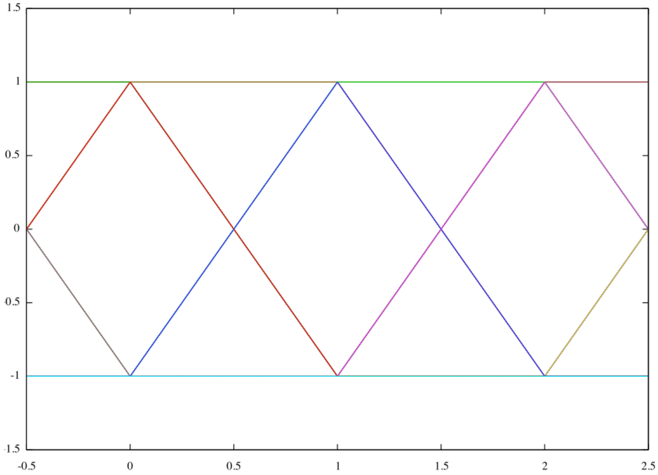
Now with $c_\ell(t) = \delta(t)$, we have, for example,

$$
\begin{aligned}
g(t) &= \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \quad \text{(Time-limited; hence, \textbf{band-unlimited!})} \\
h_\ell(t) &= g(t) \star c_\ell(t) = g(t) \\
x_\ell(t) &= h_\ell(t) \star h_\ell^*(-t) = \int_{-\infty}^{\infty} h_\ell(\tau) h_\ell^*(-(t-\tau))\, d\tau \\
&= \int_{-\infty}^{\infty} g(\tau) g(\tau - t)\, d\tau = \begin{cases} \frac{T-|t|}{T}, & |t| \le T \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

From here, you shall know the difference of using $h_\ell^*(T-t)$ and $h_\ell^*(-t)$, where the former samples at $t = T$, while the latter samples at $t = 0$.

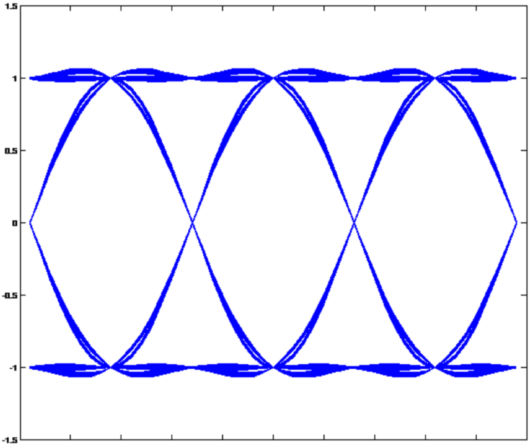
Perfect eye pattern (at Rx)

$y_\ell(t) = \sum_{n=-\infty}^{\infty} I_n x_\ell(t - nT)$ for all possible $\{I_n \in \{\pm 1\}\}_{n=-\infty}^{\infty}$.
$(y_k = y_\ell(kT) = \sum_{n=-\infty}^{\infty} I_n x_\ell(kT - nT) = I_k$; so no ISI!)



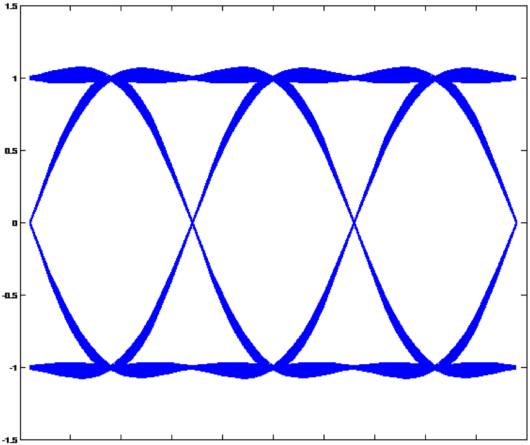
Example: BPSK with $1/T = 1K$ and $W = 3K$

Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.



