Digital Communications

Chapter 5 Carrier and Symbol Synchronization

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5.1 Signal parameters estimation

Models

Channel delay τ exists between Tx and Rx.

For AWGN + channel delay τ + carrier phase mismatch ϕ_0, given $s(t)$ transmitted, one receives

$$r(t) = s(t-\tau) + n(t).$$

As lowpass equivalent

$$s(t) = \mathbf{Re}\left[s_\ell(t)e^{\imath 2\pi f_c t}\right] \text{ but } r(t) = \mathbf{Re}\left[r_\ell(t)e^{\imath 2\pi f_c t + \imath\phi_0}\right]$$

we have

$$\begin{aligned} r(t) &= \mathbf{Re}\left[r_\ell(t)e^{\imath 2\pi f_c t + \imath\phi_0}\right] \\ &= \mathbf{Re}\left[s_\ell(t-\tau)e^{\imath 2\pi f_c (t-\tau)}\right] + \mathbf{Re}\left[n_\ell(t)e^{\imath 2\pi f_c t + \imath\phi_0}\right] \\ &= \mathbf{Re}\left\{\left[s_\ell(t-\tau)e^{\imath\phi} + n_\ell(t)\right]e^{\imath 2\pi f_c t + \imath\phi_0}\right\} \end{aligned}$$

where $\phi = -2\pi f_c \tau - \phi_0$.

$$\phi = -2\pi f_c \tau - \phi_0$$

In general, $\tau \ll T$, but

$$\left|-2\pi f_c \tau\right| \mod 2\pi$$

is far from 0 since f_c is large.

So, we should treat τ and ϕ as different random variables

$$r_\ell(t) = s_\ell(t; \phi, \tau) + n_\ell(t).$$

Notably, since the passband signal $s(t-\tau)$ is "real", it does not appear a phase mismatch ϕ for the real passband signals.

In other words, ϕ appears due to the imperfect down-conversion at the receiver.

$$r_\ell(t) \;=\; s_\ell(t;\phi,\tau) + n_\ell(t)$$

Let $\Theta = (\phi, \tau)$, and set

$$r_\ell(t) = s_\ell(t;\Theta) + n_\ell(t).$$

Let $\{\phi_{n,\ell}(t), 1 \le n \le N\}$ be a set of orthonormal functions over $[0, T_0)$, where $T_0 \ge T$, such that $r_{j,\ell} = \langle r_\ell(t), \phi_{n,\ell}(t)\rangle$ and we have a vector representation

$$\boldsymbol{r}_\ell \;=\; \boldsymbol{s}_\ell(\Theta) + \boldsymbol{n}_\ell.$$

Assuming Θ has a joint pdf $f(\Theta)$, the MAP estimate of Θ is

$$\begin{aligned}\hat{\Theta} &= \arg\max_{\Theta} f(\Theta|\boldsymbol{r}_\ell) \\ &= \arg\max_{\Theta} f(\boldsymbol{r}_\ell|\Theta)\frac{f(\Theta)}{f(\boldsymbol{r}_\ell)} = \arg\max_{\Theta} f(\boldsymbol{r}_\ell|\Theta) f(\Theta).\end{aligned}$$

Assume Θ is uniform and holds constant for an observation period of $T_0 \geq T$ (slow variation),

$$\hat{\Theta} = \arg\max_{\Theta} f(\boldsymbol{r}_\ell|\Theta) f(\Theta) = \arg\max_{\Theta} f(\boldsymbol{r}_\ell|\Theta)$$

The latter is the ML estimate of Θ.

Note that $f(\boldsymbol{r}_\ell|\Theta)$ is the likelihood function.

$$\boldsymbol{r}_\ell = \boldsymbol{s}_\ell(\Theta) + \boldsymbol{n}_\ell \quad \text{and} \quad \hat{\Theta} = \arg\max_\Theta f(\boldsymbol{r}_\ell | \Theta)$$

For a fixed $\{\phi_{j,\ell}(t)\}_{j=1}^N$ and $E[|n_{j,\ell}|^2] = \sigma_\ell^2 = 2N_0$,

$$f(\boldsymbol{r}_\ell|\Theta) = \left(\frac{1}{\pi\sigma_\ell^2}\right)^N \exp\left\{-\sum_{n=1}^N \frac{|r_{n,\ell} - s_{n,\ell}(\Theta)|^2}{\sigma_\ell^2}\right\}.$$

Text (5.1-5) uses bandpass analysis and yields

$$f(\boldsymbol{r}|\Theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left\{-\sum_{n=1}^N \frac{|r_n - s_n(\Theta)|^2}{2\sigma^2}\right\} \text{ with } \sigma^2 = \frac{N_0}{2}.$$

We will show later that both analyses yield identical result!

Assume that $\{\phi_{n,\ell}(t)\}_{j=1}^N$ is a complete orthonormal basis. Then

$$\sum_{n=1}^N |r_{n,\ell} - s_{n,\ell}(\Theta)|^2 = \int_0^{T_0} |r_\ell(t) - s_\ell(t;\Theta)|^2 \, dt.$$

The likelihood function

Given $s_\ell(t)$ known to both Tx and Rx, the ML estimate of Θ is

$$\hat{\Theta} = \arg\max_{\Theta} \Lambda(\Theta)$$

and the term

$$\Lambda(\Theta) = \exp\left\{-\frac{1}{\sigma_\ell^2}\int_0^{T_0} \left|r_\ell(t) - s_\ell(t;\Theta)\right|^2 dt\right\}$$

will be referred to as the likelihood function.

Exemplified block diagrams

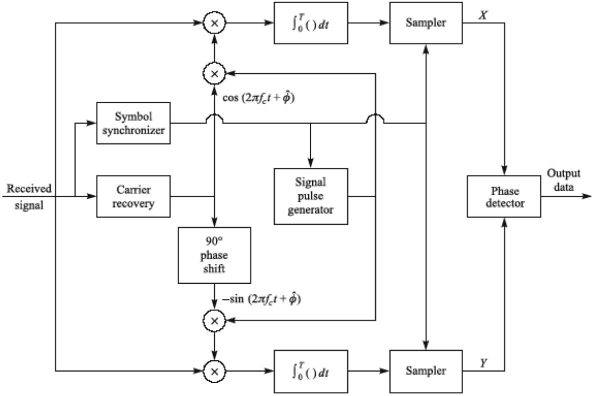


FIGURE 5.1–2
Block diagram of an M-ary PSK receiver.

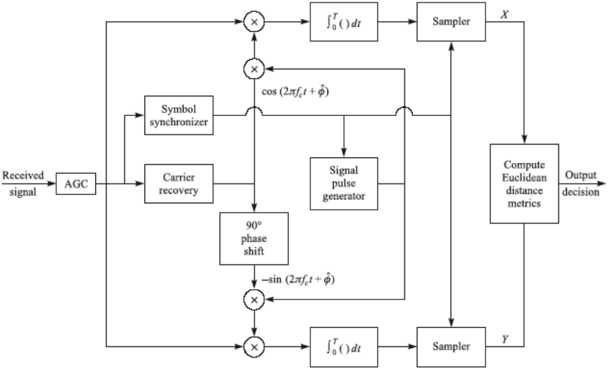


FIGURE 5.1–4
Block diagram of a QAM receiver.

5.2 Carrier phase estimation

Example 1 (DSB-SC signal where $\tau = 0$ and $\phi_0 = \tilde{\phi} - \phi$)

Assume the transmitted signal is

$$s(t) = A(t)\cos(2\pi f_c t + \phi)$$

Rx uses $c(t)$ with carrier reference $\tilde{\phi}$ to demodulate

$$c(t) = \cos\left(2\pi f_c t + \tilde{\phi}\right)$$

So even $n_\ell(t) = 0$, the down-covertion Rx gives

$$LPF\{s(t)c(t)\} = \frac{1}{2}A(t)\cos\left(\phi - \tilde{\phi}\right)$$

Performance is severely degraded due to phase error $(\phi - \tilde{\phi})$.

