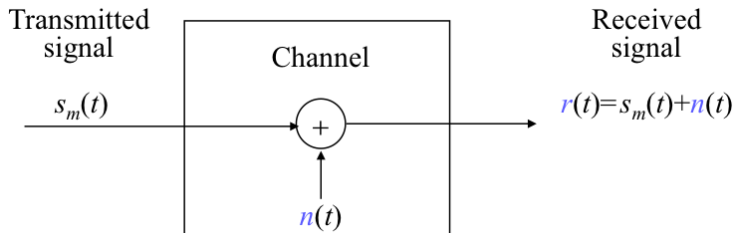# Digital Communications
Chapter 4. Optimum Receivers for AWGN Channels

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4.1 Waveform and vector channel models

System view

Transmitted signal $s_m(t)$ → Channel (+ with $n(t)$) → Received signal $r(t) = s_m(t) + n(t)$

- AWGN: Additive white Gaussian noise

$$S_n(f) = \frac{N_0}{2} \text{ (Watt/Hz); equivalently, } R_n(\tau) = \frac{N_0}{2}\delta(\tau) \text{ (Watt)}$$

Assumption

$$r(t) = s_m(t) + n(t)$$

Note: Instead of using **boldfaced** letters to denote random variables (resp. processes), we use blue-colored letters in Chapter 4, and reserve **boldfaced** blue-colored letters to denote random vectors (resp. multi-dimensional processes).

Receiver

Received signal $r(t)$ → Signal demodulator → Detector → Output decision
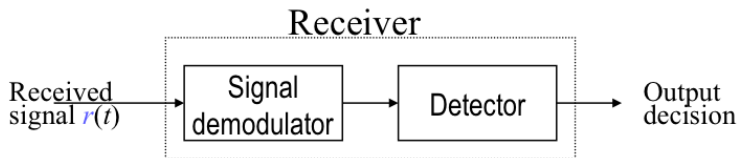
Definition 1 (Optimality)

Estimate m such that the error probability is minimized.

Models for analysis

Receiver

Received signal $r(t)$ → Signal demodulator → Detector → Output decision
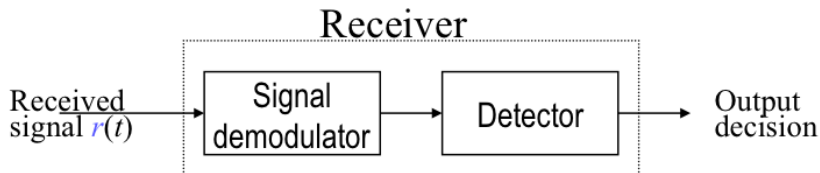
- Signal demodulator: Vectorization

$$r(t) \implies [r_1, r_2, \cdots, r_N]$$

- Detector: Minimize the probability of error in the above functional block

$$[r_1, r_2, \cdots, r_N] \implies \text{estimator } \hat{m}$$

Waveform to vector

- Let $\{\phi_i(t), 1 \leq i \leq N\}$ be a set of complete orthonormal basis for signals $\{s_m(t), 1 \leq m \leq M\}$; then define

$$
\begin{aligned}
r_i &= \langle r(t), \phi_i(t) \rangle = \int_0^T r(t)\phi_i^*(t)\, dt \\
s_{m,i} &= \langle s_m(t), \phi_i(t) \rangle = \int_0^T s_m(t)\phi_i^*(t)\, dt \\
n_i &= \langle n(t), \phi_i(t) \rangle = \int_0^T n(t)\phi_i^*(t)\, dt
\end{aligned}
$$

- Mean of n_i:

$$\mathbb{E}[n_i] = \mathbb{E}\left[\int_0^T n(t)\phi_i^*(t)\,dt\right] = 0$$

- Variance of n_i:

$$
\begin{aligned}
\mathbb{E}[|n_i|^2] &= \mathbb{E}\left[\int_0^T n(t)\phi_i^*(t)\,dt \cdot \int_0^T n^*(\tau)\phi_i(\tau)\,d\tau\right] \\
&= \int_0^T \int_0^T \mathbb{E}\left[n(t)n^*(\tau)\right]\phi_i^*(t)\,\phi_i(\tau)\,dt d\tau \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-\tau)\phi_i^*(t)\phi_i(\tau)dt\,d\tau \\
&= \frac{N_0}{2}\int_0^T \phi_i^*(\tau)\phi_i(\tau)d\tau \\
&= \frac{N_0}{2}
\end{aligned}
$$

So we have

$$n(t) = \sum_{i=1}^{N} n_i \phi_i(t) + \tilde{n}(t)$$

- Why $\tilde{n}(t)$? It is because $\{\phi_i(t), i = 1, 2, \cdots, N\}$ is not necessarily a complete basis for noise $n(t)$.

- $\tilde{n}(t)$ will not affect the error performance (it is orthogonal to $\sum_{i=1}^{N} n_i \phi_i(t)$ but could be statistically dependent on $\sum_{i=1}^{N} n_i \phi_i(t)$.) As a simple justification, the receiver can completely determine the exact value of $\tilde{n}(t)$ even if it is random in nature. So, the receiver can cleanly remove it from $r(t)$ without affecting $s_m(t)$:

$$\tilde{n}(t) = r(t) - \sum_{i=1}^{N} r_i \cdot \phi_i(t).$$

$$\Rightarrow \boxed{r(t) - \tilde{n}(t)} = \sum_{i=1}^{N} r_i \cdot \phi_i(t) = \sum_{i=1}^{N} s_{m,i} \cdot \phi_i(t) + \sum_{i=1}^{N} n_i \cdot \phi_i(t)$$

Independence and orthogonality

- Two orthogonal but possibly dependent signals, when they are summed together (i.e., when they are simultaneously transmitted), can be completely separated by communication technology (if we know the basis).

- Two independent signals, when they are summed together, cannot be completely separated with probability one (by "inner product" technology), if they are not orthogonal to each other.

- Therefore, in practice, orthogonality is more essentiall than independence.

Define

$$\boldsymbol{r} = \begin{bmatrix} r_1 & \cdots & r_N \end{bmatrix}^{\top}$$
$$\boldsymbol{s}_m = \begin{bmatrix} s_{m,1} & \cdots & s_{m,N} \end{bmatrix}^{\top}$$
$$\boldsymbol{n} = \begin{bmatrix} n_1 & \cdots & n_N \end{bmatrix}^{\top}$$

We can equivalently transform the waveform channel to a discrete channel:

$$\Rightarrow \quad \boldsymbol{r} = \boldsymbol{s}_m + \boldsymbol{n}$$

where $\boldsymbol{n}$ is zero-mean independent and identically Gaussian distributed with variance $N_0/2$.

Gaussian assumption

The joint probability density function (pdf) of $\boldsymbol{n}$ is conventionally given by

$$
f(\boldsymbol{n}) = \begin{cases} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{\|\boldsymbol{n}\|^2}{2\sigma^2}\right) & \text{if } \boldsymbol{n} \text{ real} \\ \left(\frac{1}{\pi\sigma^2}\right)^N \exp\left(-\frac{\|\boldsymbol{n}\|^2}{\sigma^2}\right) & \text{if } \boldsymbol{n} \text{ complex} \end{cases}
$$

where

$$
\mathbb{E}[\boldsymbol{n}\boldsymbol{n}^H] = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}
$$

Example.

$$\tilde{r} = r_x + \imath r_y = (s_x + n_x) + \imath (s_y + n_y) = \tilde{s} + \tilde{n},$$

where $\tilde{s} = s_x + \imath s_y$ and $\tilde{n} = n_x + \imath n_y$.

Assume n_x and n_y are independent zero-mean Gaussian with $\mathbb{E}[n_x^2] = \mathbb{E}[n_y^2]$; hence, $\mathbb{E}[|\tilde{n}|^2] = \mathbb{E}[n_x^2] + \mathbb{E}[n_y^2] = 2\mathbb{E}[n_x^2]$.

Then,

$$
\begin{aligned}
f(r_x, y_y) &= \left(\frac{1}{\sqrt{2\pi \mathbb{E}[n_x^2]}} \right)^2 e^{-\frac{(r_x - s_x)^2 + (r_y - s_y)^2}{2\,\mathbb{E}[n_x^2]}} \\
&= \left(\frac{1}{\pi \mathbb{E}[|\tilde{n}|^2]} \right)^1 e^{-\frac{\|\tilde{r} - \tilde{s}\|^2}{\mathbb{E}[|\tilde{n}|^2]}} = f(\tilde{r})
\end{aligned}
$$

Optimal decision function

- Given that the decision region for message m upon the reception of $\boldsymbol{r}$ is $\mathcal{D}_m$, i.e.,

$$g(\boldsymbol{r}) = m \qquad \text{if } \boldsymbol{r} \in \mathcal{D}_m,$$

the probability of correct decision is

$$
\begin{aligned}
P_c &= \sum_{m=1}^{M} \Pr\{\boldsymbol{s}_m \text{ sent}\} \int_{\mathcal{D}_m} f(\boldsymbol{r}|\boldsymbol{s}_m)\, d\boldsymbol{r} \\
&= \sum_{m=1}^{M} \int_{\mathcal{D}_m} f(\boldsymbol{r}) \Pr\{\boldsymbol{s}_m \text{ sent}|\boldsymbol{r} \text{ received}\}\, d\boldsymbol{r}
\end{aligned}
$$

- It implies that the optimal decision is

Maximum a posterori probability (MAP) decision

$$g_{\text{opt}}(\boldsymbol{r}) = \arg \max_{1 \le m \le M} \Pr\{\boldsymbol{s}_m \text{ sent}|\boldsymbol{r} \text{ received}\}$$

Maximum likelihood receiver

- From Bayes' rule we have

$$\Pr\{\boldsymbol{s}_m | \boldsymbol{r}\} = \frac{\Pr\{\boldsymbol{s}_m\}}{f(\boldsymbol{r})} f(\boldsymbol{r} | \boldsymbol{s}_m)$$

- If $\boldsymbol{s}_m$ are equally-likely, i.e., $\Pr\{\boldsymbol{s}_m\} = \frac{1}{M}$, then

Maximum likelihood (ML) decision

$$g_{ML}(\boldsymbol{r}) = \arg \max_{1 \leq m \leq M} f(\boldsymbol{r} | \boldsymbol{s}_m)$$

Decision region

- Given the decision function $g : \mathbb{R}^N \to \{1, \cdots, M\}$, we can define the decision region

$$\mathcal{D}_m = \left\{ \boldsymbol{r} \in \mathbb{R}^N : g(\boldsymbol{r}) = m \right\}.$$

- Symbol error probability (SER) of g is

$$\begin{aligned} P_e(= P_M \text{ in textbook}) &= \sum_{m=1}^{M} P_m \Pr\left\{ g(\boldsymbol{r}) \neq m \mid \boldsymbol{s}_m \text{ sent } \right\} \\ &= \sum_{m=1}^{M} P_m \sum_{m' \neq m} \int_{\mathcal{D}_{m'}} f\left(\boldsymbol{r}|\boldsymbol{s}_m\right) d\boldsymbol{r} \end{aligned}$$

where $P_m = \Pr\{\boldsymbol{s}_m \text{ sent}\}$.

I use P_e instead of P_M in contrast to P_c.

Bit level decision

The digital communications involve

k-bit information $\quad\longrightarrow\quad M = 2^k$ modulated signal $\boldsymbol{s}_m$

$\quad\quad\quad\quad\quad\quad\overset{+\text{noise}}{\longrightarrow}\quad \boldsymbol{r}$

$\quad\quad\quad\quad\quad\quad\overset{g}{\longrightarrow}\quad \hat{\boldsymbol{s}} = \boldsymbol{s}_{g(\boldsymbol{r})}$

$\quad\quad\quad\quad\quad\quad\longrightarrow\quad k$-bit recovering information
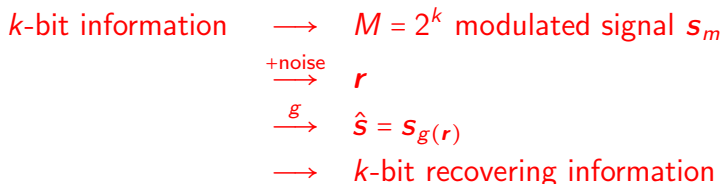
- For the ith bit $b_i \in \{0,1\}$, the *a posterori* probability of $b_i = \ell$ is
$$\Pr\{b_i = \ell | \boldsymbol{r}\} = \sum_{\boldsymbol{s}_m : b_i = \ell} \Pr\{\boldsymbol{s}_m | \boldsymbol{r}\}$$

- The MAP rule for b_i is
$$g_{MAP_i}(\boldsymbol{r}) = \arg\max_{\ell \in \{0,1\}} \sum_{\boldsymbol{s}_m : b_i = \ell} \Pr\{\boldsymbol{s}_m | \boldsymbol{r}\}$$

Bit error probability (BER)

- The decision region of b_i is

$$\begin{aligned}
\mathcal{B}_{i,0} &= \left\{ \boldsymbol{r} \in \mathbb{R}^N : g_{MAP_i}(\boldsymbol{r}) = 0 \right\} \\
\mathcal{B}_{i,1} &= \left\{ \boldsymbol{r} \in \mathbb{R}^N : g_{MAP_i}(\boldsymbol{r}) = 1 \right\}
\end{aligned}$$

- The error probability of bit b_i is

$$P_{b,i} = \sum_{\ell \in \{0,1\}} \sum_{\boldsymbol{s}_m : b_i = \ell} \Pr\{\boldsymbol{s}_m \text{ sent}\} \int_{\mathcal{B}_{i,(1-\ell)}} f(\boldsymbol{r}|\boldsymbol{s}_m) \, d\boldsymbol{r}$$

- The average bit error probability (BER) is

$$P_b = \frac{1}{k} \sum_{i=1}^{k} P_{b,i}$$

Let $\boldsymbol{e}$ be the random variable corresponding to the number of bit errors in a symbol. Then $P_b = \frac{1}{k} \mathbb{E}[\boldsymbol{e}] = \frac{1}{k} \mathbb{E}\left[\sum_{i=1}^{k} \boldsymbol{e}_i\right] = \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}[\boldsymbol{e}_i] = \frac{1}{k} \sum_{i=1}^{k} P_{b,i}$, where $\boldsymbol{e}_i = 1$ denotes the event that the ith bit is in error, and $\boldsymbol{e}_i = 0$ implies the ith bit is correctly recovered.