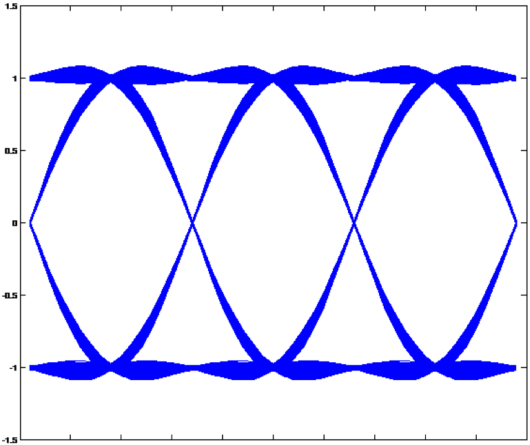
Example: BPSK with $1/T = 1K$ and $W = 1K$

Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.



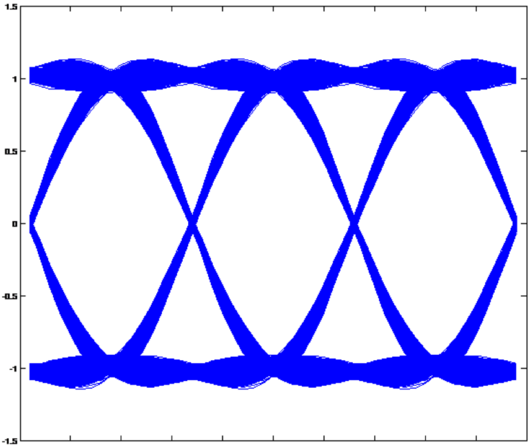
Example: BPSK with $1/T = 1K$ and $W = 0.9K$

Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.



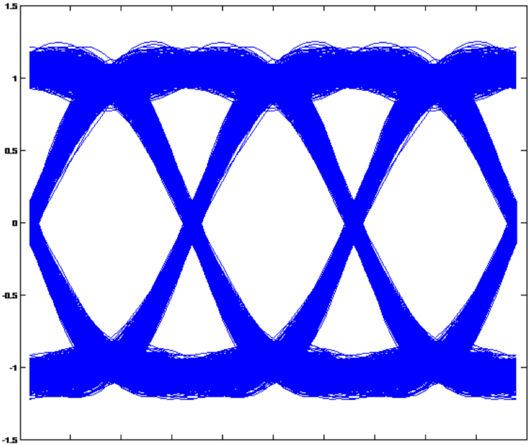
Example: BPSK with $1/T = 1K$ and $W = 0.8K$

Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.



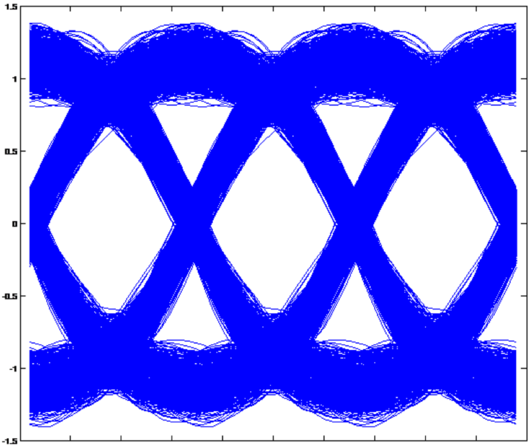
Example: BPSK with $1/T = 1K$ and $W = 0.7K$

Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.



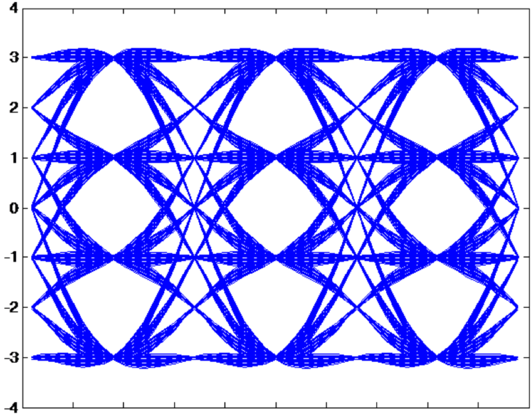
Example: BPSK with $1/T = 1K$ and $W = 0.6K$