Hence, signal power is reduced by a factor $\cos^2(\phi - \tilde{\phi})$:

- A phase error of 10° leads to 0.13 dB of signal power loss.
- A phase error of 30° leads to 1.25 dB of signal power loss.

Example 2 (QAM or PSK where $\tau = 0$ and $\phi_0 = \tilde{\phi} - \phi$)

For QAM or PSK signals, Tx sends

$$s(t) = x(t)\cos(2\pi f_c t + \phi) - y(t)\sin(2\pi f_c t + \phi)$$

and Rx uses the $c_I(t)$ and $c_Q(t)$ to demodulate

$$c_I(t) = \cos\left(2\pi f_c t + \tilde{\phi}\right) \qquad c_Q(t) = \sin\left(2\pi f_c t + \tilde{\phi}\right)$$

Hence

$$\begin{aligned} r_I(t) &= LPF\{s(t)c_I(t)\} = \frac{x(t)\cos(\phi - \tilde{\phi}) - y(t)\sin(\phi - \tilde{\phi})}{2} \\ r_Q(t) &= LPF\{s(t)c_Q(t)\} = \frac{x(t)\sin(\phi - \tilde{\phi}) + y(t)\cos(\phi - \tilde{\phi})}{2} \end{aligned}$$

Even worse, both power degradation and crosstalk occur.

5.2-1 ML carrier phase estimation

Assume $\tau = 0$ (or τ has been perfectly compensated) & estimate ϕ.

So $\Theta = \phi$. The likelihood function is

$$\begin{aligned}\Lambda(\phi) &= \exp\left\{-\frac{1}{\sigma_\ell^2}\int_0^{T_0}\left|r_\ell(t) - s_\ell(t;\phi)\right|^2 dt\right\}\\ &= \exp\left(-\frac{1}{\sigma_\ell^2}\int_0^{T_0}\left[\left|r_\ell(t)\right|^2 - 2\mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\} + \left|s_\ell(t;\phi)\right|^2\right]dt\right)\end{aligned}$$

- $|r_\ell(t)|^2$ is irrelevant to the maximization over ϕ.

For the term $|s_\ell(t;\phi)|^2$, we have

$$s_\ell(t;\phi) = s_\ell(t)e^{\imath\phi}.$$

So,

$$\left|s(t,\phi)\right|^2 = \left|s(t)\right|^2$$

Thus

$$\begin{aligned}\hat{\phi} &= \arg\max_{\phi} \exp\left\{\frac{2}{\sigma_\ell^2}\int_0^{T_0} \mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\phi)\right\}\,dt\right\}\\ &= \arg\max_{\phi} \exp\left\{\frac{4}{\sigma_\ell^2}\int_0^{T_0} r(t)s(t;\phi)\,dt\right\}\end{aligned}$$

(With $\sigma_\ell^2 = 2N_0$, this formula is the same as (5.2-8) obtained based on bandpass analysis!)

Note that from Slide 2-24,

$$\langle x(t), y(t)\rangle = \frac{1}{2}\mathbf{Re}\left\{\langle x_\ell(t), y_\ell(t)\rangle\right\}.$$

We will use one of the two above criterions (i.e., baseband or passband) to derive $\hat{\phi}$, depending on whichever is convenient.

Example

Assume

$$s(t) = A\cos(2\pi f_c t) \text{ and } r(t) = \underbrace{A\cos(2\pi f_c t + \phi)}_{s(t;\phi)} + n(t)$$

where ϕ is the unknown phase here.

$$\begin{aligned}\hat{\phi} &= \arg\max_{\phi} \exp\left\{\frac{4}{\sigma_\ell^2}\int_0^{T_0} r(t)s(t;\phi)\,dt\right\} \\ &= \arg\max_{\phi} \int_0^{T_0} r(t)s(t;\phi)\,dt\end{aligned}$$

So we seek $\hat{\phi}$ that minimizes

$$\Lambda_L(\phi) = A\int_0^{T_0} r(t)\cos(2\pi f_c t + \phi)\, dt$$

Since ϕ is continuous, a necessary condition for a minimum is that

$$\left.\frac{d\Lambda_L(\phi)}{d\phi}\right|_{\phi=\hat{\phi}} = 0$$

It yields

$$\begin{aligned}\int_0^{T_0} r(t)\sin(2\pi f_c t + \hat{\phi})\, dt &= 0\\ = \cos(\hat{\phi})\int_0^{T_0} r(t)\sin(2\pi f_c t)\, dt &+ \sin(\hat{\phi})\int_0^{T_0} r(t)\cos(2\pi f_c t)\, dt\end{aligned}$$

$$\hat{\phi} = -\tan^{-1}\frac{\int_0^{T_0} r(t)\sin(2\pi f_c t)\, dt}{\int_0^{T_0} r(t)\cos(2\pi f_c t)\, dt}$$

Performance check

Recall

$$r(t) = A\cos(2\pi f_c t + \phi) + n(t)$$

with unknown phase ϕ.

$$\mathbb{E}\left[\int_0^{T_0} r(t)\sin(2\pi f_c t)\,dt\right] = -\frac{1}{2}AT_0\sin(\phi)$$

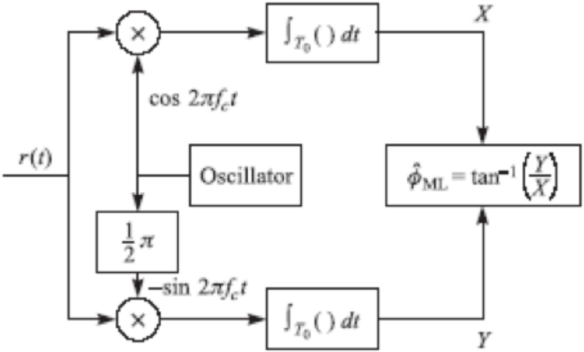
$$\mathbb{E}\left[\int_0^{T_0} r(t)\cos(2\pi f_c t)\,dt\right] = \frac{1}{2}AT_0\cos(\phi)$$

Then on the average

$$\hat{\phi} = -\tan^{-1}\frac{\mathbb{E}\left[\int_0^{T_0} r(t)\sin(2\pi f_c t)\,dt\right]}{\mathbb{E}\left[\int_0^{T_0} r(t)\cos(2\pi f_c t)\,dt\right]} = \phi$$

which is the true phase.

A one-shot ML phase estimator



5.2-2 The phase-locked loops

Instead of one-shot estimate, a phase-locked loop continuously adjusts ϕ to achieve

$$\int_0^{T_0} r(t)\sin(2\pi f_c t + \hat{\phi})\,dt = 0$$

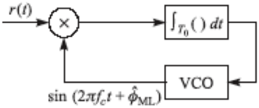


FIGURE 5.2–1
A PLL for obtaining the ML estimate of the phase of an unmodulated carrier.