Theorem 1

(If $[b_i = \hat{b}_i]$ is a marginal event of $[(b_1, \ldots, b_k) = (\hat{b}_1, \ldots, \hat{b}_k)]$, then)

$$P_b \leq P_e \leq kP_b$$

Proof:

$$
\begin{aligned}
P_b &= \frac{1}{k} \sum_{i=1}^{k} \Pr[b_i \neq \hat{b}_i] \\
&= 1 - \frac{1}{k} \sum_{i=1}^{k} \Pr[b_i = \hat{b}_i] \\
&\leq 1 - \frac{1}{k} \sum_{i=1}^{k} \Pr[(b_1, \ldots, b_k) = (\hat{b}_1, \ldots, \hat{b}_k)] \\
&= \Pr[(b_1, \ldots, b_k) \neq (\hat{b}_1, \ldots, \hat{b}_k)] = P_e \\
&\leq \sum_{i=1}^{k} \Pr[b_i \neq \hat{b}_i] = k \left(\frac{1}{k} \sum_{i=1}^{k} \Pr[b_i \neq \hat{b}_i] \right) = kP_b.
\end{aligned}
$$

$\square$

Example 1

Consider two equal-probable signals $s_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and $s_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ sending through an additive noisy channel with $\boldsymbol{n} = \begin{bmatrix} n_1 & n_2 \end{bmatrix}^\top$ with joint pdf

$$f(\boldsymbol{n}) = \begin{cases} \exp(-n_1 - n_2), & \text{if } n_1, n_2 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find MAP Rule and P_e.

Solution.

Since $P\{\boldsymbol{s}_1\} = P\{\boldsymbol{s}_2\} = \frac{1}{2}$, MAP and ML rules coincide.
Given $\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{n}$, we would choose $\boldsymbol{s}_1$ if

$$f(\boldsymbol{s}_1|\boldsymbol{r}) \geq f(\boldsymbol{s}_2|\boldsymbol{r}) \iff f(\boldsymbol{r}|\boldsymbol{s}_1) \geq f(\boldsymbol{r}|\boldsymbol{s}_2)$$

$$\iff e^{-(r_1-0)-(r_2-0)} \cdot \mathbf{1}(r_1 \geq 0, r_2 \geq 0) \geq e^{-(r_1-1)-(r_2-1)} \cdot \mathbf{1}(r_1 \geq 1, r_2 \geq 1)$$

$$\iff \mathbf{1}(r_1 \geq 0, r_2 \geq 0) \geq e^2 \cdot \mathbf{1}(r_1 \geq 1, r_2 \geq 1)$$

where $\mathbf{1}(\cdot)$ is the set indicator function. Hence

$$\mathcal{D}_2 = \{\boldsymbol{r} : r_1 \geq 1, r_2 \geq 1\} \quad \text{and} \quad \mathcal{D}_1 = \mathcal{D}_2^c$$

$$P_{e|1} = \int_{\mathcal{D}_2} f(\boldsymbol{r}|\boldsymbol{s}_1) \, d\boldsymbol{r} = \int_1^\infty \int_1^\infty e^{-r_1-r_2} \, dr_1 dr_2 = e^{-2}$$

$$P_{e|2} = \int_{\mathcal{D}_1} f(\boldsymbol{r}|\boldsymbol{s}_2) \, d\boldsymbol{r} = 0$$

$$\Rightarrow P_e = \Pr\{\boldsymbol{s}_1\}P_{e|1} + \Pr\{\boldsymbol{s}_2\}P_{e|2} = \frac{1}{2}e^{-2}$$

Z-channel

$$s_2 \xrightarrow{\quad 1 \quad} s_2$$

$$s_1 \xrightarrow{\quad 1-e^{-2} \quad} s_1$$

with the e^{-2} crossover from s_1 to s_2.

$$\begin{cases} P_{e|1} & = e^{-2} \\ P_{e|2} & = 0 \end{cases}$$
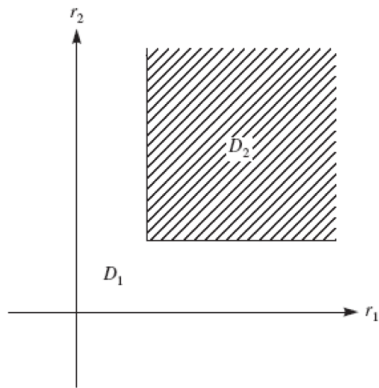
Sufficient statistics

Assuming s_m is transmitted, we receive $r = (r_1, r_2)$ with

$$f(r|s_m) = f(r_1, r_2|s_m) = f(r_1|s_m) f(r_2|r_1)$$

a Markov chain $(s_m \to r_1 \to r_2)$.

Theorem 2

Under the above assumption, the optimal decision can be made without r_2 (therefore, r_1 is called the sufficient statistics and r_2 is called the irrelevant data for detection of s_m).

Proof:

$$
\begin{aligned}
g_{opt}(r) &= \arg \max_{1 \leq m \leq M} \Pr\{s_m|r\} = \arg \max_{1 \leq m \leq M} \Pr\{s_m\} f\{r|s_m\} \\
&= \arg \max_{1 \leq m \leq M} \Pr\{s_m\} f(r_1|s_m) f(r_2|r_1) \\
&= \arg \max_{1 \leq m \leq M} \Pr\{s_m\} f(r_1|s_m)
\end{aligned}
$$

Preprocessing

$$\xrightarrow{\ s_m\ } \boxed{\text{Channel}} \xrightarrow{\ r\ } \text{Preprocessing } G \xrightarrow{\ \rho\ } \text{Detector} \xrightarrow{\ \hat{m}\ }$$

Assume $G(r) = \rho$ could be a many-to-one mapping. Then

$$
\begin{aligned}
g_{opt}(r, \rho) &= \arg \max_{1 \le m \le M} \Pr\{s_m | r, \rho\} \\
&= \arg \max_{1 \le m \le M} \Pr\{s_m\} f\{r, \rho | s_m\} \\
&= \arg \max_{1 \le m \le M} \Pr\{s_m\} f(r|s_m) f(\rho|r) \\
&= \arg \max_{1 \le m \le M} \Pr\{s_m\} f(r|s_m) \quad \text{independent of } \rho
\end{aligned}
$$

$$
\begin{aligned}
g_{opt}(\rho) &= \arg \max_{1 \le m \le M} \Pr\{s_m\} f(\rho|s_m) \\
&= \arg \max_{1 \le m \le M} \Pr\{s_m\} \int f(r, \rho | s_m) \, dr \\
&= \arg \max_{1 \le m \le M} \Pr\{s_m\} \int f(\rho|r) f(r|s_m) \, dr
\end{aligned}
$$

- In general, $g_{opt}(\boldsymbol{r}, \boldsymbol{\rho})$ (or $g_{opt}(\boldsymbol{r})$) gives a smaller error rate than $g_{opt}(\boldsymbol{\rho})$.

- They have equal performance only when pre-processing G is a bijection.

- By proceccing, data can be in a more "useful" (e.g., simpler in implementation) form but the error rate can never be reduced!

4.2-1 Optimal detection for the vector AWGN channel

Recapture Chapter 2

- Using signal space with orthonormal functions $\phi_1(t), \cdots, \phi_N(t)$, we can rewrite the waveform model

$$r(t) = s_m(t) + n(t)$$

as

$$\begin{bmatrix} r_1 & \cdots & r_N \end{bmatrix}^\top = \begin{bmatrix} s_{m,1} & \cdots & s_{m,N} \end{bmatrix}^\top + \begin{bmatrix} n_1 & \cdots & n_N \end{bmatrix}^\top$$

with

$$\mathbb{E}[n_i^2] = \mathbb{E}\left[\left|\int_0^T n(t)\phi_i(t)dt\right|^2\right] = \frac{N_0}{2}$$

The joint probability density function (pdf) of $\boldsymbol{n}$ is given by

$$f(\boldsymbol{n}) = \left(\frac{1}{\sqrt{2\pi(N_0/2)}}\right)^N \exp\left(-\frac{\|\boldsymbol{n}\|^2}{2(N_0/2)}\right)$$

$$g_{opt}(\boldsymbol{r}) \quad \left(= \quad g_{MAP}(\boldsymbol{r}) \right)$$

$$= \arg \max_{1 \le m \le M} \left[P_m f\left(\boldsymbol{r} | \boldsymbol{s}_m \right) \right]$$

$$= \arg \max_{1 \le m \le M} \left[P_m \left(\frac{1}{\sqrt{\pi N_0}} \right)^N \exp\left(-\frac{\|\boldsymbol{r} - \boldsymbol{s}_m\|^2}{N_0} \right) \right]$$

$$= \arg \max_{1 \le m \le M} \left[\log(P_m) - \frac{\|\boldsymbol{r} - \boldsymbol{s}_m\|^2}{N_0} \right]$$

$$= \arg \max_{1 \le m \le M} \left[\frac{N_0}{2} \log(P_m) - \frac{1}{2} \|\boldsymbol{r} - \boldsymbol{s}_m\|^2 \right] \text{ (Will be used later!)}$$

$$= \arg \max_{1 \le m \le M} \left[\frac{N_0}{2} \log(P_m) - \frac{1}{2} \|\boldsymbol{r}\|^2 + \boldsymbol{r}^\top \boldsymbol{s}_m - \frac{1}{2} \mathcal{E}_m \right]$$

$$= \arg \max_{1 \le m \le M} \left[\frac{N_0}{2} \log(P_m) + \boldsymbol{r}^\top \boldsymbol{s}_m - \frac{1}{2} \mathcal{E}_m \right]$$

Theorem 3 (MAP decision rule)

$$\hat{m} = \arg \max_{1 \leq m \leq M} \left[\frac{N_0}{2} \log(P_m) - \frac{1}{2}\mathcal{E}_m + \boldsymbol{r}^\top \boldsymbol{s}_m \right]$$
$$= \arg \max_{1 \leq m \leq M} \left[\eta_m + \boldsymbol{r}^\top \boldsymbol{s}_m \right]$$

where $\eta_m = \frac{N_0}{2}\log(P_m) - \frac{1}{2}\mathcal{E}_m$ is the *bias term*.

Theorem 4 (ML decision rule)

If $P_m = \frac{1}{M}$, the ML decision rule is

$$\hat{m} = \arg \max_{1 \leq m \leq M} \left[\frac{N_0}{2} \log(P_m) - \frac{1}{2} \|\boldsymbol{r} - \boldsymbol{s}_m\|^2 \right]$$

$$= \arg \min_{1 \leq m \leq M} \|\boldsymbol{r} - \boldsymbol{s}_m\|^2 = \arg \min_{1 \leq m \leq M} \|\boldsymbol{r} - \boldsymbol{s}_m\|$$

also known as *minimum distance decision rule*.

When signals are both equally likely and of equal energy, i.e.

$$P_m = \frac{1}{M} \quad \text{and} \quad \|\boldsymbol{s}_m\|^2 = \mathcal{E},$$

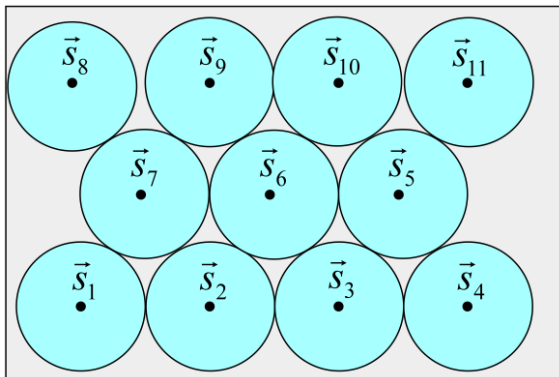the bias term η_m is independent of m, and the ML decision rule is simplified to

$$\hat{m} = \arg \max_{1 \le m \le M} \boldsymbol{r}^\top \boldsymbol{s}_m.$$

This is called correlation rule since

$$\boldsymbol{r}^\top \boldsymbol{s}_m = \int_0^T r(t) s_m(t) \, dt.$$

Signal space diagram for ML decision maker for one kind of signal assignment

Example

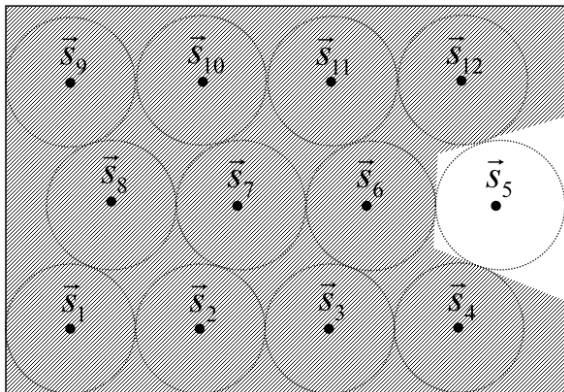Erroneous decision region $\mathcal{D}_5$ for the 5th signal
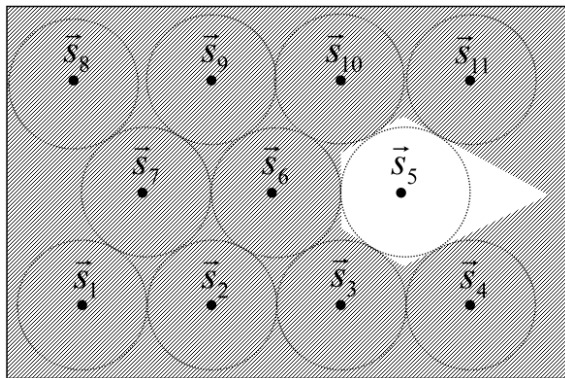
Example

Alternative signal space assignment
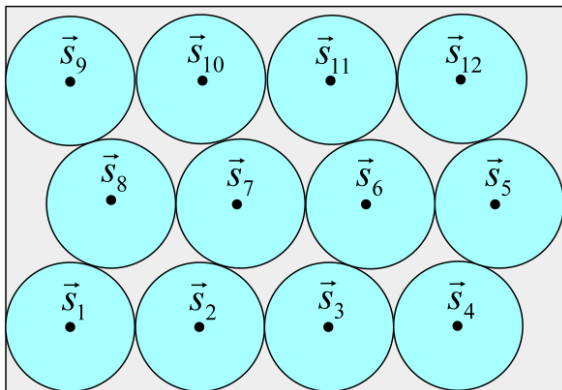
Example

Erroneous decision region $\mathcal{D}_5$ for the 5th signal

- There are two factors that determine the error probability.