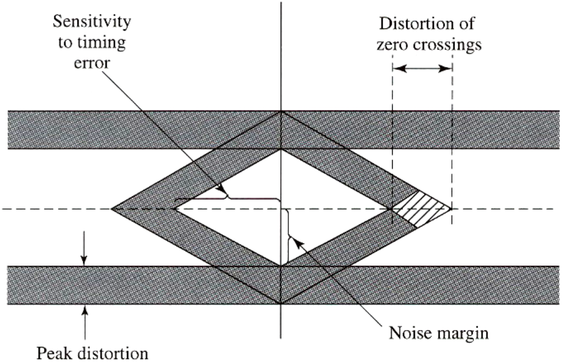
Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.



Example: 4PAM with $1/T = 1K$ and $W = 1K$

Output $y_\ell(t)$ with $c_\ell(t)$ being ideal lowpass filter of bandwidth W.





Conclusion: The smaller W

- Horizontal: Decision is more sensible to timing error.
- Vertical: Decision is more sensible to noise.

Example (revisited)

Now changing to $c_\ell(t) = \delta(t) + \delta(t - T)$, we have, for example,

$$\begin{aligned}
g(t) &= \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \quad \text{(Time-limited; hence, \textbf{band-unlimited!})} \\
h_\ell(t) &= g(t) \star c_\ell(t) = g(t) + g(t - T) = g(t/2) \\
x_\ell(t) &= h_\ell(t) \star \underbrace{h_\ell^*(2T - t)}_{\text{causal}} = \int_{-\infty}^{\infty} h_\ell(\tau) h_\ell^*(2T - (t - \tau)) d\tau \\
&= \int_{-\infty}^{\infty} g(\tau/2) g((2T + \tau - t)/2) d\tau \\
&= \begin{cases} \frac{2T - |t - 2T|}{T}, & |t - 2T| \le 2T \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$y_k = y_\ell(kT) = \sum_{n=-\infty}^{\infty} I_n x_\ell(kT - nT) + z_\ell(kT) = 2I_{k-2} + \underbrace{I_{k-1} + I_{k-3}}_{\text{ISI}} + z_k$$

The question next to be asked is that how to remove the ISI?

9.2-1 Design of band-limited signals for no intersymbol interference - The Nyquist criterion

Intersymbol interference channel

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

Design goal

Given W, T and $c_\ell(t)$, we would like to design $g(t)$ such that

$$x_{k-n} = \delta_{k-n}$$

and that

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k = I_k + z_k$$

Here, δ_k is the Kronecker delta function, defined as

$$\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

Nyquist rate

Theorem 1 (Nyquist criterion)

Let $x_\ell(t)$ be a band-limited signal with band $[-W, W]$ and $x_\ell(0) = 1$. Sample at rate $\frac{1}{T}$ such that

$$x_k = x_\ell(kT) = \int_{-\infty}^{\infty} x_\ell(t)\delta(t - kT)\, dt = \delta_k$$

if and only if

$$\sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = T$$

where $X_\ell(f) = \mathcal{F}\{x_\ell(t)\}$.

Proof:

The condition of $x_k = x_\ell(kT) = \delta_k$ is equivalent to

$$x_\ell(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x_\ell(kT)\delta(t - kT) = \delta(t). \quad (1)$$

We however have (from Lemma 2 in the next slide)

$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t - kT)\right\} = \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \text{ and } \mathcal{F}\{\delta(t)\} = 1.$$

Taking Fourier transform on both sides of (1) gives

$$X_\ell(f) \star \left[\frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = 1,$$

and proves the theorem. □

Proof of key lemma

Lemma 2

$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty}\delta(t-kT)\right\}=\frac{1}{T}\sum_{m=-\infty}^{\infty}\delta\left(f-\frac{m}{T}\right)$$