We can then change the sign of sine function to facilitate the follow-up analysis.

$$\int_0^{T_0} r(t)\left(-\sin(2\pi f_c t + \hat{\phi})\right)\,dt = 0$$

The analysis of the PLL can be visioned in a simplified basic diagram:

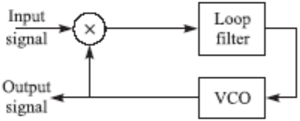
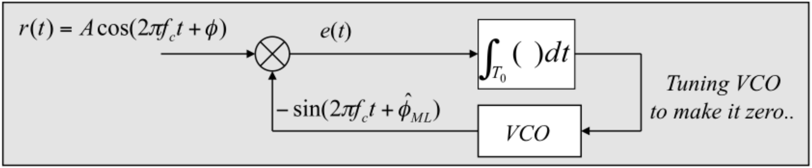
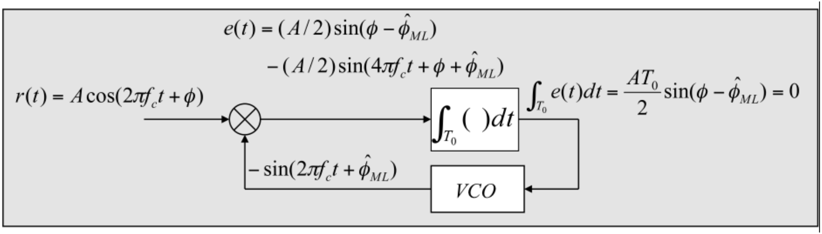


FIGURE 5.2–3
Basic elements of a phase-locked loop (PLL).

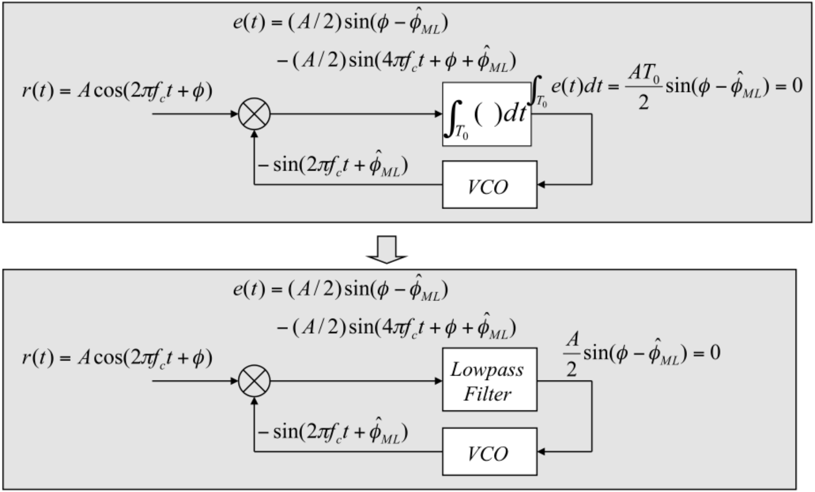




If T_0 is a multiple of $1/(2f_c)$, then

$$\begin{aligned}
&\int_0^{T_0} e(t)dt \\
&= \frac{A}{2}\int_0^{T_0} \sin(\phi - \hat{\phi}_{\text{ML}})dt - \frac{A}{2}\int_0^{T_0} \sin(4\pi f_c t + \phi + \hat{\phi}_{\text{ML}})dt \\
&= \frac{AT_0}{2}\sin(\phi - \hat{\phi}_{\text{ML}}) - 0
\end{aligned}$$

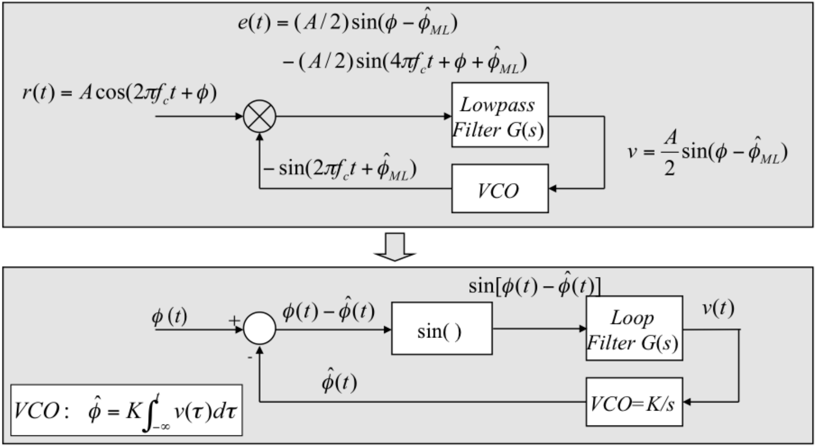
- The effect of integration is similar to a lowpass filter.


$e(t) = (A/2)\sin(\phi - \hat{\phi}_{ML})$
$-(A/2)\sin(4\pi f_c t + \phi + \hat{\phi}_{ML})$
$r(t) = A\cos(2\pi f_c t + \phi)$
$\int_{T_0} e(t)dt = \frac{AT_0}{2}\sin(\phi - \hat{\phi}_{ML}) = 0$
$\int_{T_0} (\ )dt$
$-\sin(2\pi f_c t + \hat{\phi}_{ML})$
VCO
$e(t) = (A/2)\sin(\phi - \hat{\phi}_{ML})$
$-(A/2)\sin(4\pi f_c t + \phi + \hat{\phi}_{ML})$
$r(t) = A\cos(2\pi f_c t + \phi)$
$\frac{A}{2}\sin(\phi - \hat{\phi}_{ML}) = 0$
Lowpass Filter
$-\sin(2\pi f_c t + \hat{\phi}_{ML})$
VCO

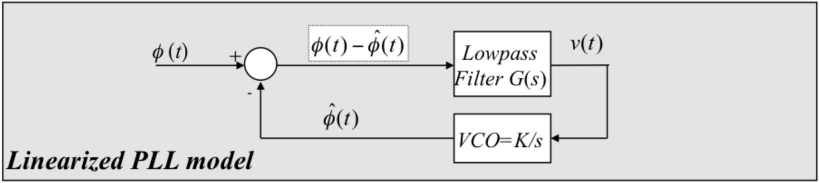
- Effective VCO output $\hat{\phi}(t)$ can be modeled as

$$\hat{\phi}(t) = K \int_{-\infty}^{t} v(\tau) d\tau$$

where $v(\cdot)$ is the input of the VCO.



The nonlinear $\sin(\cdot)$ causes difficulty in analysis. Hence, we may simplify it using $\sin(x) \approx x$.



In terms of Laplacian transform technique, we then derive the close-loop system transfer function.

$$
\begin{aligned}
H(s) &= \frac{\hat{\phi}(s)}{\phi(s)} = \frac{\hat{\phi}(s)}{[\phi(s)-\hat{\phi}(s)]+\hat{\phi}(s)} \\
&= \frac{(KG(s)/s)[\phi(s)-\hat{\phi}(s)]}{[\phi(s)-\hat{\phi}(s)]+(KG(s)/s)[\phi(s)-\hat{\phi}(s)]} \\
&= \frac{KG(s)/s}{1+KG(s)/s}
\end{aligned}
$$

Second-order loop transfer function

$$G(s) = \frac{1+\tau_2 s}{1+\tau_1 s},$$ where $\tau_1 \gg \tau_2$ for a lowpass filter.