  1. The Euclidean distances among signal vectors.

     Generally speaking, the larger the Euclidean distance among signal vectors, the smaller the error probability.

  2. The positions of the signal vectors.

     The two signal space diagrams in Slides 4-30~4-33 have the same pair-wise Euclidean distance among signal vectors!

Realization of ML rule

$$
\begin{aligned}
\hat{m} &= \arg \min_{1 \le m \le M} \| \boldsymbol{r} - \boldsymbol{s}_m \|^2 \\
&= \arg \min_{1 \le m \le M} \left(\| \boldsymbol{r} \|^2 - 2 \boldsymbol{r}^\top \boldsymbol{s}_m + \| \boldsymbol{s}_m \|^2 \right) \\
&= \arg \max_{1 \le m \le M} \left(\boldsymbol{r}^\top \boldsymbol{s}_m - \frac{1}{2} \| \boldsymbol{s}_m \|^2 \right) \\
&= \arg \max_{1 \le m \le M} \left(\int_0^T r(t) s_m(t) dt - \frac{1}{2} \int_0^T |s_m(t)|^2 dt \right)
\end{aligned}
$$

- The 1st term $=$ Projection of received signal onto each channel symbols.

- The 2nd term $=$ Compensation for channel symbols with unequal powers, such as PAM.
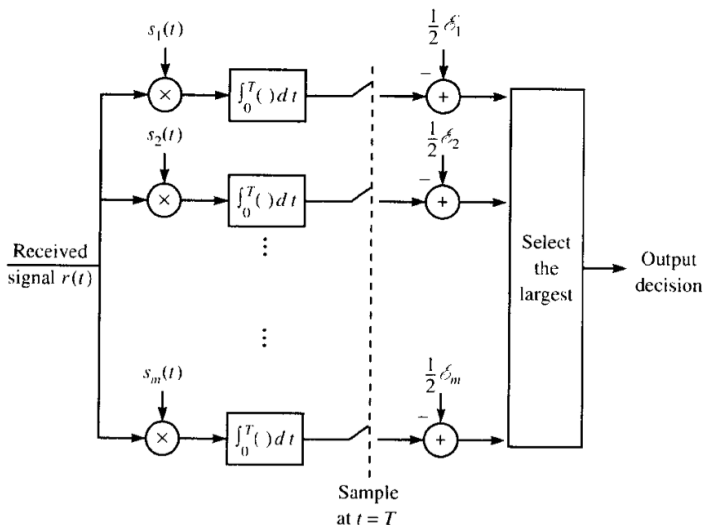
Optimal detection for binary antipodal signaling

Under AWGN, consider

- $s_1(t) = s(t)$ and $s_2(t) = -s(t)$;
- $\Pr\{s_1(t)\} = p$ and $\Pr\{s_2(t)\} = 1 - p$
- Let s_1 and s_2 be respectively signal space representation of $s_1(t)$ and $s_2(t)$ using $\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}$
- Then we have $s_1 = \sqrt{\mathcal{E}_s}$ and $s_2 = -\sqrt{\mathcal{E}_s}$ with $\mathcal{E}_s = \mathcal{E}_b$

Decision region for s_1 is

$$
\begin{aligned}
\mathcal{D}_1 &= \left\{ r \in \mathbb{R} : \eta_1 + r \cdot s_1 > \eta_2 + r \cdot s_2 \right\} \\
&= \left\{ r \in \mathbb{R} : \frac{N_0}{2} \log(p) - \frac{\mathcal{E}_b}{2} + r\sqrt{\mathcal{E}_b} > \frac{N_0}{2} \log(1-p) - \frac{\mathcal{E}_b}{2} - r\sqrt{\mathcal{E}_b} \right\} \\
&= \left\{ r \in \mathbb{R} : r > \frac{N_0}{4\sqrt{\mathcal{E}_b}} \log \frac{1-p}{p} \right\}
\end{aligned}
$$

Threshold detection

To detect binary antipodal signaling,

$$g_{opt}(r) = \begin{cases} 1, & \text{if } r > r_{th} \\ \text{tie}, & \text{if } r = r_{th} \\ 2, & \text{if } r < r_{th} \end{cases} , \quad \text{where } r_{th} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \log \frac{1-p}{p}$$

Error probability of binary antipodal signaling

$$
\begin{aligned}
P_e &= \sum_{m=1}^{2} P_m \sum_{m' \neq m} \int_{\mathcal{D}_{m'}} f(r|s_m)\, dr \\
&= p \int_{\mathcal{D}_2} f\left(r \,\middle|\, s = \sqrt{\mathcal{E}_b}\right) dr + (1-p) \int_{\mathcal{D}_1} f\left(r \,\middle|\, s = -\sqrt{\mathcal{E}_b}\right) dr \\
&= p \int_{-\infty}^{r_{th}} f\left(r \,\middle|\, s = \sqrt{\mathcal{E}_b}\right) dr + (1-p) \int_{r_{th}}^{\infty} f\left(r \,\middle|\, s = -\sqrt{\mathcal{E}_b}\right) dr \\
&= p \Pr\left\{ \mathcal{N}\left(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) < r_{th} \right\} + (1-p) \Pr\left\{ \mathcal{N}\left(-\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) > r_{th} \right\} \\
&= p \Pr\left\{ \mathcal{N}\left(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) < r_{th} \right\} + (1-p) \Pr\left\{ \mathcal{N}\left(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) < -r_{th} \right\} \\
&= p\, Q\left(\frac{\sqrt{\mathcal{E}_b} - r_{th}}{\sqrt{N_0/2}} \right) + (1-p)\, Q\left(\frac{\sqrt{\mathcal{E}_b} + r_{th}}{\sqrt{N_0/2}} \right)
\end{aligned}
$$

$$
\boxed{\Pr\left\{ \mathcal{N}(m, \sigma^2) < r \right\} = Q\left(\frac{m-r}{\sigma} \right)}
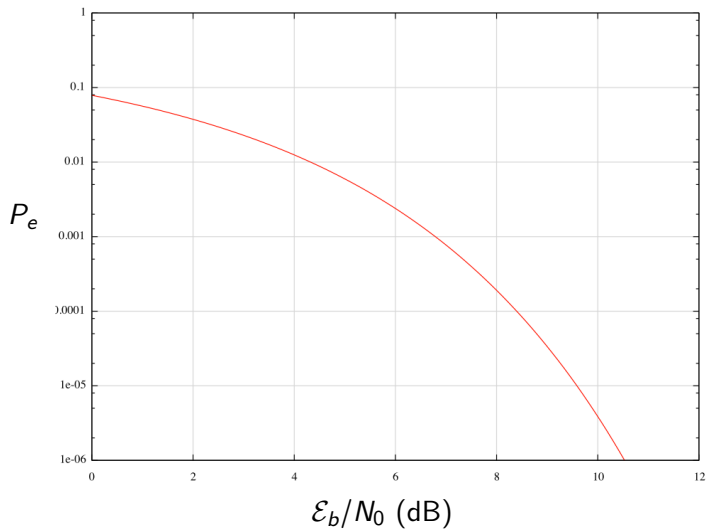$$

ML decision for binary antipodal signaling

For ML Detection, we have $p = 1 - p = \frac{1}{2}$.

$$g_{ML}(r) = \begin{cases} 1, & \text{if } r > r_{th} \\ 2, & \text{if } r < r_{th} \end{cases} \quad , \quad \text{where } r_{th} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = 0$$

and

$$P_e = p Q\left(\frac{\sqrt{\mathcal{E}_b} - r_{th}}{\sqrt{N_0/2}}\right) + (1-p) Q\left(\frac{\sqrt{\mathcal{E}_b} + r_{th}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

Binary equal-probable orthogonal signaling scheme

- For binary antipodal signals, $s_1(t)$ and $s_2(t)$ are not orthogonal! How about using two orthogonal signals.
- Assume $\Pr\{s_1(t)\} = \Pr\{s_2(t)\} = \frac{1}{2}$
- Assume tentatively that $s_1(t)$ is orthogonal to $s_2(t)$; so we need two orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$
- Signal space representation $s_1(t) \mapsto \boldsymbol{s}_1$ and $s_2(t) \mapsto \boldsymbol{s}_2$

Under AWGN, the ML decision region for $\boldsymbol{s}_1$ is

$$\mathcal{D}_1 = \left\{ \boldsymbol{r} \in \mathbb{R}^2 \ : \ \|\boldsymbol{r} - \boldsymbol{s}_1\| < \|\boldsymbol{r} - \boldsymbol{s}_2\| \right\}$$

i.e., the minimum distance decision rule.

Computing error probability concerns the following integrations:

$$\int_{\mathcal{D}_1} f(\boldsymbol{r}|\boldsymbol{s}_2)\, d\boldsymbol{r} \quad \text{and} \quad \int_{\mathcal{D}_2} f(\boldsymbol{r}|\boldsymbol{s}_1)\, d\boldsymbol{r}.$$

For $\mathcal{D}_1$, given $\boldsymbol{r} = \boldsymbol{s}_2 + \boldsymbol{n}$ we have

$$
\begin{aligned}
\|\boldsymbol{r} - \boldsymbol{s}_1\| < \|\boldsymbol{r} - \boldsymbol{s}_2\| &\implies \|\boldsymbol{s}_2 + \boldsymbol{n} - \boldsymbol{s}_1\| < \|\boldsymbol{n}\| \\
&\implies \|\boldsymbol{s}_2 - \boldsymbol{s}_1 + \boldsymbol{n}\|^2 < \|\boldsymbol{n}\|^2 \\
&\implies \|\boldsymbol{s}_2 - \boldsymbol{s}_1\|^2 + \|\boldsymbol{n}\|^2 + 2(\boldsymbol{s}_2 - \boldsymbol{s}_1)^\top \boldsymbol{n} < \|\boldsymbol{n}\|^2 \\
&\implies (\boldsymbol{s}_2 - \boldsymbol{s}_1)^\top \boldsymbol{n} < -\frac{1}{2}\|\boldsymbol{s}_2 - \boldsymbol{s}_1\|^2
\end{aligned}
$$

Recall $\boldsymbol{n}$ is Gaussian with covariance matrix $K_{\boldsymbol{n}} = \frac{N_0}{2} I_2$; hence $(\boldsymbol{s}_2 - \boldsymbol{s}_1)^\top \boldsymbol{n}$ is Gaussian with variance

$$\mathbb{E}\left[(\boldsymbol{s}_2 - \boldsymbol{s}_1)^\top \boldsymbol{n} \boldsymbol{n}^\top (\boldsymbol{s}_2 - \boldsymbol{s}_1)\right] = \frac{N_0}{2}\|\boldsymbol{s}_2 - \boldsymbol{s}_1\|^2$$

Setting

$$d_{12}^2 = \|s_2 - s_1\|^2$$

we obtain

$$\int_{\mathcal{D}_1} f(r|s_2) \, dr = \Pr\left\{ \mathcal{N}\left(0, \frac{N_0}{2}d_{12}^2\right) < -\frac{1}{2}d_{12}^2 \right\}$$

$$= Q\left(\frac{\frac{d_{12}^2}{2}}{d_{12}\sqrt{\frac{N_0}{2}}} \right) = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}} \right)$$

Similarly, we can show

$$\int_{\mathcal{D}_2} f(r|s_1) \, dr = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}} \right)$$

The derivation from Slide 4-42 to this page remains solid even if $s_1(t)$ and $s_2(t)$ are not orthogonal! So, it can be applied as well to binary antipodal signals.

Example 2 (Binary antipodal)

In this case we have $\boldsymbol{s}_1 = \sqrt{\mathcal{E}_b}$ and $\boldsymbol{s}_2 = -\sqrt{\mathcal{E}_b}$, so

$$d_{12}^2 = \left| 2\sqrt{\mathcal{E}_b} \right|^2 = 4\mathcal{E}_b$$

Hence

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}} \right) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right)$$

Example 3 (General equal-energy binary)

In this case we have $\|\boldsymbol{s}_1\|^2 = \|\boldsymbol{s}_2\|^2 = \mathcal{E}_b$, so

$$d_{12}^2 = \|\boldsymbol{s}_2 - \boldsymbol{s}_1\|^2 = \|\boldsymbol{s}_2\|^2 + \|\boldsymbol{s}_1\|^2 - 2\langle \boldsymbol{s}_2, \boldsymbol{s}_1 \rangle = 2\mathcal{E}_b(1 - \rho)$$

where $\rho = \langle \boldsymbol{s}_2, \boldsymbol{s}_1 \rangle / (\|\boldsymbol{s}_2\|\|\boldsymbol{s}_1\|)$. Hence

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{(1-\rho)\frac{\mathcal{E}_b}{N_0}}\right)$$

* The error rate is minimized by taking $\rho = -1$ (i.e., antipodal).

Binary antipodal vs orthogonal

For binary antipodal such as BPSK

$$P_{b,BPSK} = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

and for binary orthogonal such as BFSK

$$P_{b,BFSK} = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

we see

- BPSK is 3 dB (specifically, $10\log_{10}(2) = 3.010$) better than BFSK in error performance.
- The term $\frac{\mathcal{E}_b}{N_0}$ is commonly referred to as signal-to-noise ratio per information bit.