Proof: Consider the function

$$\alpha(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty}\delta\left(f-\frac{m}{T}\right)$$

which is periodic with period $\frac{1}{T}$. From Fourier series, we have

$$\alpha(f) = \frac{1}{1/T}\sum_{n=-\infty}^{\infty}c_n e^{-\imath 2\pi nf/(1/T)} = \sum_{n=-\infty}^{\infty}e^{-\imath 2\pi nfT}$$

where

$$c_n=\int_{-\frac{1}{2T}}^{\frac{1}{2T}}\alpha(f)e^{\imath 2\pi nf/(1/T)}df=\int_{-\frac{1}{2T}}^{\frac{1}{2T}}\frac{1}{T}\delta(f)e^{\imath 2\pi nf/(1/T)}df=\frac{1}{T}$$

by following the replication property of $\delta(f)$.

Next by linearity of Fourier transform, we have

$$\begin{aligned}\mathcal{F}\left\{\sum_{k=-\infty}^{\infty}\delta(t-kT)\right\} &= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(t-kT)e^{-\imath 2\pi f t}\,dt \\ &= \sum_{k=-\infty}^{\infty} e^{-\imath 2\pi f k T}.\end{aligned}$$

□

Implication of Nyquist criterion

$$x_k = \delta_k \quad \text{iff} \quad \sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = T \qquad (2)$$

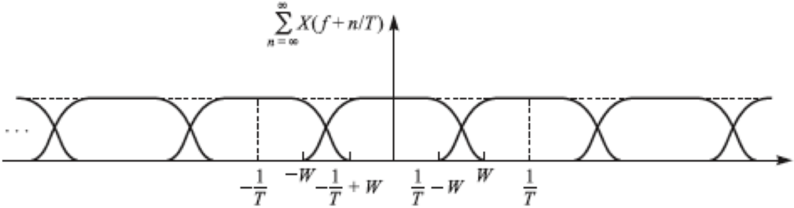
1. $2W < \frac{1}{T}$:
   $X_\ell\left(f - \frac{m}{T}\right)$ and $X_\ell\left(f - \frac{m'}{T}\right)$ do not overlap for $m \neq m'$.
   Hence (2) is impossible!

2. $2W = \frac{1}{T}$:
   This means $X_\ell(f) = T\text{rect}(Tf)$, hence $x_\ell(t) = \text{sinc}\left(\frac{t}{T}\right)$.
   Theoretically ok but physically impossible!

3. So we need $2W > \frac{1}{T}$

Channel bandwidth must be larger than **sampling rate**.

$$x_k = \delta_k \quad \text{iff} \quad \sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = T$$



$$2W > \frac{1}{T} \quad \left(\text{from } W > \frac{1}{T} - W\right)$$

Raised cosine pulse

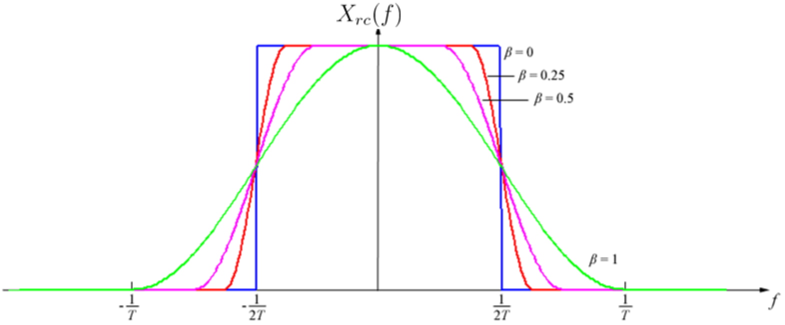
Definition 1

$$X_{rc}(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\}, & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0, & otherwise \end{cases}$$