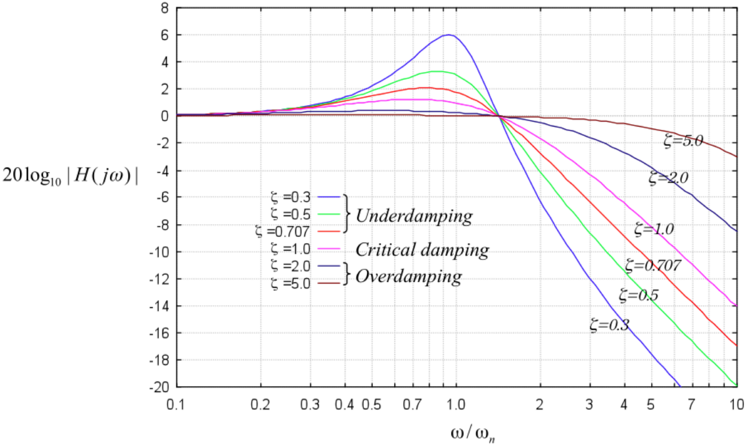
$$\Longrightarrow \quad H(s) = \frac{\overbrace{(2\zeta - \omega_n/K)}^{=\tau_2\omega_n}(s/\omega_n)+1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n)+1}$$

where

$$\begin{cases} \omega_n = \sqrt{K/\tau_1} & \text{natural frequency of the loop} \\ \zeta = \omega_n(\tau_2 + 1/K)/2 & \text{loop damping factor} \end{cases}$$

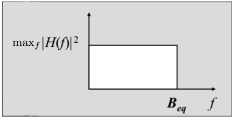
Assume $2\zeta - \frac{\omega_n}{K} \approx 2\zeta$ (i.e., $\frac{\omega_n/K}{2\zeta} = \frac{1}{K\tau_2+1} \ll 1$). Hence,

$$H(s) \approx \frac{2\zeta(s/\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}.$$



- Damping factor=response speed (for changes)

Noise-equivalent bandwidth of $H(f)$



- Definition: One-sided noise-equivalent bandwidth of $H(f)$

$$B_{\text{eq}} = \frac{1}{\max_f |H(f)|^2} \int_0^\infty |H(f)|^2 df = \frac{1 + (\tau_2 \omega_n)^2}{8\zeta/\omega_n} \approx \frac{1 + 4\zeta^2}{8\zeta}\omega_n$$

where we use $\tau_2 \omega_n = 2\zeta - \frac{\omega_n}{K} \approx 2\zeta$.

- Tradeoff in parameter selection in PLL
    - It is desirable to have a larger PLL bandwidth ω_n in order to track any time variation in the phase of the received carrier.
    - However, with a larger PLL bandwidth, more noise will be passed into the loop; hence, the phase estimate is less accurate.

5.2-3 Effect of additive noise on the phase estimate

Assume that

$$r(t) = A_c \cos(2\pi f_c t + \phi(t)) + n(t)$$

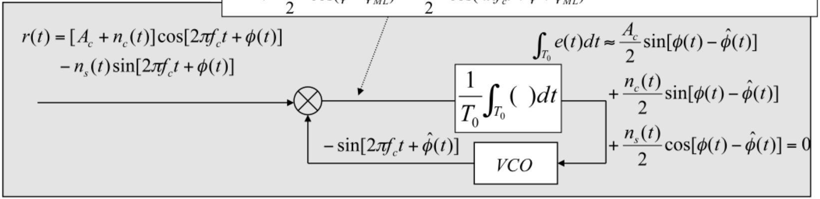
where

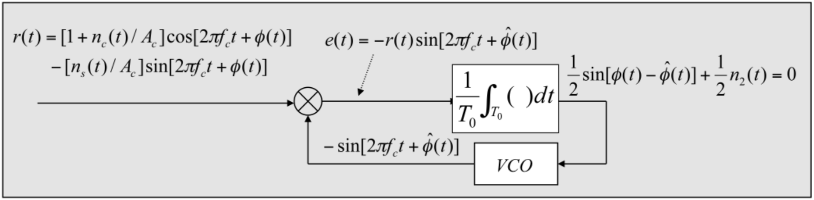
$$n(t) = n_c(t)\cos(2\pi f_c t + \phi(t)) - n_s(t)\sin(2\pi f_c t + \phi(t))$$

and $n_c(t)$ and $n_s(t)$ are independent Gaussian random processes.

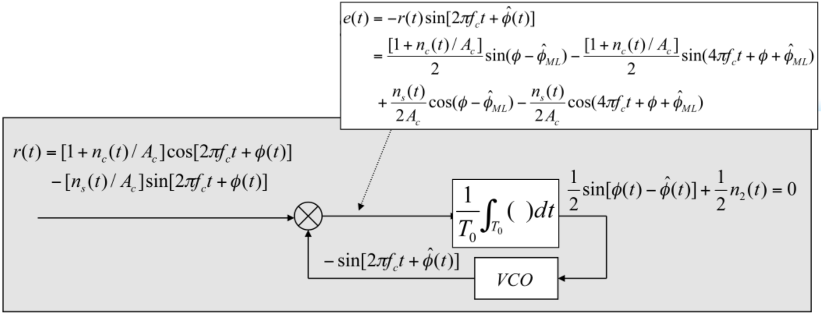
For convenience, we abbreviate $\phi(t)$ and $\hat{\phi}(t)$ as ϕ and ϕ_{ML} in the derivation.

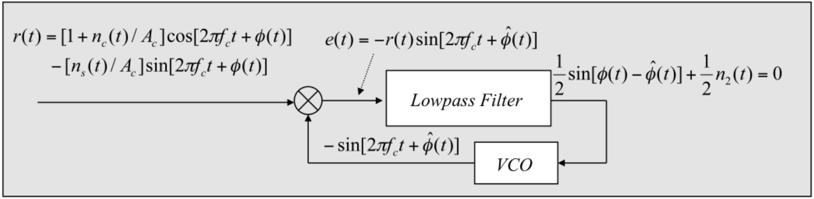
$$
\begin{aligned}
e(t) &= -r(t)\sin[2\pi f_c t + \hat{\phi}(t)] \\
&= -[A_c + n_c(t)]\cos(2\pi f_c t + \phi)\sin(2\pi f_c t + \hat{\phi}_{ML}) + n_s(t)\sin(2\pi f_c t + \phi)\sin(2\pi f_c t + \hat{\phi}_{ML}) \\
&= \frac{[A_c + n_c(t)]}{2}\sin(\phi - \hat{\phi}_{ML}) - \frac{[A_c + n_c(t)]}{2}\sin(4\pi f_c t + \phi + \hat{\phi}_{ML}) \\
&\quad + \frac{n_s(t)}{2}\cos(\phi - \hat{\phi}_{ML}) - \frac{n_s(t)}{2}\cos(4\pi f_c t + \phi + \hat{\phi}_{ML})
\end{aligned}
$$

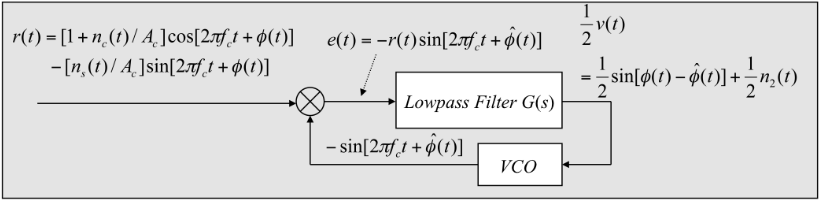




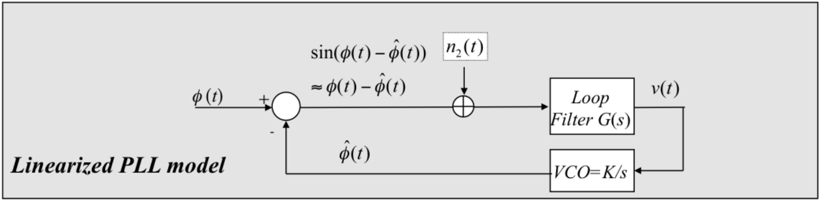
Let $n_2(t) = \frac{1}{A_c}(n_c(t)\sin[\phi(t) - \hat{\phi}(t)] + n_s(t)\cos[\phi(t) - \hat{\phi}(t)])$.