P_e

$\frac{\mathcal{E}_b}{N_0}$ (dB)
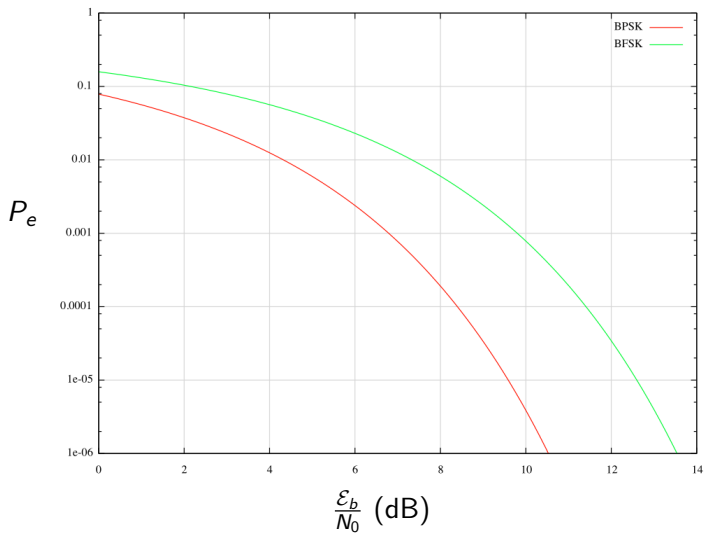
4.2-2 Implementation of optimal receiver for AWGN channels

Recall that the optimal decision rule is

$$g_{opt}(\boldsymbol{r}) = \arg \max_{1 \leq m \leq M} \left[\eta_m + \boldsymbol{r}^\top \boldsymbol{s}_m \right]$$

where we note that

$$\boldsymbol{r}^\top \boldsymbol{s}_m = \int_0^T r(t) s_m(t) \, dt$$

This suggests a correlation receiver.
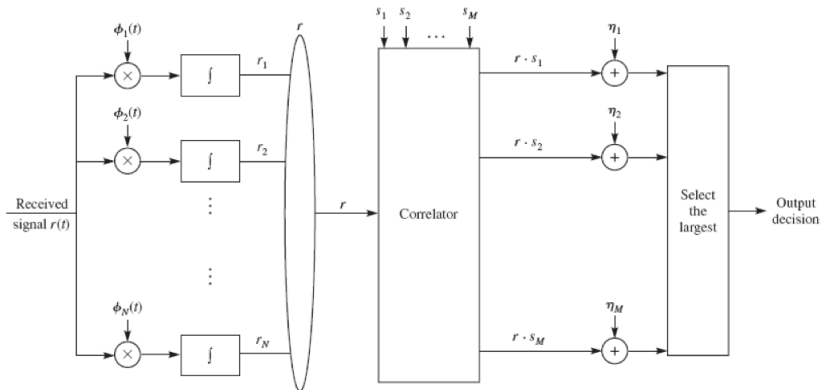
Correlation receiver
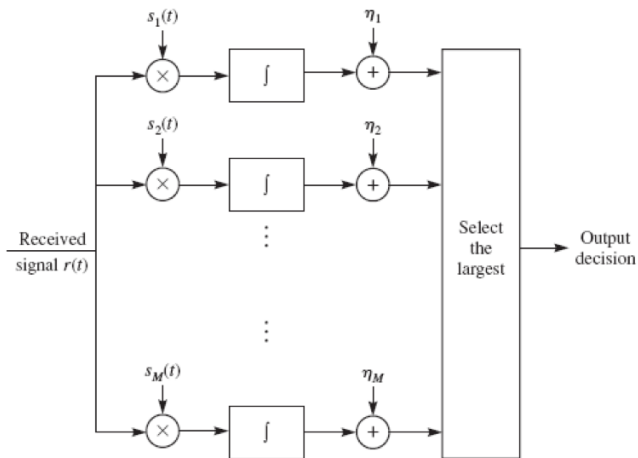
Matched filter receiver

$$r^\top s_m = \int_0^T r(t) s_m(t) \, dt$$

On the other hand, we could define a filter h with impulse response

$$h_m(t) = s_m(T - t)$$

such that

$$r(t) \star h_m(t)\big|_{t=T} = \int_{-\infty}^\infty r(\tau) h_m(T-\tau) \, d\tau = \int_0^T r(t) s_m(t) \, dt$$

This gives the matched filter receiver (that directly generates $r^\top s_m$).

Optimality of matched filter

Assume that we use a filter $h(t)$ to process the incoming signal

$$r(t) = s(t) + n(t).$$

Then

$$y(t) = h(t) \star r(t) = h(t) \star s(t) + h(t) \star n(t) = h(t) \star s(t) + z(t)$$

Hence the noiseless signal at $t = T$ is

$$h(t) \star s(t)|_{t=T} = \left. \int_{-\infty}^{\infty} H(f) S(f) e^{i 2\pi f t} df \right|_{t=T}$$

The noise variance $\sigma_z^2 = E[z^2(T)] = R_Z(0)$ of $z(t)|_{t=T}$ is

$$\sigma_z^2 = \int_{-\infty}^{\infty} S_N(f) |H(f)|^2 df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df.$$

Optimality of matched filter

Thus the output SNR is

$$\text{SNR}_O = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{\imath 2\pi fT}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df}$$

$$\leq \frac{\int_{-\infty}^{\infty}|H(f)|^2\,df \cdot \int_{-\infty}^{\infty}\left|S(f)e^{\imath 2\pi fT}\right|^2\,df}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2\,df}$$

$$= \frac{2}{N_0}\int_{-\infty}^{\infty}\left|S(f)e^{\imath 2\pi fT}\right|^2\,df.$$

The Cauchy-Schwartz inequality holds with equality iff

$$H(f) = \alpha \cdot S^*(f)e^{-\imath 2\pi fT} \implies h(t) = \alpha\, s^*(T-t)$$

$$\int_{-\infty}^{\infty} s^*(T-t)e^{-\imath 2\pi ft}dt = \left(\int_{-\infty}^{\infty} s(T-t)e^{\imath 2\pi ft}dt\right)^*$$
$$= \left(\int_{-\infty}^{\infty} s(t')e^{-\imath 2\pi f(t'-T)}dt'\right)^* = S^*(f)e^{-\imath 2\pi fT}$$

4.2-3 A union bound on the probability of error of maximum likelihood detection

Recall for ML decoding

$$P_e = \frac{1}{M} \sum_{m=1}^{M} \int_{\mathcal{D}_m^c} f(\mathbf{r}|\mathbf{s}_m) \, d\mathbf{r}$$

where the ML decoding rule is

$$g_{ML}(\mathbf{r}) = \arg \max_{1 \leq m \leq M} f(\mathbf{r}|\mathbf{s}_m)$$

$$\mathcal{D}_m = \{\mathbf{r} : f(\mathbf{r}|\mathbf{s}_m) > f(\mathbf{r}|\mathbf{s}_k) \text{ for all } k \neq m'.\}$$

and

$$\begin{aligned}
\mathcal{D}_m^c &= \{\mathbf{r} : f(\mathbf{r}|\mathbf{s}_m) \leq f(\mathbf{r}|\mathbf{s}_k) \text{ for some } k \neq m.\} \\
&= \{\mathbf{r} : f(\mathbf{r}|\mathbf{s}_m) \leq f(\mathbf{r}|\mathbf{s}_1) \text{ or } \cdots \text{ or } f(\mathbf{r}|\mathbf{s}_m) \leq f(\mathbf{r}|\mathbf{s}_{m-1}) \\
&\qquad \text{ or } f(\mathbf{r}|\mathbf{s}_m) \leq f(\mathbf{r}|\mathbf{s}_{m+1}) \text{ or } \cdots \text{ or } f(\mathbf{r}|\mathbf{s}_m) \leq f(\mathbf{r}|\mathbf{s}_M)\}
\end{aligned}$$

Define the error event $\mathcal{E}_{m \to m'}$

$$\mathcal{E}_{m \to m'} = \{ \mathbf{r} : f(\mathbf{r}|\mathbf{s}_m) \le f(\mathbf{r}|\mathbf{s}_{m'}) \}$$

Then we note

$$\mathcal{D}_m^c = \bigcup_{\substack{1 \le m' \le M \\ m' \ne m}} \mathcal{E}_{m \to m'}.$$

Hence, by union inequality (i.e., $P(A \cup B) \le P(A) + P(B)$),

$$
\begin{aligned}
P_e &= \frac{1}{M} \sum_{m=1}^{M} \int_{\mathcal{D}_m^c} f(\mathbf{r}|\mathbf{s}_m) \, d\mathbf{r} \\
&= \frac{1}{M} \sum_{m=1}^{M} P_{\mathbf{r}|\mathbf{s}_m} \{ \mathcal{D}_m^c \} \\
&= \frac{1}{M} \sum_{m=1}^{M} P_{\mathbf{r}|\mathbf{s}_m} \left\{ \bigcup_{\substack{1 \le m' \le M \\ m' \ne m}} \mathcal{E}_{m \to m'} \right\} \\
&\le \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{1 \le m' \le M \\ m' \ne m}} P_{\mathbf{r}|\mathbf{s}_m} \{ \mathcal{E}_{m \to m'} \} \quad \text{(Union bound)}
\end{aligned}
$$

Appendix: Good to know!

- Union bound (Boole's inequality): $\Pr\left(\bigcup_{k=1}^{N} A_k\right) \le \sum_{k=1}^{N} \Pr(A_k)$.

- Reverse union bound: $\Pr\left(A - \bigcup_{k=1}^{N} A_k\right) \ge \Pr(A)\left[1 - \sum_{k=1}^{N} \Pr(A_k|A)\right]$.

Proof:

$$
\begin{aligned}
\Pr\left(A - \bigcup_{k=1}^{N} A_k\right) &= \Pr\left(A - \bigcup_{k=1}^{N}(A \cap A_k)\right) \\
&\ge \Pr(A) - \Pr\left(\bigcup_{k=1}^{N}(A \cap A_k)\right) \\
&\ge \Pr(A) - \sum_{k=1}^{N} \Pr(A \cap A_k) \quad \text{(Alternative form)} \\
&= \Pr(A) - \Pr(A)\sum_{k=1}^{N} \frac{\Pr(A \cap A_k)}{\Pr(A)}. \qquad \square
\end{aligned}
$$

Appendix: Good to know!

- Union bound is a special case of Bonferroni inequalities:

Let

$$
\begin{aligned}
S_1 &= \sum_{i=1}^{N} \Pr(A_i) \\
&\vdots \\
S_k &= \sum_{i_1 < i_2 < \cdots < i_k} \Pr(A_{i_1} \cap \cdots \cap A_{i_k}) \\
&\vdots
\end{aligned}
$$

Then for any $2u_1 - 1 \leq N$ and $2u_2 \leq N$,

$$
\underbrace{\sum_{i=1}^{2u_2} (-1)^{i-1} S_i}_{= S_1 - S_2 + \cdots + S_{2u_2-1} - S_{2u_2}} \leq \Pr\left(\bigcup_{i=1}^{N} A_i\right) \leq \underbrace{\sum_{i=1}^{2u_1-1} (-1)^{i-1} S_i}_{= S_1 - S_2 + \cdots - S_{2u_2-2} + S_{2u_1-1}}
$$

Pairwise error probability for AWGN channel
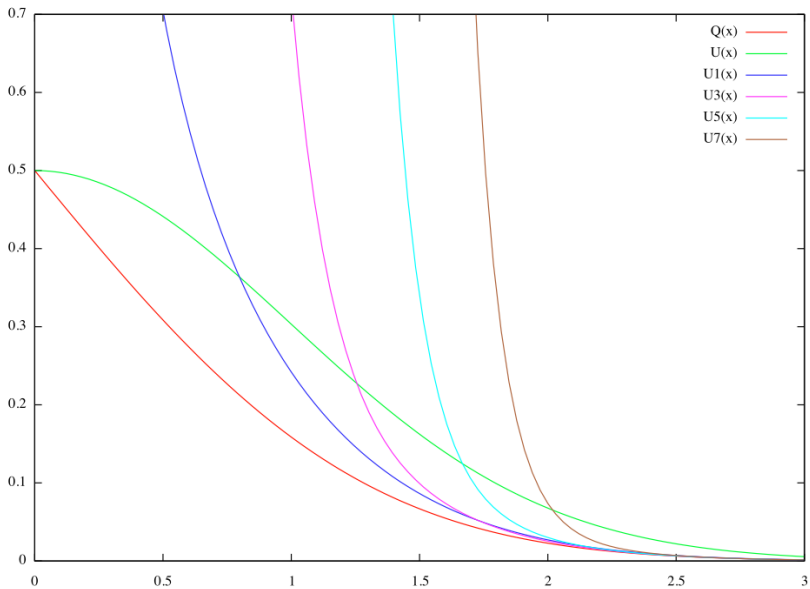
For AWGN channel, we have

$$P_{\mathbf{r}|\mathbf{s}_m}\left\{\mathcal{E}_{m\to m'}\right\} \;=\; \int_{\mathcal{E}_{m\to m'}} f(\mathbf{r}|\mathbf{s}_m)\,d\mathbf{r} \;=\; Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}}\right)$$
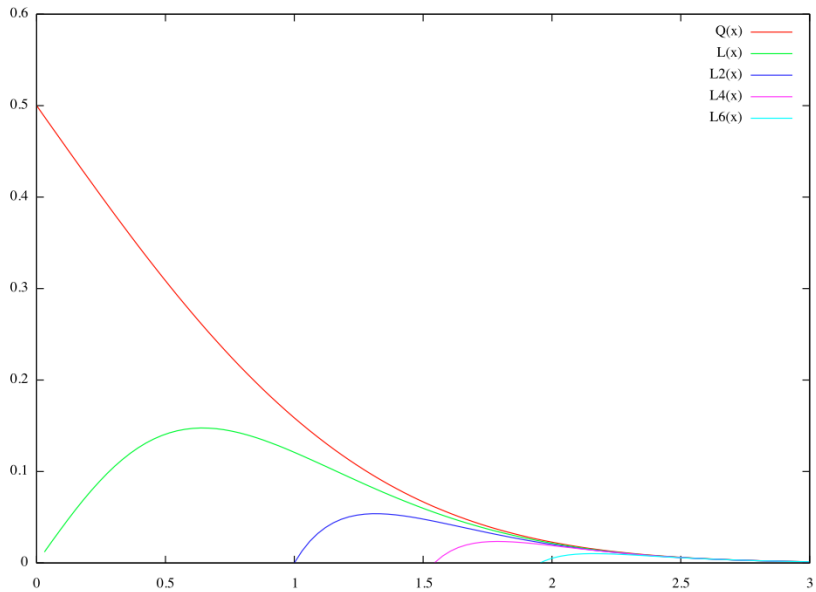
where $d_{m,m'} = \|\mathbf{s}_m - \mathbf{s}_{m'}\|$.