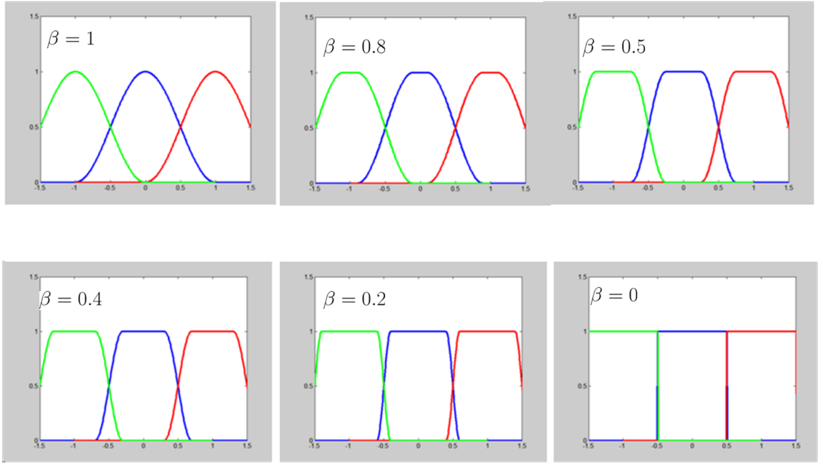
where $0 \le \beta \le 1$ *is the roll-off factor.*

$$x_{rc}(t) = sinc\left(\frac{t}{T}\right)\frac{\cos(\pi\beta t/T)}{1 - 4\beta^2 t^2/T^2}$$

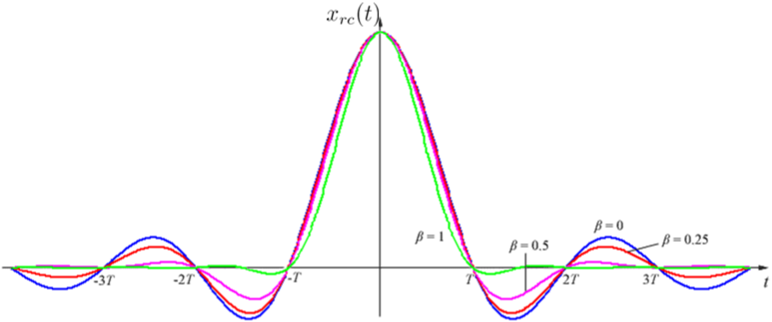
$$X_{rc}(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\}, & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{cases}$$



$$X_{rc}\left(f - \frac{m}{T}\right) \text{ with } m = -1, 0, 1$$



$$x_{rc}(t) = \operatorname{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\pi\beta t/T\right)}{1-4\beta^2t^2/T^2}$$



Example.

Assuming

$$C_\ell(f) = 1$$

- Let $G_T(f) = \mathcal{F}\{g(t)\}$.
- Let $G_R(f) = \mathcal{F}\{h_\ell^*(-t)\}$, where $h_\ell(t) = g(t) \star c_\ell(t) = g(t)$.

This gives

$$G_R(f) = \mathcal{F}\{g^*(-t)\} = G_T^*(f)$$

and

$$X_{rc}(f) = G_T(f)C_\ell(f)G_R(f) = G_T(f)G_R(f) = |G_T(f)|^2.$$

- Note $|x_{rc}(t)| \approx \frac{1}{t^3}$ for large $|t|$. So it can be truncated at $\pm t_0$ for some t_0 large.

$$x_{rc}(t) = \text{sinc}\left(\frac{t}{T}\right)\frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2}$$

So we can set

$$G_T(f) = \sqrt{X_{rc}(f)}e^{-\imath 2\pi f t_0} \quad \text{and} \quad G_R(f) = G_T^*(f).$$

9.2-2 Design of band-limited signals with controlled ISI: Partial response signals

Duobinary pulse

We relax the no-ISI condition so that

$$x_k = \delta_k + \delta_{k-1} \implies y_k = I_k + I_{k-1} + z_k$$

Following similar arguments (see the next slide), it shows

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell\left(f - \frac{m}{T}\right) = 1 + e^{-\imath 2\pi f T}.$$

Setting $2W = \frac{1}{T}$ (with W being the bandwidth of $X_\ell(f)$), we have

$$\begin{aligned} X_\ell(f) &= T\left(1 + e^{-\imath 2\pi f T}\right)\text{rect}\left(\frac{f}{2W}\right) \\ x_\ell(t) &= \text{sinc}\,(2Wt) + \text{sinc}\left[2\left(Wt - \frac{1}{2}\right)\right] \end{aligned}$$

This is called duobinary pulse.