$e(t) = -r(t)\sin[2\pi f_c t + \hat{\phi}(t)]$
$= \frac{[1+n_c(t)/A_c]}{2}\sin(\phi - \hat{\phi}_{ML}) - \frac{[1+n_c(t)/A_c]}{2}\sin(4\pi f_c t + \phi + \hat{\phi}_{ML})$
$+ \frac{n_s(t)}{2A_c}\cos(\phi - \hat{\phi}_{ML}) - \frac{n_s(t)}{2A_c}\cos(4\pi f_c t + \phi + \hat{\phi}_{ML})$
$r(t) = [1+n_c(t)/A_c]\cos[2\pi f_c t + \phi(t)]$
$-[n_s(t)/A_c]\sin[2\pi f_c t + \phi(t)]$
$\frac{1}{T_0}\int_{T_0}(\ )dt$
$\frac{1}{2}\sin[\phi(t) - \hat{\phi}(t)] + \frac{1}{2}n_2(t) = 0$
$-\sin[2\pi f_c t + \hat{\phi}(t)]$
VCO


$r(t) = [1+n_c(t)/A_c]\cos[2\pi f_c t + \phi(t)]$
$-[n_s(t)/A_c]\sin[2\pi f_c t + \phi(t)]$
$e(t) = -r(t)\sin[2\pi f_c t + \hat{\phi}(t)]$
Lowpass Filter
$\frac{1}{2}\sin[\phi(t) - \hat{\phi}(t)] + \frac{1}{2}n_2(t) = 0$
$-\sin[2\pi f_c t + \hat{\phi}(t)]$
VCO



|| $n_2(t)$ **white** with $\Phi_{n_1}(f) = \dfrac{N_0}{2A_c^2}$.



As a result,

$$
\begin{aligned}
&\frac{\hat{\phi}(s)}{\phi(s)+n_2(s)} \\
&= \frac{\hat{\phi}(s)}{[\phi(s)-\hat{\phi}(s)+n_2(s)]+\hat{\phi}(s)} \\
&= \frac{(KG(s)/s)[(\phi(s)-\hat{\phi}(s))+n_2(s)]}{[\phi(s)-\hat{\phi}(s)+n_2(s)]+(KG(s)/s)[(\phi(s)-\hat{\phi}(s))+n_2(s)]} \\
&= \frac{KG(s)/s}{1+KG(s)/s} = H(s)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \hat{\phi}(s) = [\phi(s)+n_2(s)]H(s) \\
\Rightarrow \quad & \hat{\phi}(t) = [\phi(t)+n_2(t)]\star h(t) = \phi(t)\star h(t) + \underbrace{n_2(t)\star h(t)}_{\text{noise}}
\end{aligned}
$$

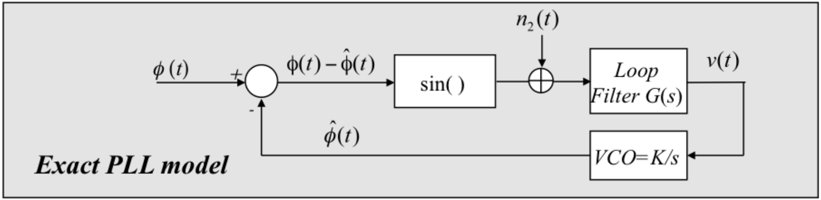
Let's calculate noise variance:

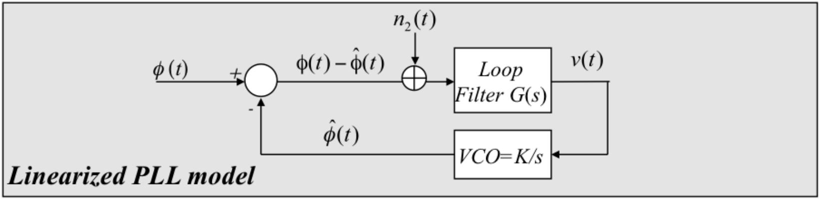
$$\begin{aligned}\sigma^2_{\hat{\phi}} &= \int_{-\infty}^{\infty} \Phi_{n_2}(f)|H(f)|^2 df \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2}\frac{1}{A_c^2}|H(f)|^2 df \\ &= \frac{N_0 B_{\text{eq}}}{A_c^2}\max_f |H(f)|^2\end{aligned}$$

$\sigma^2_{\hat{\phi}}$ is proportional to B_{eq}; since the signal power is fixed as 1/2, SNR is inversely proportional to B_{eq}.

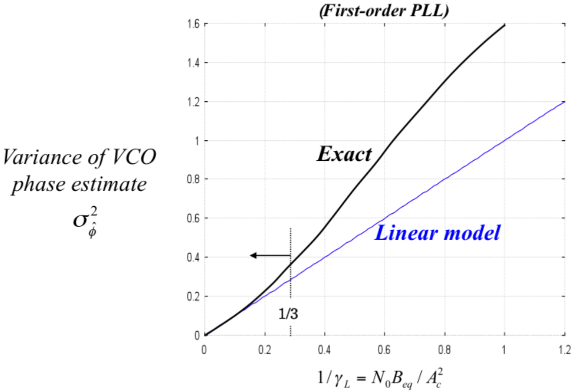
Subject to that the bandwidth B_{eq} of the "equivalent (ideal) filter" is large enough to pass all the input power, the "signal power" is equal to $\int_{-\infty}^{\infty} S_\phi(f)|H(f)|^2 df = \max_f |H(f)|^2 \cdot \int_{-\infty}^{\infty} S_\phi(f) df$; hence, SNR γ_L is proportional to $\frac{A_c^2}{N_0 B_{\text{eq}}}$.

Exact PLL model versus linearlized PLL model





It turns out that when $G(s) = 1$, the $\sigma^2_{\hat{\phi}}$ of the exact PLL model is tractable (Vitebi 1966). The linear model gives $\sigma^2_{\hat{\phi}} = 1/\gamma_L = \frac{N_0 B_{eq}}{A_c^2}$.



- The linear model well approximates the exact model when $\gamma_L = A_c^2/(N_0 B_{eq}) > 3 \approx 4.77$ dB.

5.2-4 Decision directed loops

For general modulation scheme, let

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT);$$

then

$$s(t;\phi) = \mathbf{Re}\left\{s_\ell(t)e^{\imath\phi}e^{\imath 2\pi f_c t}\right\}$$

Hence, by letting $T_0 = KT$,

$$\begin{aligned}\Lambda_L(\phi) &= \int_0^{T_0} r(t)s(t;\phi)\,dt \\ &= \int_0^{T_0} r(t)\mathbf{Re}\left\{\sum_{n=-\infty}^{\infty} I_n g(t-nT)e^{\imath\phi}e^{\imath 2\pi f_c t}\right\} dt \\ &= \int_0^{T_0} r(t)\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n g(t-nT)e^{\imath\phi}e^{\imath 2\pi f_c t}\right\} dt\end{aligned}$$

$$\Lambda_L(\phi) = \int_0^{T_0} r(t)\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n g(t-nT)e^{\imath\phi}e^{\imath 2\pi f_c t}\right\} dt$$

$$\begin{aligned}
\Lambda_L(\phi) &= \mathbf{Re}\left\{e^{\imath\phi}\sum_{n=0}^{K-1} I_n \int_0^{T_0} r(t)g(t-nT)e^{\imath 2\pi f_c t}\,dt\right\} \\
&= \mathbf{Re}\left\{e^{\imath\phi}\sum_{n=0}^{K-1} I_n \underbrace{\int_{nT}^{(n+1)T} r(t)g(t-nT)e^{\imath 2\pi f_c t}\,dt}_{y_n}\right\} \\
&= \mathbf{Re}\left\{e^{\imath\phi}\sum_{n=0}^{K-1} I_n y_n\right\} \\
&= \mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}\cos(\phi) - \mathbf{Im}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}\sin(\phi)
\end{aligned}$$

$$\Lambda_L(\phi) = \mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}\cos(\phi) - \mathbf{Im}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}\sin(\phi)$$

Now

$$\frac{d\Lambda_L(\phi)}{d\phi} = -\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}\sin(\phi) - \mathbf{Im}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}\cos(\phi)$$

and the optimal estimate $\hat{\phi}$ is given by

$$\hat{\phi} = -\tan^{-1}\left(\frac{\mathbf{Im}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}}{\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}}\right)$$

This is called decision directed estimation of ϕ.