A famous approximation to Q function:

$$Q(x) = \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}\left(1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4} - \frac{1\cdot 3\cdot 5}{x^6} + \cdots\right) \text{ for } x \geq 0.$$

$$\left.\begin{array}{c} L2(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) \\ L4(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4} - \frac{1\cdot 3\cdot 5}{x^6}\right) \\ L(x) = \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}\left(\frac{x^2}{1+x^2}\right) \end{array}\right\} \leq Q(x) \leq \left\{\begin{array}{l} U1(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}x} \\ U3(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4}\right) \\ U(x) = \frac{1}{2}e^{-x^2/2} \end{array}\right.$$

Four union bounds

Using for simplicity, we employ $Q(x) \leq U(x) = \frac{1}{2}e^{-\frac{x^2}{2}}$ and obtain

$$P_e \leq \underbrace{\frac{1}{M}\sum_{m=1}^{M}\sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}}\right)}_{\text{bound 1}} \leq \underbrace{\frac{1}{2M}\sum_{m=1}^{M}\sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \exp\left(-\frac{d_{m,m'}^2}{4N_0}\right)}_{\text{bound 2}}$$

Define $\qquad d_{\min} = \min_{m \neq m'} d_{m,m'} = \min_{m \neq m'} \|\boldsymbol{s}_m - \boldsymbol{s}_{m'}\|$

Then we have

$$Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}}\right) \leq Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \text{ and } \exp\left(-\frac{d_{m,m'}^2}{4N_0}\right) \leq \exp\left(-\frac{d_{\min}^2}{4N_0}\right)$$

and hence

$$P_e \leq \underbrace{(M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)}_{\text{bound 3: minimum distance bound}} \leq \underbrace{\frac{M-1}{2}\exp\left(-\frac{d_{\min}^2}{4N_0}\right)}_{\text{bound 4: minimun distance bound}} .$$

Define the distance enumerator function as

$$T(X) = \sum_{m=1}^{M} \sum_{\substack{1 \le m' \le M \\ m' \ne m}} X^{d_{m,m'}^2} = \sum_{\text{all distinct } d} a_d X^{d^2},$$

where a_d is the number of $d_{m,m'}$ being equal to d.

Then,

$$\text{bound 2} = \underbrace{\frac{1}{2M} T\left(e^{-1/(4N_0)}\right)}_{\text{bound 5}}.$$

Lower bound on P_e

$$
\begin{aligned}
P_e &= \frac{1}{M} \sum_{m=1}^{M} \int_{\mathcal{D}_m^c} f(\boldsymbol{r}|\boldsymbol{s}_m) \, d\boldsymbol{r} \\
&\geq \frac{1}{M} \sum_{m=1}^{M} \max_{\substack{1 \leq m' \leq M \\ m' \neq m}} \int_{\mathcal{E}_{m \to m'}} f(\boldsymbol{r}|\boldsymbol{s}_m) \, d\boldsymbol{r} \\
&= \frac{1}{M} \sum_{m=1}^{M} \max_{\substack{1 \leq m' \leq M \\ m' \neq m}} Q\left(\sqrt{\frac{d_{m,m'}^2}{2N_0}} \right) \\
&= \frac{1}{M} \sum_{m=1}^{M} Q\left(\sqrt{\frac{(d_{\min,m})^2}{2N_0}} \right) \quad \text{where } d_{\min,m} = \min_{\substack{1 \leq m' \leq M \\ m' \neq m}} d_{m,m'} \\
&\geq \frac{N_{\min}}{M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}} \right)
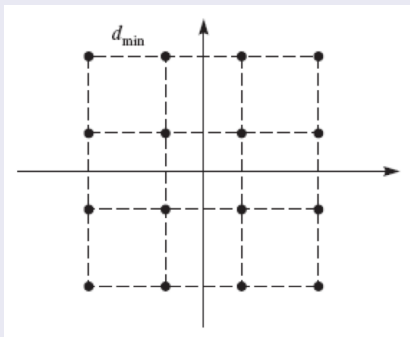\end{aligned}
$$

where $N_{\min} \leq M$ is the number of m (in $1 \leq m \leq M$) such that $d_{\min,m} = d_{\min}$ (the textbook uses $d_{\min}^m$ instead of $d_{\min,m}$).

Example 4 (16QAM)

Among $2\binom{16}{2} = 240$ distances,

there are
$\begin{cases}
48 \ d_{\min} \\
36 \ \sqrt{2}d_{\min} \\
32 \ 2d_{\min} \\
48 \ \sqrt{5}d_{\min} \\
16 \ \sqrt{8}d_{\min} \\
16 \ 3d_{\min} \\
24 \ \sqrt{10}d_{\min} \\
16 \ \sqrt{13}d_{\min} \\
4 \ \sqrt{18}d_{\min}
\end{cases}$

$$T(X) = 48X^{d_{\min}^2} + 36X^{2d_{\min}^2} + 32X^{4d_{\min}^2} + 48X^{5d_{\min}^2} + 16X^{8d_{\min}^2}$$
$$+ 16X^{9d_{\min}^2} + 24X^{10d_{\min}^2} + 16X^{13d_{\min}^2} + 4X^{18d_{\min}^2}$$

$$P_e \leq \frac{1}{2M} T\left(e^{-1/(4N_0)}\right) = \frac{1}{32} T\left(e^{-1/(4N_0)}\right).$$

Example 5 (16QAM)

For 16QAM, $N_{\min} = M$; hence,

$$P_e \geq \frac{N_{\min}}{M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right).$$

From Slide 3-29, we have

$$\mathcal{E}_{bavg} = \frac{M-1}{3\log_2 M}\mathcal{E}_g \quad and \quad d_{\min} = \sqrt{2\mathcal{E}_g}$$

So, $d_{\min} = \sqrt{2\mathcal{E}_g} = \sqrt{\frac{6\log_2 M}{M-1}\mathcal{E}_{bavg}} = \sqrt{\frac{8}{5}\mathcal{E}_{bavg}}$,

$$P_e \geq Q\left(\sqrt{\frac{4\mathcal{E}_{bavg}}{5N_0}}\right)$$

Example 6 (16QAM)

The exact P_e for 16QAM can be derived, which is

$$P_e = 3Q\left(\sqrt{\frac{4\mathcal{E}_{bavg}}{5N_0}}\right) - \frac{9}{4}\left[Q\left(\sqrt{\frac{4\mathcal{E}_{bavg}}{5N_0}}\right)\right]^2$$

Hint: (Will be introduced in Section 4.3)

- Derive the error rate for m-ary PAM:

$$P_{e,m\text{-ary PAM}} = \frac{2(m-1)}{m}Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)$$

- The error rate for $M = m^2$-ary QAM:

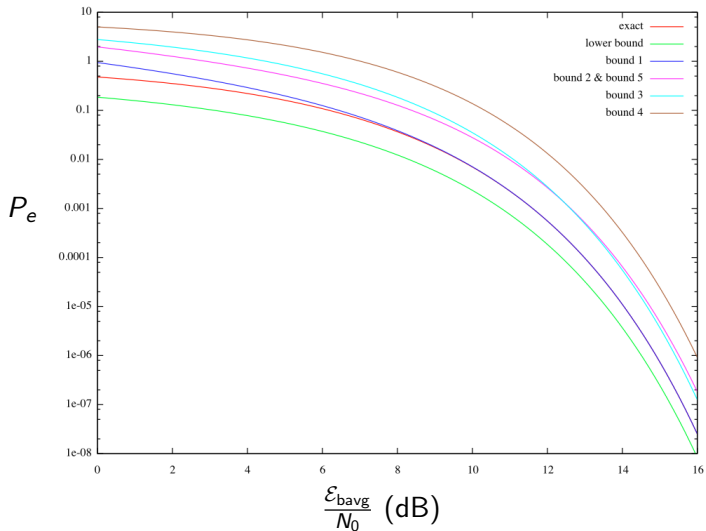$$P_e = 1 - (1 - P_{e,m\text{-ary PAM}})^2 = 2P_{e,m\text{-ary PAM}} - P_{e,m\text{-ary PAM}}^2$$

16QAM

$$\underline{\text{bound 1}} = \frac{1}{16}\left(48Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) + 36Q\left(\sqrt{2\frac{d_{\min}^2}{2N_0}}\right) + 32Q\left(\sqrt{4\frac{d_{\min}^2}{2N_0}}\right)\right.$$

$$+ 48Q\left(\sqrt{5\frac{d_{\min}^2}{2N_0}}\right) + 16Q\left(\sqrt{8\frac{d_{\min}^2}{2N_0}}\right) + 16Q\left(\sqrt{9\frac{d_{\min}^2}{2N_0}}\right)$$

$$\left. + 24Q\left(\sqrt{10\frac{d_{\min}^2}{2N_0}}\right) + 16Q\left(13\sqrt{\frac{d_{\min}^2}{2N_0}}\right) + 4Q\left(\sqrt{18\frac{d_{\min}^2}{2N_0}}\right)\right)$$

$$\underline{\text{bound 2}} = \frac{1}{32}\left(48e^{-d_{\min}^2/(4N_0)} + 36e^{-2d_{\min}^2/(4N_0)} + 32e^{-4d_{\min}^2/(4N_0)}\right.$$

$$+ 48e^{-5d_{\min}^2/(4N_0)} + 16e^{-8d_{\min}^2/(4N_0)} + 16e^{-9d_{\min}^2/(4N_0)}$$

$$\left. + 24e^{-10d_{\min}^2/(4N_0)} + 16e^{-13d_{\min}^2/(4N_0)} + 4e^{-18d_{\min}^2/(4N_0)}\right)$$
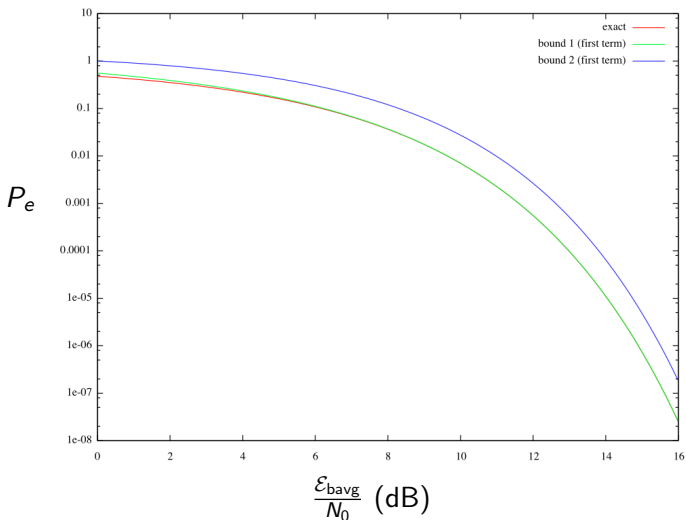
$$\underline{\text{bound 3}} = 15Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right), \quad \underline{\text{bound 4}} = \frac{15}{2}\exp\left(-\frac{d_{\min}^2}{4N_0}\right), \quad \frac{d_{\min}^2}{N_0} = \frac{8}{5}\frac{\mathcal{E}_{\text{bavg}}}{N_0}$$

16QAM bounds

16QAM approximations

Suppose we only take the first terms of bound 1 and bound 2. Then, approximations (instead of bounds) are obtained.
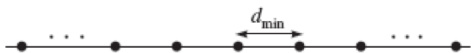
4.3 Optimal detection and error probability for bandlimited signaling

ASK or PAM signaling

- Let $d_{\min}$ be the minimum distance between adjacent PAM constellations.
- Consider the signal constellations

$$\mathcal{S} = \left\{ \pm\frac{1}{2}d_{\min}, \pm\frac{3}{2}d_{\min}, \cdots, \pm\frac{M-1}{2}d_{\min} \right\}$$

- The average bit signal energy is

$$\mathcal{E}_{\text{bavg}} = \frac{1}{\log_2(M)}\mathbb{E}[|\mathbf{s}|^2] = \frac{M^2-1}{12\log_2(M)}d_{\min}^2$$

The average bit signal energy of m^2-QAM should be equal to that of m-PAM. From Slide 3-29, $\mathcal{E}_{\text{bavg},m^2\text{-QAM}} = \frac{m^2-1}{3\log_2(m^2)}\mathcal{E}_g = \frac{m^2-1}{3\log_2(m^2)}\left(\frac{1}{2}d_{\min}^2\right)$.

There are two types of error events (under AWGN):

- Inner points with error probability P_{ei}

$$
\begin{aligned}
P_{ei} &= \Pr\left\{|\boldsymbol{n}| > \frac{d_{\min}}{2}\right\} = 2\Pr\left\{\boldsymbol{n} < -\frac{d_{\min}}{2}\right\} \\
&= 2Q\left(\frac{0 - (-d_{\min}/2)}{\sqrt{N_0/2}}\right) = 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)
\end{aligned}
$$

- Outer points with error probability P_{eo}: only one end causes errors:

$$
P_{eo} = \Pr\left\{\boldsymbol{n} > \frac{d_{\min}}{2}\right\} = \Pr\left\{\boldsymbol{n} < -\frac{d_{\min}}{2}\right\} = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)
$$

Symbol error probability of PAM

The symbol error probability is given by

$$
\begin{aligned}
P_e &= \frac{1}{M} \sum_{m=1}^{M} \Pr\{\text{error}|m \text{ sent}\} \\
&= \frac{1}{M}\left[(M-2)\cdot 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) + 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right] \\
&= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \quad \left(\text{Note } \mathcal{E}_{\text{bavg}} = \frac{M^2-1}{12\log_2(M)}d_{\min}^2.\right) \\
&= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2(M)}{(M^2-1)}\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right)
\end{aligned}
$$

Efficiency

- To increase rate by 1 bit (i.e., $M \to 2M$), we need to double M.
- To keep (almost) the same P_e, we need $\mathcal{E}_{\text{bavg}}$ to quadruple.