Proof: The condition of $x_k = \delta_k + \delta_{k-1}$ is equivalent to

$$x_\ell(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} x_\ell(kT)\delta(t-kT) = \delta(t)+\delta(t-T). \quad (3)$$

We however have

$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t-kT)\right\} = \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \text{ and } \mathcal{F}\{\delta(t)\} = 1.$$

Taking Fourier transform on both sides of (3) gives

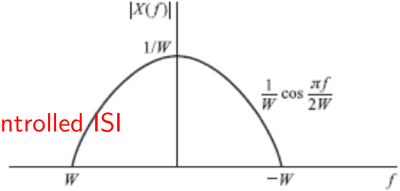
$$X_\ell(f) \star \left[\frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f-\frac{m}{T}\right)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell\left(f-\frac{m}{T}\right) = 1 + e^{-\imath 2\pi f T},$$

and proves the theorem.

When $W = \frac{1}{2T}$,

$$X_\ell(f) = \begin{cases} T\left(1 + e^{-\imath 2\pi f T}\right)\mathrm{rect}\left(\frac{f}{2W}\right), & \text{with controlled ISI} \\ T\mathrm{rect}\left(\frac{f}{2W}\right), & \text{without controlled ISI} \end{cases}$$

$X_\ell(f)$ with controlled ISI



The duobinary filter is apparently more physically realizable!

Recall that with no controlled ISI, we require $W > \frac{1}{2T}$ because channel bandwidth $W = \frac{1}{2T}$ is not physically realizable. But now with controlled ISI, $W = \frac{1}{2T}$ becomes physically realizable.

9.2-3 Data detection for controlled ISI

Precoding for duobinary pulses

Received signal for duobinary shaping is

$$y_k = I_k + I_{k-1} + z_k.$$

We could **precode** the sequence $\{I_k\}$ to simplify detection.

Example 1 (Binary case - Differential encoding)

Given the binary bit (information) stream $\{b_k\}$

- *Define* $P_k = b_k \oplus P_{k-1} \in \{0,1\}$.
- *Set* $I_k = 2P_k - 1 \in \{\pm 1\}$.
- *Receive* $y_k = I_k + I_{k-1} + z_k$.

$$I_k + I_{k-1} = \begin{cases} \pm 2, & \text{if } b_k = 0 \\ 0, & \text{if } b_k = 1. \end{cases}$$

- $\hat{b}_k = 0$ *if* $|y_k| > 1$ *and* $\hat{b}_k = 1$ *if* $|y_k| \le 1$

Summary of precoding system

Compare $\begin{cases} \frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell\left(f-\frac{m}{T}\right)=1 \\ \frac{1}{T}\sum_{m=-\infty}^{\infty} X_\ell^{(p)}\left(f-\frac{m}{T}\right)=1+e^{-\imath 2\pi fT}. \end{cases}$

We can say $X_\ell^{(p)}(f)=X_\ell(f)\left(1+e^{-\imath 2\pi fT}\right)$ and

$$\begin{cases} y_\ell(t)=\sum_{n=-\infty}^{\infty} I_n x_\ell(t-nT)+z_\ell(t) \\ y_\ell^{(p)}(t)=\sum_{n=-\infty}^{\infty} I_n x_\ell^{(p)}(t-nT)+z_\ell(t) \end{cases}$$

and

$$\begin{cases} y_k=I_k+z_k \\ y_k^{(p)}=I_k+I_{k-1}+z_k \end{cases} \Rightarrow \begin{cases} y_k \lesseqgtr 0 \\ |y_k^{(p)}| \lesseqgtr 1 \end{cases}$$

Note that $\|x_\ell(t)\|^2$ does not decide the transmission energy!