Note that from Slide 2-24,

$$\langle x(t), y(t)\rangle = \frac{1}{2}\mathbf{Re}\left\{\langle x_\ell(t), y_\ell(t)\rangle\right\}.$$

Hence,

$$\begin{aligned}
\mathbf{Re}\{I_n y_n\} &= \int_{nT}^{(n+1)T} r(t)\cdot \mathbf{Re}\left\{I_n g(t-nT)e^{\imath 2\pi f_c t}\right\} dt \\
&= \frac{1}{2}\mathbf{Re}\left\{\int_{nT}^{(n+1)T} r_\ell(t) I_n^* g^*(t-nT)\, dt\right\} \\
&= \frac{1}{2}\mathbf{Re}\left\{I_n^* \underbrace{\int_{nT}^{(n+1)T} r_\ell(t) g^*(t-nT)\, dt}_{y_{n,\ell}}\right\} \\
&= \frac{1}{2}\mathbf{Re}\left\{I_n^* y_{n,\ell}\right\}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{Im}\{I_n y_n\} &= \int_{nT}^{(n+1)T} r(t)\cdot \mathbf{Im}\left\{I_n g(t-nT)e^{\imath 2\pi f_c t}\right\} dt \\
&= \int_{nT}^{(n+1)T} r(t)\cdot \mathbf{Re}\left\{(-\imath) I_n g(t-nT)e^{\imath 2\pi f_c t}\right\} dt \\
&= \frac{1}{2}\mathbf{Re}\left\{\int_{nT}^{(n+1)T} r_\ell(t)\cdot \imath\, I_n^* g^*(t-nT)\, dt\right\} \\
&= -\frac{1}{2}\mathbf{Im}\left\{I_n^* y_{n,\ell}\right\}
\end{aligned}
$$

$$
\hat{\phi} = \tan^{-1}\left(\frac{\mathbf{Im}\left\{\sum_{n=0}^{K-1} I_n^* y_{n,\ell}\right\}}{\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n^* y_{n,\ell}\right\}}\right)
$$

Final note: The formula (5.2-38) in text has an extra "−" sign because the text (inconsistently to (5.1-2)) assumes

$s(t;\phi) = \mathbf{Re}\left\{s_\ell(t)e^{-\imath\phi}e^{\imath 2\pi f_c t}\right\}$; but we assume

$s(t;\phi) = \mathbf{Re}\left\{s_\ell(t)e^{+\imath\phi}e^{\imath 2\pi f_c t}\right\}$ as (5.1-2) did.

5.2-5 Non-decision-directed loops

For carrier phase estimation with $\sigma_\ell^2 = 2N_0$, we have shown that

$$\begin{aligned}\hat{\phi} &= \arg\max_{\phi} \exp\left\{\frac{2}{N_0}\int_0^{T_0} r(t)s(t;\phi)\,dt\right\} \\ &= -\tan^{-1}\left(\frac{\mathbf{Im}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}}{\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n y_n\right\}}\right)\end{aligned}$$

When $\{I_n\}_{n=0}^{K-1}$ is unavailable, we take the expectation with respect to $\{I_n\}_{n=0}^{K-1}$ instead:

$$\begin{aligned}\hat{\phi} &= \arg\max_{\phi} \mathbb{E}\left[\exp\left\{\frac{2}{N_0}\int_0^{T_0} r(t)s(t;\phi)\,dt\right\}\right] \\ &= \arg\max_{\phi} \mathbb{E}\left[\exp\left\{\frac{2}{N_0}\int_0^{T_0} r(t)\mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n g(t-nT)e^{\imath\phi}e^{\imath 2\pi f_c t}\right\}\,dt\right\}\right] \\ &= \arg\max_{\phi} \mathbb{E}\left[\exp\left\{\frac{2}{N_0}\sum_{n=0}^{K-1} I_n y_n(\phi)\right\}\right]\end{aligned}$$

where we assume both $\{I_n\}$ and $g(t)$ are real and

$y_n(\phi) = \int_{nT}^{(n+1)T} r(t)g(t-nT)\cos(2\pi f_c t + \phi)dt.$

If $\{I_n\}$ i.i.d. and equal-probable over $\{-1, 1\}$,

$$\begin{aligned}
\hat{\phi} &= \arg\max_{\phi} \prod_{n=0}^{K-1} \mathbb{E}\left[\exp\left\{\frac{2}{N_0} I_n y_n(\phi)\right\}\right] \\
&= \arg\max_{\phi} \prod_{n=0}^{K-1} \left(\exp\left\{-\frac{2}{N_0} y_n(\phi)\right\} + \exp\left\{\frac{2}{N_0} y_n(\phi)\right\}\right) \\
&= \arg\max_{\phi} \prod_{n=0}^{K-1} \cosh\left(\frac{2}{N_0} y_n(\phi)\right) \\
&= \arg\max_{\phi} \sum_{n=0}^{K-1} \log\cosh\left(\frac{2}{N_0} y_n(\phi)\right)
\end{aligned}$$

We may then determine the optimal $\hat{\phi}$ by deriving

$$\frac{\partial \sum_{n=0}^{K-1} \log\cosh\left(\frac{2}{N_0} y_n(\phi)\right)}{\partial \phi} = 0.$$

- For $|x| \ll 1$ (low SNR), $\log\cosh(x) \approx \frac{x^2}{2}$ (By Taylor exaponsion).
- For $|x| \gg 1$ (high SNR), $\log\cosh(x) \approx |x|$.

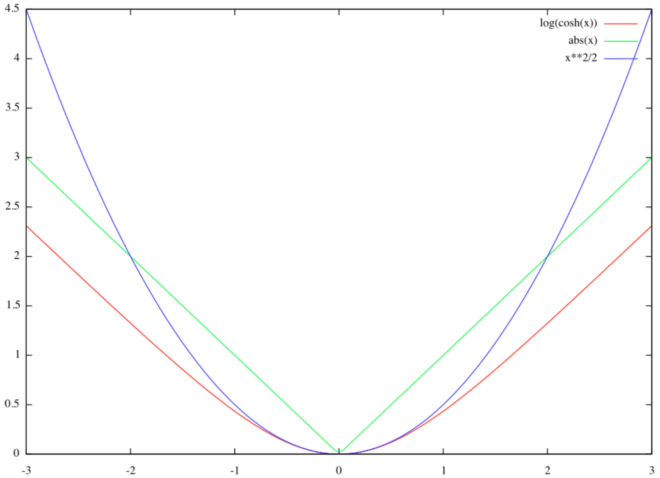
$$
\begin{aligned}
\hat{\phi} &= \arg\max_{\phi} \sum_{n=0}^{K-1} \log\cosh\left(\frac{2}{N_0} y_n(\phi)\right) \\
&\approx \begin{cases} \arg\max_{\phi} \sum_{n=0}^{K-1} \frac{2}{N_0^2} y_n^2(\phi) & N_0 \text{ large} \\ \arg\max_{\phi} \sum_{n=0}^{K-1} \frac{2}{N_0} |y_n(\phi)| & N_0 \text{ small} \end{cases} \\
&= \begin{cases} \arg\max_{\phi} \sum_{n=0}^{K-1} y_n^2(\phi) & N_0 \text{ large} \\ \arg\max_{\phi} \sum_{n=0}^{K-1} |y_n(\phi)| & N_0 \text{ small} \end{cases}
\end{aligned}
$$