M	2	4	8	16
$\frac{6\log_2(M)}{(M^2-1)}$	2	$\frac{4}{5}$	$\frac{2}{7}$	$\frac{8}{85}$

$2\to4$	$4\to8$	$8\to16$	$\cdots$	$M \to 2M$ as M large
2.5	2.8	3.0	$\cdots$	4

$$\frac{6\log_2(M)}{(M^2-1)}\frac{\mathcal{E}_{\text{bavg}}}{N_0} \approx \frac{6\log_2(2M)}{((2M)^2-1)}\frac{\mathcal{E}_{\text{bavg}}^{(\text{new})}}{N_0} \Rightarrow \mathcal{E}_{\text{bavg}}^{(\text{new})} \approx 4\mathcal{E}_{\text{bavg}}$$
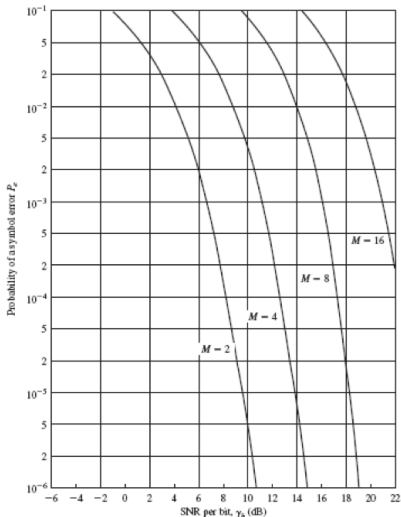
- Increase rate by 1 bit $\implies$ increase $\mathcal{E}_{\text{bavg}}$ by 6 dB

PAM performance

The larger the M is,
the worse the symbol performance!

At small M, increasing by 1 bit
only requires additional 4 dB.

The true winner will be more
"clear" from BER vs. $\mathcal{E}_b/N_0$ plot.

PSK signaling

Signal constellation for M-ary PSK is

$$\mathcal{S} = \left\{ \boldsymbol{s}_k = \sqrt{\mathcal{E}}\left(\cos\left(\frac{2\pi k}{M}\right), \sin\left(\frac{2\pi k}{M}\right)\right) \ : \ k = 0, 1, \cdots, M-1 \right\}$$

- By symmetry we can assume $\boldsymbol{s}_0 = \left(\sqrt{\mathcal{E}}, 0\right)$ was transmitted.
- The received signal vector $\boldsymbol{r}$ is

$$\boldsymbol{r} = \left(\sqrt{\mathcal{E}} + \boldsymbol{n}_1, \boldsymbol{n}_2\right)^{\top}$$

- Assume Gaussian random process with $R_{\boldsymbol{n}}(\tau) = \frac{N_0}{2}\delta(\tau)$

$$f(n_1, n_2) = \left(\frac{1}{\sqrt{2\pi(N_0/2)}}\right)^2 \exp\left(-\frac{n_1^2 + n_2^2}{2(N_0/2)}\right)$$

- Thus we have

$$f(\boldsymbol{r} = (r_1, r_2)|\boldsymbol{s}_0) = \frac{1}{\pi N_0} \exp\left(-\frac{(r_1 - \sqrt{\mathcal{E}})^2 + r_2^2}{N_0}\right)$$

- Define $V = \sqrt{r_1^2 + r_2^2}$ and $\Theta = \arctan\dfrac{r_2}{r_1}$

$$f(v, \theta|\boldsymbol{s}_0) = \frac{v}{\pi N_0} \exp\left(-\frac{v^2 + \mathcal{E} - 2\sqrt{\mathcal{E}}v\cos\theta}{N_0}\right)$$

The ML decision region for s_0 is

$$\mathcal{D}_0 \;=\; \left\{ r \;:\; -\frac{\pi}{M} < \theta < \frac{\pi}{M} \right\}$$
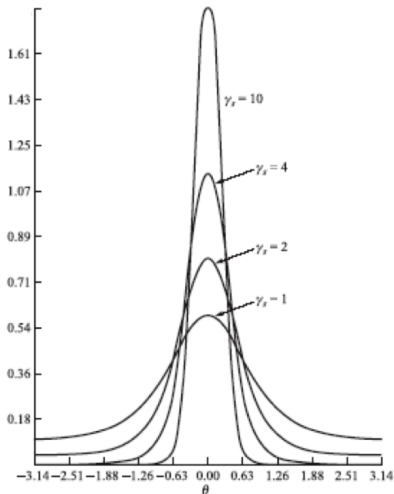
The probability of erroneous decision given s_0 is

$$
\begin{aligned}
& \Pr\left\{\text{error}|s_0\right\} \\
&= \; 1 - \iint_{\mathcal{D}_0} f(v, \theta | s_0) \, dv \, d\theta \\
&= \; 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \underbrace{\int_0^\infty \frac{v}{\pi N_0} \exp\left(-\frac{v^2 + \mathcal{E} - 2\sqrt{\mathcal{E}}\, v \cos\theta}{N_0} \right) dv}_{f(\theta|s_0)} \, d\theta
\end{aligned}
$$

$$
\begin{aligned}
f(\theta|\boldsymbol{s}_0) &= \int_0^\infty \frac{v}{\pi N_0} \exp\left(-\frac{v^2 + \mathcal{E} - 2\sqrt{\mathcal{E}}\, v \cos\theta}{N_0} \right) dv \\
&= \int_0^\infty \frac{v}{\pi N_0} \exp\left(-\frac{(v - \sqrt{\mathcal{E}} \cos\theta)^2 + \mathcal{E} \sin^2\theta}{N_0} \right) dv \\
&= \frac{1}{2\pi} \exp\left(-\gamma_s \sin^2\theta \right) \int_0^\infty t \exp\left(-\frac{(t - \sqrt{2\gamma_s} \cos\theta)^2}{2} \right) dt,
\end{aligned}
$$

where $\gamma_s = \mathcal{E}/N_0$ and $t = v/\sqrt{N_0/2}$.

The larger the γ_s,
the narrower the $f(\theta|\boldsymbol{s}_0)$,
and the smaller the P_e.

$$P_e = 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} f(\theta|\boldsymbol{s}_0)\,d\theta$$

- When $M = 2$, binary PSK is antipodal ($\mathcal{E} = \mathcal{E}_b$):

$$P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

- When $M = 4$, it is QPSK ($\mathcal{E} = 2\mathcal{E}_b$).

$$P_e = 1 - \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2$$

- When $M > 4$, no simple Q-function expression for P_e! However, we can obtain a good approximation.

$$f(\theta|\boldsymbol{s}_0)$$

$$= \frac{1}{2\pi} e^{-\gamma_s \sin^2 \theta} \int_0^\infty t \exp\left(-\frac{(t - \sqrt{2\gamma_s} \cos \theta)^2}{2}\right) dt$$

$$= \frac{1}{2\pi} e^{-\gamma_s \sin^2 \theta} \int_{-\sqrt{2\gamma_s} \cos \theta}^\infty (x + \sqrt{2\gamma_s} \cos \theta) e^{-x^2/2} dx$$

$$\quad \text{(Let } x = t - \sqrt{2\gamma_s} \cos \theta.\text{)}$$

$$= \frac{1}{2\pi} e^{-\gamma_s \sin^2 \theta} \left(\int_{-\sqrt{2\gamma_s} \cos \theta}^\infty x e^{-x^2/2} dx \right.$$

$$\left. \quad\quad + \sqrt{4\pi\gamma_s} \cos \theta \int_{-\sqrt{2\gamma_s} \cos \theta}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)$$

$$= \frac{1}{2\pi} e^{-\gamma_s \sin^2 \theta} \left(e^{-\gamma_s \cos^2 \theta} + \sqrt{4\pi\gamma_s} \cos \theta \left[1 - Q\left(\sqrt{2\gamma_s} \cos \theta\right) \right] \right)$$

$$\geq \frac{1}{2\pi} e^{-\gamma_s \sin^2 \theta} \left(e^{-\gamma_s \cos^2 \theta} + \sqrt{4\pi\gamma_s} \cos \theta \left[1 - \frac{1}{\sqrt{4\pi\gamma_s} \cos(\theta)} e^{-\gamma_s \cos^2 \theta} \right] \right)$$

where we have used $Q(u) \leq U1(u) = \frac{1}{\sqrt{2\pi}u} e^{-u^2/2}$ for $u \geq 0$.

$$f(\theta|\mathbf{s}_0)$$

$$\geq \frac{1}{2\pi}e^{-\gamma_s \sin^2\theta}\left(e^{-\gamma_s \cos^2\theta} + \sqrt{4\pi\gamma_s}\cos\theta\left[1 - \frac{1}{\sqrt{4\pi\gamma_s}\cos(\theta)}e^{-\gamma_s \cos^2\theta}\right]\right)$$

$$= \sqrt{\frac{\gamma_s}{\pi}}e^{-\gamma_s \sin^2\theta}\cos\theta.$$

Thus

$$
\begin{aligned}
P_e &= 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} f(\theta|\mathbf{s}_0)d\theta \\
&\leq 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \sqrt{\frac{\gamma_s}{\pi}}e^{-\gamma_s \sin^2\theta}\cos\theta\, d\theta \\
&= 1 - \int_{-\sqrt{2\gamma_s}\sin(\pi/M)}^{\sqrt{2\gamma_s}\sin(\pi/M)} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}du \quad (u = \sqrt{2\gamma_s}\sin\theta) \\
&= 2Q\left(\sqrt{2\gamma_s}\sin(\pi/M)\right)
\end{aligned}
$$

Efficiency of PSK

For large M, we can approximate $\sin(\pi/M) \leq \pi/M$ and $\gamma_s = \frac{\mathcal{E}}{N_0} = \log_2(M)\frac{\mathcal{E}_b}{N_0}$,

$$P_e \approx \left(\left\{ \begin{matrix} \not{\leq} \\ \not{\geq} \end{matrix} \right. \right) 2Q\left(\sqrt{2\pi^2 \frac{\log_2 M}{M^2} \frac{\mathcal{E}_b}{N_0}} \right)$$

- To increase rate by 1 bit, we need to double M.
- To keep (almost) the same P_e, we need $\mathcal{E}_{\mathrm{bavg}}$ to quadruple.

M	2	4	8	16
$\frac{\log_2(M)}{M^2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{64}$	$\frac{1}{64}$

2→4	4→8	8→16	$\cdots$	$M \to 2M$ as M large
2	2.67	3	$\cdots$	4

- Increase rate by 1 bit $\implies$ increase $\mathcal{E}_{\mathrm{bavg}}$ by 6 dB as M large
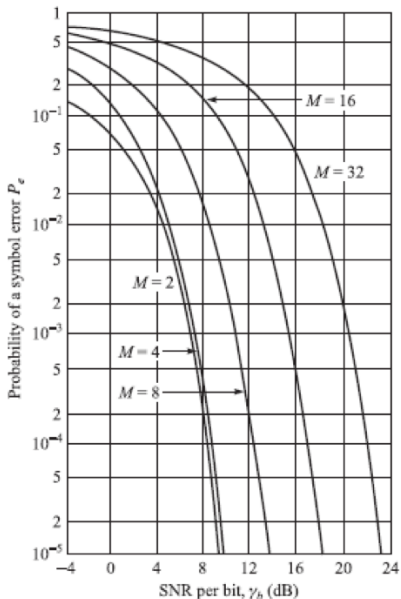
PSK performance

Same as PAM:

The larger the M is,
the worse the symbol performance!

At small M, increasing by 1 bit
only requires additional 4 dB
(such as $M = 4 \rightarrow 8$).

Difference from $M = 2$ to 4 is
very limited!

The true winner will be more
"clear" from BER vs. $\mathcal{E}_b/N_0$ plot.

M-ary (rectangular) QAM signaling

- M is usually a product number, $M = M_1 M_2$
- M-ary QAM is composed of two independent M_i-ary PAM (because the noise is white)

$$
\begin{aligned}
\mathcal{S}_{PAM_i} &= \left\{ \pm\frac{1}{2}d_{\min}, \pm\frac{3}{2}d_{\min}, \cdots, \pm\frac{M_i - 1}{2}d_{\min} \right\} \\
\mathcal{S}_{QAM} &= \left\{ (x, y) \,:\, x \in \mathcal{S}_{PAM_1} \text{ and } y \in \mathcal{S}_{PAM_2} \right\}
\end{aligned}
$$

From Slide 4-75, we have

$$
\mathbb{E}[|x|^2] = \frac{M_1^2 - 1}{12}d_{\min}^2 \quad \text{and} \quad \mathbb{E}[|y|^2] = \frac{M_2^2 - 1}{12}d_{\min}^2
$$

Thus for M-ary QAM we have

$$
\mathcal{E}_{\text{bavg}} = \frac{\mathbb{E}[|x|^2] + \mathbb{E}[|y|^2]}{\log_2(M)} = \frac{(M_1^2 - 1) + (M_2^2 - 1)}{12\log_2 M}d_{\min}^2
$$

Hence

$$
\begin{aligned}
P_{e,M\text{-QAM}} &= 1 - \left(1 - P_{e,M_1\text{-PAM}}\right)\left(1 - P_{e,M_2\text{-PAM}}\right) \\
&= P_{e,M_1\text{-PAM}} + P_{e,M_2\text{-PAM}} - P_{e,M_1\text{-PAM}}P_{e,M_2\text{-PAM}}
\end{aligned}
$$

Since (cf. Slide 4-77)

$$
P_{e,M_i\text{-PAM}} = 2\left(1 - \frac{1}{M_i}\right)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \leq 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)
$$

we have

$$
\begin{aligned}
P_{e,M\text{-QAM}} &\leq P_{e,M_1\text{-PAM}} + P_{e,M_2\text{-PAM}} \leq 4Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \\
&= 4Q\left(\sqrt{\left(\frac{6\log_2 M}{M_1^2 + M_2^2 - 2}\right)\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right)
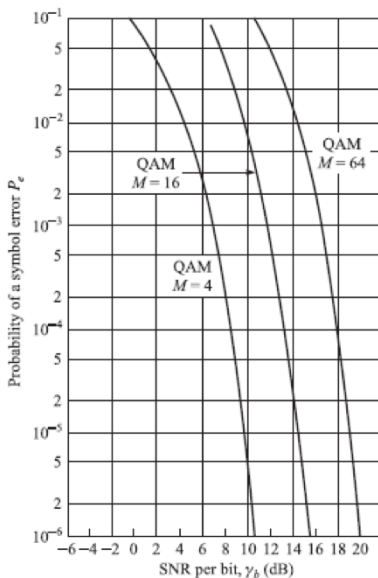\end{aligned}
$$

Efficiency of QAM

When $M_1 = M_2$,

$$P_{e,M\text{-QAM}} \leq 4Q\left(\sqrt{\left(\frac{3\log_2 M}{M-1}\right)\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right)$$

- To increase rate by 2 bit, we need to quadruple M.
- To keep (almost) the same P_e, we need $\mathcal{E}_{\text{bavg}}$ to double.