Summary of precoding system

$$\Rightarrow \begin{cases} P_e = Q\left(\sqrt{\frac{1}{\mathbb{E}[z_k^2]}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_{b,\ell}/\|g_T(t)\|^2}{N_0\|g_R(t)\|^2}}\right), & \text{error rate for } I_k \\ \quad = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}\frac{1}{\|g_T(t)\|^2\|g_R(t)\|^2}}\right) = Q(\sqrt{2a\gamma_b}) & \\ P_e^{(p)} = \frac{3}{2}Q(\sqrt{2a\gamma_b}) - \frac{1}{2}Q(3\sqrt{2a\gamma_b}), & \text{error rate for } b_k \end{cases}$$

where

$$\begin{cases} I_k \in \{\pm 1\} \\ \text{the (lowpass) transmission energy per bit } \mathcal{E}_{b,\ell} = \|g_T(t)\|^2 = 2\mathcal{E}_b \\ \text{the lowpass noise } \mathbb{E}[z_k^2] = \sigma_\ell^2\|g_R(t)\|^2 \\ \qquad \text{with } n_\ell(t) \text{ having two-sided PSD } \sigma_\ell^2 = N_0 \\ a = \frac{1}{\|g_T(t)\|^2\|g_R(t)\|^2} \text{ (subject to } g_T(t) \star c_\ell(t) \star g_R(t) = x_\ell(t)) \end{cases}$$

Hence, pre-coding technique provides a better spectrum efficiency at the price of performance degradation.

9.2-4 Signal design for channel with distortion

In general, we have $C_\ell(f) \neq \text{rect}\left(\frac{f}{2W}\right)$, and in this case

$$\begin{aligned}
s_\ell(t) &= \sum_{n=-\infty}^{\infty} I_n g_T(t - nT) \\
r_\ell(t) &= c_\ell(t) \star s_\ell(t) + n_\ell(t) \\
y_\ell(t) &= r_\ell(t) \star g_R(t) \;=\; s_\ell(t) \star c_\ell(t) \star g_R(t) + z_\ell(t) \\
y_k &= y_\ell(kT) \;=\; I_k + z_k \quad \text{(We hope there is no ISI!)}
\end{aligned}$$

where

- $g_T(t)$ transmit filter
- $c_\ell(t)$ channel impulse response
- $g_R(t)$ receive filter

Hence

$$\begin{cases} x_{rc}(t) = & g_T(t) \star c_\ell(t) \star g_R(t) \\ X_{rc}(f) = & G_T(f) C_\ell(f) G_R(f) \\ S_{z_\ell}(f) = & N_0 |G_R(f)|^2 \text{ because we assume } \{I_k\} \text{ real} \end{cases}$$

If $c_\ell(t)$ is known to Tx, then we may choose to "pre-equalize" the channel effect at Tx:

$$|G_T(f)| = \frac{\sqrt{X_{rc}(f)}}{|C_\ell(f)|} \quad \text{and} \quad |G_R(f)| = \sqrt{X_{rc}(f)}$$

Then, ISI is avoided; also, the noise power remains

$$\mathbb{E}[z_k^2] = \int_{-\infty}^{\infty} \underbrace{S_{z_\ell}(f)}_{=N_0|G_R(f)|^2} df = N_0 \int_{-\infty}^{\infty} X_{rc}(f)\, df = N_0 x_{rc}(0) = N_0.$$

$I_n \in \{\pm d\}$ and $X_{rc}(f) = G_T(f)C_\ell(f)G_R(f)$

Signal power

$$P_{av,\ell} = \frac{d^2 \left\| g_T(t) \right\|^2}{T} = \frac{d^2}{T}\int_{-\infty}^{\infty} \frac{X_{rc}(f)}{\left|C_\ell(f)\right|^2}\,df$$