When x small,

$$\begin{aligned}\log(\cosh(x)) &= \log \frac{e^{-x}+e^{x}}{2} \\ &= \log \frac{\left[1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+O(x^4)\right]+\left[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+O(x^4)\right]}{2} \\ &= \log\left(1+\frac{1}{2}x^2+O(x^4)\right) \\ &= \frac{1}{2}x^2+O(x^4)\end{aligned}$$

and

$$\lim_{x\to\infty}\frac{\log(\cosh(x))}{x}=\lim_{x\to\infty}\tanh(x)=\lim_{x\to\infty}\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=1.$$


log(cosh(x))
abs(x)
x**2/2
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-3
-2
-1
0
1
2
3

When $K = 1$, which covers the case of Example 5.2-2 (p. 308) in text, the optimal decision becomes irrelevant to N_0:

$$
\begin{aligned}
\hat{\phi} &= \begin{cases} \arg\max_\phi y_0^2(\phi) & N_0 \text{ large} \\ \arg\max_\phi |y_0(\phi)| & N_0 \text{ small} \end{cases} \\
&= \arg\max_\phi \left| \int_0^T r(t)g(t)\cos(2\pi f_c t + \phi)\,dt \right| \\
&= \arg\max_\phi \left| \cos(\phi) \int_0^T r(t)g(t)\cos(2\pi f_c t)\,dt \right. \\
&\qquad\qquad \left. - \sin(\phi) \int_0^T r(t)g(t)\sin(2\pi f_c t)\,dt \right| \\
&= \arg\max_\phi |\cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)| = \arg\max_\phi |\cos(\phi + \theta)|
\end{aligned}
$$

where $\tan(\theta) = \frac{\int_0^T r(t)g(t)\sin(2\pi f_c t)dt}{\int_0^T r(t)g(t)\cos(2\pi f_c t)dt}$. So the optimal $\hat{\phi}$ should make

$$
\hat{\phi} = -\theta = -\tan^{-1} \frac{\int_0^T r(t)g(t)\sin(2\pi f_c t)\,dt}{\int_0^T r(t)g(t)\cos(2\pi f_c t)\,dt}.
$$

5.3 Symbol timing estimation

Assume $\phi = 0$ (or ϕ has been perfectly compensated) & estimate τ.

In such case,

$$r_\ell(t) = s_\ell(t;\tau) + n_\ell(t) = s_\ell(t-\tau) + n_\ell(t).$$

We could rewrite the likelihood function (cf. Slide 5-8 with $\sigma_\ell^2 = 2N_0$) as

$$\begin{aligned}\Lambda(\tau) &= \exp\left\{-\frac{1}{2N_0}\int_0^{T_0} |r_\ell(t) - s_\ell(t;\tau)|^2\, dt\right\} \\ &= \exp\left(-\frac{1}{2N_0}\int_0^{T_0} \left[|r_\ell(t)|^2 - 2\mathbf{Re}\left\{r_\ell(t)s_\ell^*(t;\tau)\right\} + |s_\ell(t;\tau)|^2\right] dt\right)\end{aligned}$$

Same as before, the 1st term is independent of τ and can be ignored. But

$$\int_0^{T_0} |s_\ell(t;\tau)|^2 dt = \int_0^{T_0} |s_\ell(t-\tau)|^2 dt$$

could be a function of τ.

So, we would "say" when $\tau \ll T$, the 3rd term is nearly independent of τ and can also be ignored.

This gives:

$$\Lambda_L(\tau) = \mathbf{Re}\left\{\int_0^{T_0} r_\ell(t)s_\ell^*(t;\tau)dt\right\}$$

Assume

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$$

Then as $\tau \ll T$,

$$\begin{aligned}\Lambda_L(\tau) &\approx \mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n^* \int_0^{T_0} r_\ell(t)g^*(t-nT-\tau)dt\right\} \\ &= \mathbf{Re}\left\{\sum_{n=0}^{K-1} I_n^* \tilde{y}_{n,\ell}(\tau)\right\}\end{aligned}$$

where

$$\tilde{y}_{n,\ell}(\tau) = \int_0^{T_0} r_\ell(t)g^*(t-nT-\tau)dt$$

Decision directed estimation

- The text assumes that both $\{I_n\}$ and $g(t)$ are real; hence $r_\ell(t)$ can be made real by eliminating the complex part, and

$$\Lambda_L(\tau) = \sum_{n=0}^{K-1} I_n y_{n,\ell}(\tau)$$

where

$$y_{n,\ell}(\tau) = \int_0^{T_0} \mathbf{Re}\{r_\ell(t)\} g(t - nT - \tau) dt.$$

- Then the optimal decision is the $\hat{\tau}$ such that

$$\frac{d\Lambda_L(\tau)}{d\tau} = \sum_{n=0}^{K-1} I_n \frac{dy_{n,\ell}(\tau)}{d\tau} = 0$$

- Likewise, it is called **decision directed estimation**.

Non-decision directed estimation

Consider again the case of BPSK, i.e. $I_n = \pm 1$ equal-probable; then because the complex noise can be excluded, we have

$$\begin{aligned}
\Lambda(\tau) &= \exp\left(\frac{1}{N_0}\int_0^{T_0} \mathbf{Re}\{r_\ell(t)\} s_\ell(t;\tau)\,dt\right) \\
&= \exp\left(\frac{1}{N_0}\sum_{n=0}^{K-1} I_n \int_0^{T_0} \mathbf{Re}\{r_\ell(t)\} g(t-nT-\tau)\,dt\right) \\
&= \prod_{n=0}^{K-1} \exp\left(\frac{1}{N_0} I_n y_{n,\ell}(\tau)\right)
\end{aligned}$$

$$\bar{\Lambda}(\tau) = \mathbb{E}\left[\Lambda(\tau)\right] = \prod_{n=0}^{K-1} \cosh\left(\frac{1}{N_0} y_{n,\ell}(\tau)\right)$$

Thus

$$\log \bar{\Lambda}(\tau) = \sum_{n=0}^{K-1} \log\cosh\left(\frac{1}{N_0} y_{n,\ell}(\tau)\right)$$

For low to moderate SNR, people simplify $\log\cosh(x)$ to $\frac{1}{2}x^2$,

$$\log \bar{\Lambda}(\tau) \;\approx\; \frac{1}{2N_0^2} \sum_{n=0}^{K-1} y_{n,\ell}^2(\tau).$$

Taking derivative, we see a necessary condition for $\hat{\tau}$ is

$$N_0^2 \frac{d \log \bar{\Lambda}(\tau)}{d\tau}\bigg|_{\tau=\hat{\tau}} = \sum_{n=0}^{K-1} y_{n,\ell}(\hat{\tau})\; y'_{n,\ell}(\hat{\tau}) = \sum_{n=0}^{K-1} y_{n,\ell}^2(\hat{\tau}) \frac{y'_{n,\ell}(\hat{\tau})}{y_{n,\ell}(\hat{\tau})} = 0$$

For high SNR, people simplify $\log\cosh(x)$ to $|x|$,

$$\log \bar{\Lambda}(\tau) \approx \frac{1}{N_0}\sum_{n=0}^{K-1} |y_{n,\ell}(\tau)|.$$

Taking derivative, we see a necessary condition for $\hat{\tau}$ is

$$N_0 \frac{d\log\bar{\Lambda}(\tau)}{d\tau}\bigg|_{\tau=\hat{\tau}} = \sum_{n=0}^{K-1} \text{sgn}(y_{n,\ell}(\hat{\tau}))\ y'_{n,\ell}(\hat{\tau}) = \sum_{n=0}^{K-1} |y_{n,\ell}(\hat{\tau})| \frac{y'_{n,\ell}(\hat{\tau})}{y_{n,\ell}(\hat{\tau})} = 0$$