M	4	16	64
$\frac{\log_2(M)}{M-1}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{2}{21}$

4→16	16→64	$\cdots$	$M \to 4M$ as M large
2.5	2.8	$\cdots$	4

- Equivalently, increase rate by 1 bit $\implies$ increase $\mathcal{E}_{\text{bavg}}$ by 3 dB as M large.
- QAM is more power efficient than PAM and PSK.

QAM Performance

Comparison between M-PSK and M-QAM

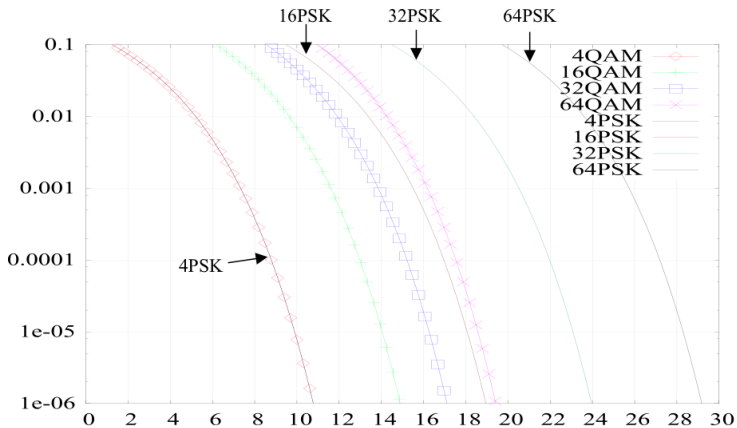$$P_{e,M\text{-PSK}} \le 2Q\left(\sqrt{2\sin^2(\pi/M)\log_2(M)\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right)$$

$$P_{e,M\text{-QAM}} \le 4Q\left(\sqrt{\frac{3}{(M-1)}\log_2(M)\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right).$$

Since (from the two upper bounds)

$$\frac{3/(M-1)}{2\sin^2(\pi/M)} > 1 \text{ for } M \ge 4 \text{ (and } 1 < M \le 3),$$

M-QAM (anticipatively) performs better than M-PSK.

M	4	8	16	32	64
$10\log_{10}(3/[2(M-1)\sin^2(\pi/M)])$	0	1.65	4.20	7.02	9.95

- 4PSK = 4QAM
- 16PSK is 4dB poorer than 16QAM
- 32PSK performs 7dB poorer than 32QAM
- 64PSK performs 10dB poorer than 64QAM

4.3-4 Demodulation and detection

For the bandpass signals, we use two basis functions

$$\phi_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}} \, g(t) \cos\left(2\pi f_c t\right)$$

$$\phi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}} \, g(t) \sin\left(2\pi f_c t\right)$$

for $0 \le t < T$.

Note:

- We usually "transform" the bandpass signal to its lowpass equivalent signal, and then "vectorize" the lowpass equivalent signal.
- This section shows that we can actually "vectorize" the bandpass signal directly.

Transmission of PAM signals

We use the (bandpass) constellation set $\mathcal{S}_{PAM}$

$$\mathcal{S}_{PAM} = \left\{ \pm\frac{1}{2}d_{\min}, \pm\frac{3}{2}d_{\min}, \cdots, \pm\frac{M-1}{2}d_{\min} \right\}$$

where

$$d_{\min} = \sqrt{\frac{12\log_2 M}{M^2 - 1}\mathcal{E}_{\text{bavg}}}$$

Hence the (bandpass) M-ary PAM waveforms are

$$\mathcal{S}_{PAM}(t) = \left\{ \pm\frac{d_{\min}}{2}\phi_1(t), \pm\frac{3d_{\min}}{2}\phi_1(t), \cdots, \pm\frac{(M-1)d_{\min}}{2}\phi_1(t) \right\}$$

Demodulation and detection of PAM

Assuming (bandpass) $s_m(t) \in \mathcal{S}_{PAM}(t)$ was transmitted, the received signal is

$$r(t) = s_m(t) + n(t)$$

Define

$$r = \langle r(t), \phi_1(t) \rangle = \int_0^T r(t)\phi_1^*(t)\, dt$$

The (bandpass) MAP rule (cf. Slide 4-27) is

$$\hat{m} = \arg \max_{1 \le m \le M} \left[r \cdot s_m + \frac{N_0}{2} \log P_m - \frac{1}{2}\left| s_m \right|^2 \right]$$

where $P_m = \Pr\{s_m\}$.

Now we in turn "implement" the MAP rule in baseband!

Alternative description of transmission of PAM

Define the set of baseband PAM waveforms

$$\mathcal{S}_{PAM,\ell}(t) = \{s_{m,\ell}(t) = s_{m,\ell}\phi_{1,\ell}(t) : s_m \in \mathcal{S}_{PAM}\}$$

where $\phi_{1,\ell}(t) = \sqrt{\frac{1}{\mathcal{E}_g}}\, g(t)$ and $s_{m,\ell} = \sqrt{2}s_m$.

Then the bandpass signals are

$$\mathcal{S}_{PAM}(t) = \left\{\mathbf{Re}\left[s_{m,\ell}(t)e^{\imath 2\pi f_c t}\right] : s_{m,\ell}(t) \in \mathcal{S}_{PAM,\ell}(t)\right\}$$

For (ideally bandlimited) baseband signals (cf. Slide 2-102), we have

$$
\begin{aligned}
\langle x(t), y(t) \rangle &= \left\langle \mathbf{Re}\left\{ x_\ell(t) e^{\imath 2\pi f_c t} \right\}, \mathbf{Re}\left\{ y_\ell(t) e^{\imath 2\pi f_c t} \right\} \right\rangle \\
&= \frac{1}{2} \mathbf{Re}\left\{ \langle x_\ell(t), y_\ell(t) \rangle \right\}.
\end{aligned}
$$

Hence, the baseband MAP rule is

$$
\begin{aligned}
\hat{m} &= \arg \max_{1 \le m \le M} \left[2r \cdot s_m + N_0 \log P_m - |s_m|^2 \right] \quad \text{(passband rule)} \\
&= \arg \max_{1 \le m \le M} \left[\mathbf{Re}\left\{ r_\ell \cdot s_{m,\ell} \right\} + N_0 \log P_m - \frac{1}{2} |s_{m,\ell}|^2 \right] \\
&= \arg \max_{1 \le m \le M} \left[\mathbf{Re}\left\{ \int_{-\infty}^{\infty} r_\ell(t) s_{m,\ell}^*(t) \, dt \right\} + N_0 \log P_m \right. \\
&\qquad\qquad\qquad \left. - \frac{1}{2} \int_{-\infty}^{\infty} |s_{m,\ell}(t)|^2 \, dt \right]
\end{aligned}
$$

Transmission of PSK signals

(Bandpass) Signal constellation of M-ary PSK is

$$\mathcal{S}_{PSK} = \left\{ \boldsymbol{s}_k = \sqrt{\mathcal{E}} \left[\cos\left(\frac{2\pi k}{M}\right), \sin\left(\frac{2\pi k}{M}\right) \right]^{\top} : k \in \mathbb{Z}_M \right\},$$

where $\mathbb{Z}_M = \{0, 1, 2, \ldots, M-1\}$.

Hence the (bandpass) M-ary PSK waveforms are

$$\mathcal{S}_{PSK}(t)$$
$$= \left\{ s_m(t) = \sqrt{\mathcal{E}} \left[\cos\left(\frac{2\pi k}{M}\right) \phi_1(t) + \sin\left(\frac{2\pi k}{M}\right) \phi_2(t) \right] : k \in \mathbb{Z}_M \right\}$$

where $\phi_1(t)$ and $\phi_2(t)$ are defined in Slide 4-96.

Alternative description of transmission of PSK

Down to the baseband PSK signals:

$$\mathcal{S}_{PSK,\ell} = \left\{ s_{k,\ell} = \sqrt{2\mathcal{E}} e^{\imath \frac{2\pi k}{M}} : k \in \mathbb{Z}_M \right\}$$

The set of baseband PSK waveforms is

$$\mathcal{S}_{PSK,\ell}(t) = \left\{ s_{k,\ell}(t) = \sqrt{2\mathcal{E}} e^{\imath \frac{2\pi k}{M}} \underbrace{\frac{g(t)}{\|g(t)\|}}_{\text{basis}} : k \in \mathbb{Z}_M \right\}$$

Note: It is a one-dimensional signal in "complex" domain, but a two-dimensional signal in "real" domain!

Then the bandpass PSK waveforms are

$$\mathcal{S}_{PSK}(t) = \left\{ \mathbf{Re}\left[s_{k,\ell}(t) e^{\imath 2\pi f_c t} \right] : s_{k,\ell}(t) \in \mathcal{S}_{PSK,\ell}(t) \right\}$$

Demodulation and detection of PSK signals

Given (bandpass) $s_m(t) \in \mathcal{S}_{PSK}(t)$ was transmitted, the bandpass received signal is

$$r(t) = s_m(t) + n(t)$$

Let $r_\ell(t)$ be the lowpass equivalent (received) signal

$$r_\ell(t) = s_{m,\ell}(t) + n_\ell(t)$$

Then compute

$$r_\ell = \left\langle r_\ell(t), \frac{g(t)}{\|g(t)\|} \right\rangle = \int_0^\top r_\ell(t) \frac{g(t)^*}{\|g(t)\|}\, dt.$$

The baseband MAP rule is

$$\begin{aligned}
\hat{m} &= \arg \max_{1 \le m \le M} \left\{ \mathbf{Re}\{r_\ell s_{m,\ell}^*\} + N_0 \ln P_m - \frac{1}{2}(2\mathcal{E}) \right\} \\
&= \arg \max_{1 \le m \le M} \left\{ \mathbf{Re}\{r_\ell s_{m,\ell}^*\} + N_0 \ln P_m \right\}
\end{aligned}$$

Transmission of QAM

(Bandpass) Signal constellation of M-ary QAM with $M = M_1 M_2$ is

$$
\begin{aligned}
\mathcal{S}_{PAM_i} &= \left\{ \pm \frac{1}{2} d_{\min}, \pm \frac{3}{2} d_{\min}, \cdots, \pm \frac{M_i - 1}{2} d_{\min} \right\} \\
\mathcal{S}_{QAM} &= \left\{ (x, y) : x \in \mathcal{S}_{PAM_1} \text{ and } y \in \mathcal{S}_{PAM_2} \right\}
\end{aligned}
$$

where from Slide 4-90

$$
d_{\min} = \sqrt{\frac{12 \log_2 M}{M_1^2 + M_2^2 - 2} \mathcal{E}_{\text{bavg}}}
$$

Hence the M-ary QAM waveforms are

$$
\mathcal{S}_{QAM}(t) = \left\{ x\phi_1(t) + y\phi_2(t) : x \in \mathcal{S}_{PAM_1} \text{ and } y \in \mathcal{S}_{PAM_2} \right\}
$$

Demodulation of QAM is similar to that of PSK; hence we omit it.

In summary: Theory for lowpass MAP detection

- Let $\mathcal{S}_\ell = \{\boldsymbol{s}_{1,\ell}, \cdots, \boldsymbol{s}_{M,\ell}\} \subset \mathbb{C}^N$ be the signal constellation of a certain modulation scheme with respect to the lowpass basis functions $\{\phi_{n,\ell}(t) : n = 1, 2, \cdots, N\}$ (Dimension = N).

- The lowpass equivalent signals are

$$s_{m,\ell}(t) = \sum_{n=1}^{N} s_{m,n,\ell} \phi_{n,\ell}(t)$$

where $\boldsymbol{s}_{m,\ell} = \begin{bmatrix} s_{m,1,\ell} & \cdots & s_{m,N,\ell} \end{bmatrix}^\top$.