Error probability

$$\begin{aligned} P_b &= Q\left(\sqrt{\frac{d_{12}^2}{4\mathbb{E}[z_k^2]}}\right) \\ &= Q\left(\frac{d}{\sqrt{\mathbb{E}[z_k^2]}}\right) = Q\left(\sqrt{\frac{P_{av,\ell}T}{N_0\left[\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{\left|C_\ell(f)\right|^2}\,df\right]}}\right) \end{aligned}$$

If $c_\ell(t)$ only known to Rx

We can only equalize the "channel effect" at Rx:

$$|G_T(f)| = \sqrt{X_{rc}(f)} \quad \text{and} \quad |G_R(f)| = \frac{\sqrt{X_{rc}(f)}}{|C_\ell(f)|}.$$

Signal power

$$P_{av,\ell} = \frac{d^2}{T}\|g_T(t)\|^2 = \frac{d^2}{T}\int_{-\infty}^{\infty} X_{rc}(f)\,df = \frac{d^2}{T}$$

Noise power

$$\mathbb{E}[z_k^2] = \int_{-\infty}^{\infty} S_{z_\ell}(f)\,df = N_0\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|^2}\,df$$

Error probability

$$P_b = Q\left(\frac{d}{\sqrt{\mathbb{E}[z_k^2]}}\right) = Q\left(\sqrt{\frac{P_{av,\ell}T}{N_0\left[\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|^2}\,df\right]}}\right)$$

If however $c_\ell(t)$ known to both Tx and Rx

We may design:

$$|G_T(f)| = \sqrt{\frac{X_{rc}(f)}{|C_\ell(f)|}} \quad \text{and} \quad |G_R(f)| = \sqrt{\frac{X_{rc}(f)}{|C_\ell(f)|}}$$

Signal power

$$P_{av,\ell} = \frac{d^2}{T}\|g_T(t)\|^2 = \frac{d^2}{T}\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|}\,df$$

Noise power

$$\mathbb{E}[z_k^2] = \int_{-\infty}^{\infty} S_{z_\ell}(f)\,df = N_0\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|}\,df$$

Error probability

$$P_b = Q\left(\frac{d}{\sqrt{\mathbb{E}[z_k^2]}}\right) = Q\left(\sqrt{\frac{P_{av,\ell}\,T}{N_0\left[\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|}\,df\right]^2}}\right)$$

Either Tx or Rx knows $c_\ell(t)$

$$P_{b,T} = P_{b,R} = Q\left(\sqrt{\frac{P_{av,\ell}T}{N_0\left[\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|^2}\,df\right]}}\right)$$

Both Tx and Rx know $c_\ell(t)$

$$P_{b,TR} = Q\left(\sqrt{\frac{P_{av,\ell}T}{N_0\left[\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|}\,df\right]^2}}\right)$$

Note from Cauchy-Schwartz inequality

$$\left[\int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|}\,df\right]^2 = \left|\left\langle\sqrt{X_{rc}(f)}, \frac{\sqrt{X_{rc}(f)}}{|C_\ell(f)|}\right\rangle\right|^2$$

$$\leq \left\|\sqrt{X_{rc}(f)}\right\|^2 \left\|\frac{\sqrt{X_{rc}(f)}}{|C_\ell(f)|}\right\|^2 = \int_{-\infty}^{\infty}\frac{X_{rc}(f)}{|C_\ell(f)|^2}\,df$$

This shows $P_{b,TR} \leq P_{b,T} = P_{b,R}$. "$=$" holds iff $|C_\ell(f)| = 1$.

What you learn from Chapter 9

- Match filter to input pulse shaping and channel impulse response
  - (Good to know) Eye pattern to examine ISI
- Nyquist criterion
  - Sampling rate $<$ channel bandwidth for no ISI (I.e., increasing sampling rate will give more samples for perhaps better performance, but adjacent samples will be eventually "interfered" to each other)
  - Since ISI is unavoidable for high sampling rate, let's accept and face it, and just use controlled ISI.
- A better performance is resulted when both Tx and Rx know the channel.