Note that here, we use

$$\frac{\partial |f(x)|}{\partial x} = \begin{Bmatrix} f'(x), & f(x) > 0 \\ -f'(x), & f(x) < 0 \end{Bmatrix} = \text{sgn}(f(x)) f'(x).$$

5.4 Joint estimation of carrier phase and symbol timing

The likelihood function is

$$\Lambda(\phi, \tau) = \exp\left\{-\frac{1}{2N_0}\int_0^{T_0} \left|r_\ell(t) - s_\ell(t;\tau,\phi)\right|^2 dt\right\}$$

Assuming

$$s_\ell(t) = \sum_{n=-\infty}^{\infty} \left(I_n g(t - nT - \tau) + \imath J_n w(t - nT - \tau)\right)$$

we have

$$s_\ell(t;\phi,\tau) = \sum_{n=-\infty}^{\infty} \left(I_n g(t - nT - \tau) + \imath J_n w(t - nT - \tau)\right) e^{-\imath\phi}$$

Here, I use $e^{-\imath\phi}$ in order to "synchronize" with the textbook.

- PAM: I_n real and $J_n = 0$
- QAM and PSK: I_n complex and $J_n = 0$
- OQPSK: $w(t) = g(t - T/2)$

Along similar technique used before, we rewrite $\Lambda(\phi,\tau)$ as

$$
\begin{aligned}
&\log\Lambda(\phi,\tau)\\
&= \mathbf{Re}\left\{\int_0^{T_0} r_\ell(t)s_\ell^*(t;\phi,\tau)\,dt\right\}\\
&= \mathbf{Re}\left\{e^{\imath\phi}\sum_{n=0}^{K-1}\int_0^{T_0} r_\ell(t)\left(I_n^* g^*(t-nT-\tau)-\imath J_n^* w^*(t-nT-\tau)\right)dt\right\}\\
&= \mathbf{Re}\left\{e^{\imath\phi}\sum_{n=0}^{K-1}\left(I_n^* y_{n,\ell}(\tau)-\imath J_n^* x_{n,\ell}(\tau)\right)\right\}\\
&= \mathbf{Re}\left\{e^{\imath\phi}\left(A(\tau)+\imath B(\tau)\right)\right\} = A(\tau)\cos(\phi)-B(\tau)\sin(\phi)
\end{aligned}
$$

where

$$
\begin{cases}
y_{n,\ell}(\tau) = \int_0^{T_0} r_\ell(t)g^*(t-nT-\tau)dt\\
x_{n,\ell}(\tau) = \int_0^{T_0} r_\ell(t)w^*(t-nT-\tau)dt\\
A(\tau)+\imath B(\tau) = \sum_{n=0}^{K-1}\left(I_n^* y_{n,\ell}(\tau)-\imath J_n^* x_{n,\ell}(\tau)\right)
\end{cases}
$$

The necessary conditions for $\hat{\phi}$ and $\hat{\tau}$ are

$$\left.\frac{\partial \log \Lambda(\phi,\tau)}{\partial \tau}\right|_{\tau=\hat{\tau}} = 0 \quad \text{and} \quad \left.\frac{\partial \log \Lambda(\phi,\tau)}{\partial \phi}\right|_{\phi=\hat{\phi}} = 0$$

Finally solving jointly the above two equations, we have the optimal estimates given by

$$\hat{\tau} \text{ satisfies } A(\tau)\frac{\partial A(\tau)}{\partial \tau} + B(\tau)\frac{\partial B(\tau)}{\partial \tau} = 0$$

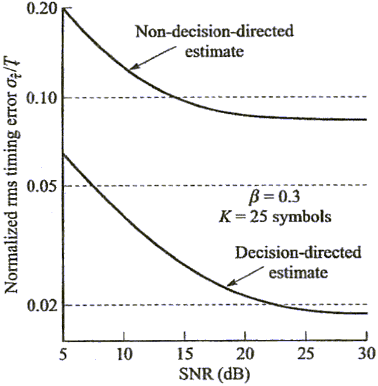
$$\hat{\phi} = -\tan^{-1}\frac{B(\hat{\tau})}{A(\hat{\tau})}$$

5.5 Performance characteristics of ML estimators

Comparison between decision-directed (DD) and non-decision-directed (NDD) estimators

Comparison between symbol timing (i.e., τ) DD and NDD estimates with raised-cosine(-spectrum) pulse shape.

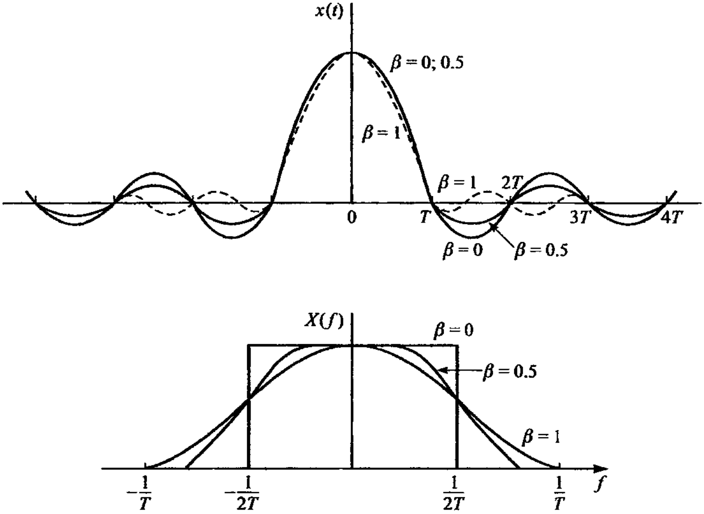
- β is a parameter of the raised-cosine pulse



Raise-cosine spectrum

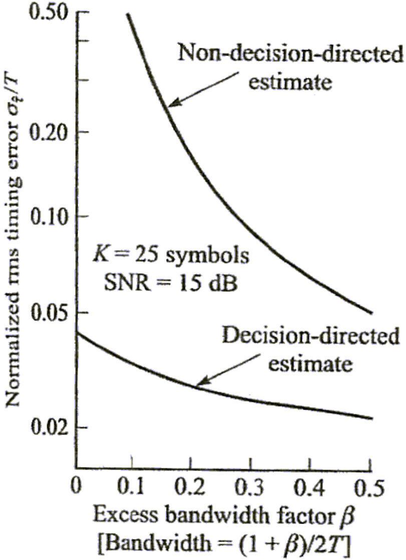
$$X_{\text{rc}}(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\beta}{2T}; \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\}, & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T}; \\ 0, & \text{otherwise} \end{cases}$$

- $\beta \in [0, 1]$ roll-off factor
- $\beta/(2T)$ bandwidth beyond the Nyquist bandwidth $1/(2T)$ is called the **excess bandwidth**.


x(t)
β = 0; 0.5
β = 1
β = 1
2T
0
T
3T
4T
β = 0
β = 0.5
X(f)
β = 0
β = 0.5
β = 1
−1/T
−1/2T
1/2T
1/T
f

Comparison between symbol timing (i.e., τ) DD and NDD estimates with raised-cosine(-spectrum) pulse shape.

- The larger the excess bandwidth, the better the estimate.
- NDD variance may go without bound when β small.



What you learn from Chapter 5

- MAP/ML estimate of τ and ϕ based on likelihood ratio function and known signals
- Phase lock loop
  - Linear model analysis and its transfer function with and without additive noise
- Decision-directed (or decision-feedback) loop
- Non-decision-directed loop
  - Take expectation on a quantity, proportional to probability.