- The corresponding bandpass signals are

$$s_m(t) = \mathbf{Re}\left\{ s_{m,\ell}(t) e^{\imath 2\pi f_c t} \right\}$$

Note

$$\mathcal{E}_m = \|s_m(t)\|^2 = \frac{1}{2}\|s_{m,\ell}(t)\|^2 = \frac{1}{2}\|\boldsymbol{s}_{m,\ell}\|^2 = \frac{1}{2}\mathcal{E}_{m,\ell}$$

Given $s_m(t)$ was transmitted, the bandpass received signal is

$$r(t) = s_m(t) + n(t).$$

Let $r_\ell(t)$ be the lowpass equivalent signal

$$r_\ell(t) = s_{m,\ell}(t) + n_\ell(t)$$

Set for $1 \leq n \leq N$,

$$
\begin{aligned}
r_{n,\ell} &= \langle r_\ell(t), \phi_{n,\ell}(t) \rangle = \int_0^T r_\ell(t) \phi_{n,\ell}^*(t) \, dt \\
n_{n,\ell} &= \langle n_\ell(t), \phi_{n,\ell}(t) \rangle = \int_0^T n_\ell(t) \phi_{n,\ell}^*(t) \, dt
\end{aligned}
$$

Hence we have

$$\boldsymbol{r}_\ell = \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell$$

The lowpass equivalent MAP detection then seeks to find

$$\hat{m} = \arg\max_{1 \le m \le M} \left\{ \mathbf{Re}\left\{ \boldsymbol{r}_\ell \boldsymbol{s}_{m,\ell}^* \right\} + N_0 \log P_m - \frac{1}{2} \left\| \boldsymbol{s}_{m,\ell} \right\|^2 \right\}$$

Since $\boldsymbol{n}_\ell$ is complex (cf. Slide 4-11), the multiplicative constant a on σ^2 (in the below equation) is equal to 1:

$$\hat{m} = \arg\max_{1 \le m \le M} P_m f(\boldsymbol{r}_\ell | \boldsymbol{s}_{m,\ell}) = \arg\max_{1 \le m \le M} P_m \exp\left(-\frac{\left\| \boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell} \right\|^2}{a\sigma^2} \right)$$

$$= \arg\max_{1 \le m \le M} \left(\mathbf{Re}\{ \boldsymbol{r}_\ell \boldsymbol{s}_{m,\ell}^* \} + \frac{1}{2} a\sigma^2 \log P_m - \frac{1}{2} \left\| \boldsymbol{s}_{m,\ell} \right\|^2 \right)$$

where $\sigma^2 = 2N_0$ (for baseband noise) and $E[\boldsymbol{n}_\ell \boldsymbol{n}_\ell^H] = \begin{bmatrix} \sigma^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^2 \end{bmatrix}$.

Same derivation can be done for passband signal with $a = 2$ and $\sigma^2 = \frac{N_0}{2}$ (cf. Slide 4-27). This coincides with the filtered white noise derivation on Slide 2-72.

Appendix: Should we use $E[|n_\ell|^2] = \sigma^2 = 2N_0$ when doing baseband simulation?

- $\mathcal{E}_b/N_0$ is an essential index in the performance evaluation of a communication system.

- In general, $\mathcal{E}_b$ should be the passband transmission energy per information bit, and N_0 should be from the passband noise.

- So, for example, for BPSK passband transmission,

$$r(t) = \pm\sqrt{2\mathcal{E}_b}\frac{g(t)}{\|g(t)\|}\cos(2\pi f_c t) + n(t) \text{ with } S_n(f) = \frac{N_0}{2}.$$

   - Vectorization using $\phi(t) = \sqrt{2}\frac{g(t)}{\|g(t)\|}\cos(2\pi f_c t)$ yields

$$r = \langle r(t), \phi(t)\rangle = \left\langle \pm\sqrt{2\mathcal{E}_b}\cos(2\pi f_c t), \phi(t)\right\rangle + \langle n(t), \phi(t)\rangle$$

$$= \pm\sqrt{\mathcal{E}_b} + n \text{ with } \mathbb{E}[n^2] = \int_0^T\int_0^T \frac{N_0}{2}\delta(t-s)\phi(t)\phi(s)\,dt\,ds = \frac{N_0}{2}.$$

- This is equivalent to

$$
\begin{aligned}
r_\ell &= \left\langle r_\ell(t), \frac{g(t)}{\|g(t)\|} \right\rangle = \left\langle r_\ell(t), \frac{g(t)}{\|g(t)\|} \right\rangle + \left\langle n_\ell(t), \frac{g(t)}{\|g(t)\|} \right\rangle \\
&= s_{m,\ell} + n_\ell = \pm\sqrt{2\mathcal{E}_b} + n_\ell
\end{aligned}
$$

Equivalently since only real part contains info,

$$
\mathbf{Re}\left\{ \frac{1}{\sqrt{2}} r_\ell \right\} = \pm\sqrt{\mathcal{E}_b} + \frac{1}{\sqrt{2}} n_{x,\ell}, \quad \text{as } n_\ell = n_{x,\ell} + \imath\, n_{y,\ell}
$$

where $\mathbb{E}[(\frac{n_{x,\ell}}{\sqrt{2}})^2] = \frac{1}{2}\mathbb{E}[n_{x,\ell}^2] = \frac{1}{2}\left(\frac{1}{2}\mathbb{E}[|n_\ell|^2]\right) = \frac{1}{2}\left(\frac{1}{2}(2N_0)\right) = \frac{N_0}{2}$.

- In most cases, we will use $\boxed{r = \pm\sqrt{\mathcal{E}_b} + n}$ directly in both analysis and simulation (in our technical papers) but not the baseband equivalent system $\boxed{r_\ell = \pm\sqrt{2\mathcal{E}_b} + n_\ell}$.

- For QPSK, the simulated system should be

$$r_x + \imath\, r_y = \left\{ \pm\sqrt{\mathcal{E}},\ \pm \imath\, \sqrt{\mathcal{E}} \right\} + (n_x + \imath\, n_y)$$

with $\mathbb{E}[n_x^2] = \mathbb{E}[n_y^2] = \frac{N_0}{2}$, where n_x and n_y are the **passband projection noises**.

  - Rotating 45 degree does not change the noise statistics and yields

$$r_x + \imath\, r_y = \pm\sqrt{\frac{\mathcal{E}}{2}} \pm \imath\, \sqrt{\frac{\mathcal{E}}{2}} + (n_x + \imath\, n_y) = \pm\sqrt{\mathcal{E}_b} \pm \imath\, \sqrt{\mathcal{E}_b} + (n_x + \imath\, n_y).$$

4.4 Optimal detection and error probability for power limited signaling

Orthogonal (FSK) signaling

(Bandpass) Signal constellation of M-ary orthogonal signaling (OS) is

$$\mathcal{S}_{OS} = \left\{ \boldsymbol{s}_1 = [\sqrt{\mathcal{E}}, 0, \cdots, 0]^\top, \cdots, \boldsymbol{s}_M = \left[0, \cdots, 0, \sqrt{\mathcal{E}}\right]^\top \right\},$$

where the dimension N is equal to M.

Given $\boldsymbol{s}_1$ transmitted, the received signal vector is

$$\boldsymbol{r} = \boldsymbol{s}_1 + \boldsymbol{n}$$

with ($\boldsymbol{n}$ being the bandpass projection noise and)

$$
\begin{aligned}
r_1 &= \sqrt{\mathcal{E}} + n_1 \\
r_2 &= n_2 \\
&\vdots \quad \vdots \quad \vdots \\
r_M &= n_M
\end{aligned}
$$

By assuming the signals s_m are equiprobable, the (bandpass) MAP/ML decision is

$$\hat{m} = \arg \max_{1 \leq m \leq M} r^\top s_m$$

Hence, given s_1 transmitted, we need for correct decision

$$\langle r, s_1 \rangle = \mathcal{E} + \sqrt{\mathcal{E}}\, n_1 > \langle r, s_m \rangle = \sqrt{\mathcal{E}}\, n_m \quad \text{for } 2 \leq m \leq M$$

It means

$$\Pr\left\{\text{Correct}|s_1\right\} = \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M\right\}.$$

By symmetry, we have
$\Pr\left\{\text{Correct}|s_1\right\} = \Pr\left\{\text{Correct}|s_2\right\} = \cdots = \Pr\left\{\text{Correct}|s_M\right\}$;
hence

$$\Pr\left\{\text{Correct}\right\} = \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M\right\}.$$

$$
\begin{aligned}
P_c &= \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M\right\} \\
&= \int_{-\infty}^{\infty} \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M \Big| n_1\right\} f(n_1) \, dn_1 \\
&= \int_{-\infty}^{\infty} \left(\Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2 \Big| n_1\right\}\right)^{M-1} f(n_1) \, dn_1 \\
&\left(= \int_{-\infty}^{\infty} \left[Q\left(\frac{0 - (n_1 + \sqrt{\mathcal{E}})}{\sqrt{\frac{N_0}{2}}}\right)\right]^{M-1} f(n_1) \, dn_1\right) \\
&= \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}}\right)\right]^{M-1} f(n_1) \, dn_1
\end{aligned}
$$

$$
\boxed{\Pr\left\{\mathcal{N}\left(m, \sigma^2\right) < r\right\} = Q\left(\frac{m-r}{\sigma}\right)}
$$

Hence

$$
\begin{aligned}
P_e &= 1 - P_c \\
&= 1 - \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} f(n_1) \, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} \right) \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_1^2}{N_0}} \, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q(x) \right]^{M-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sqrt{2k\gamma_b})^2}{2}} \, dx
\end{aligned}
$$

where $x = \frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}}$, and $\gamma_b = \mathcal{E}_b / N_0$, and $k = \log_2(M)$.

Due to the complete symmetry of (binary) orthogonal signaling, the bit error rate P_b (see the red-color equation below) has a closed-form formula.

$$\begin{cases} \Pr\{\hat{m} = i\} = P_c & \text{if } i = m \quad (e = \boxed{0 \text{ bit error}}) \\ \Pr\{\hat{m} = i\} = \frac{P_e}{M-1} & \text{if } i \neq m \quad (e = \boxed{1 \sim k \text{ bits in error}}) \end{cases}$$

where $k = \log_2(M)$.

We then have

$$\begin{aligned} P_b &= \frac{E[e]}{k} \\ &= \frac{1}{k} \sum_{e=1}^{k} e \cdot \binom{k}{e} \frac{P_e}{M-1} \\ &= \frac{1}{2} \cdot \frac{2^k}{2^k - 1} P_e \approx \frac{1}{2} P_e \end{aligned}$$

Different from PAM/PSK:

The larger the M is,
the better the performance!!!

For example, to achieve $P_b = 10^{-5}$,
one needs $\gamma_b = 12$ dB for $M = 2$;

but it only requires $\gamma_b = 6$ dB
for $M = 64$;

a 6 dB save in transmission power!
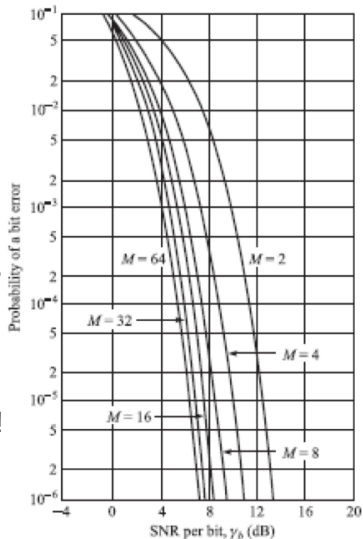
Error probability upper bound

Since P_b decreases with respect to M, is it possible that

$$\lim_{M \to \infty} P_e = \lim_{M \to \infty} P_b = 0?$$

Shannon limit of the AWGN channel:

1. If $\gamma_b > \log(2) \approx -1.6$ dB, then
   $\lim_{M \to \infty} P_e = \inf_{M \geq 1} P_e = 0$.

2. If $\gamma_b < \log(2) \approx -1.6$ dB, then $\inf_{M \geq 1} P_e > 0$.

- For item 1, we can adopt the derivation in **Section 6.6: Achieving channel capacities with orthogonal signals** to prove it directly.

For $x_0 > 0$, we use

$$1 - [1 - Q(x)]^{M-1} \leq \begin{cases} 1 & x < x_0 \\ (M-1)Q(x) & x \geq x_0 \end{cases} \leq \begin{cases} 1 & x < x_0 \\ Me^{-x^2/2} & x \geq x_0 \end{cases}$$

Proof: For $0 \leq u \leq 1$, by induction, $1 - (1-u)^n \leq nu$ implies
$1 - (1-u)^{n+1} = (1-u)(1-(1-u)^n) + u \leq (1-u) \cdot nu + u \leq nu + u$.

Then,

$$
\begin{aligned}
P_e &= \int_{-\infty}^{\infty} \left(1 - [1 - Q(x)]^{M-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \\
&= \int_{-\infty}^{x_0} \left(1 - [1 - Q(x)]^{M-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \\
&\quad + \int_{x_0}^{\infty} \left(1 - [1 - Q(x)]^{M-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \\
&\leq \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx + \int_{x_0}^{\infty} (M-1)Q(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx
\end{aligned}
$$
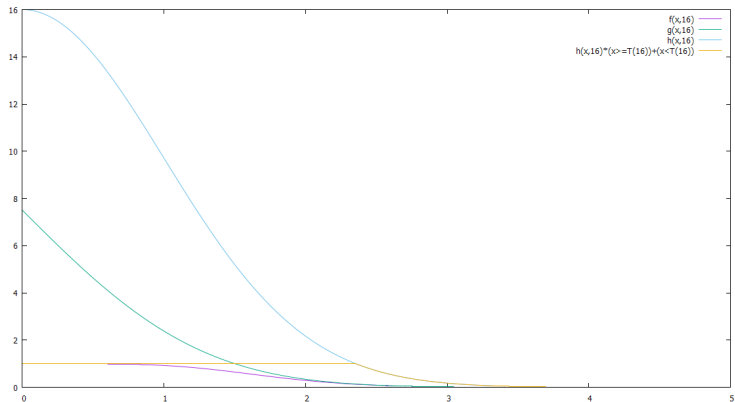
$$\begin{cases} f(x, M) = 1 - [1 - Q(x)]^{M-1} \\ g(x, M) = (M - 1)Q(x) \\ h(x, M) = Me^{-x^2/2} \end{cases} \quad \text{with } T(M) = \sqrt{2k \log(2)} = \sqrt{2 \log(M)}$$

Hence,

$$P_e \le \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx + M \int_{x_0}^{\infty} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx$$

$$\Rightarrow P_e \le \frac{1}{\sqrt{2\pi}} \min_{x_0 > 0} \left(\int_{-\infty}^{x_0} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx + M \int_{x_0}^{\infty} e^{-x^2/2} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \right)$$

$$\frac{\partial}{\partial x_0} \left(\int_{-\infty}^{x_0} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx + M \int_{x_0}^{\infty} e^{-x^2/2} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \right)$$

$$= e^{-\frac{(x_0-\sqrt{2k\gamma_b})^2}{2}} - M e^{-x_0^2/2} e^{-\frac{(x_0-\sqrt{2k\gamma_b})^2}{2}} \begin{cases} > 0, & x > x_0^* \\ < 0, & 0 < x < x_0^* \end{cases}$$

where $x_0^* = \sqrt{2k \log(2)}$.

$$
\begin{aligned}
P_e &\leq \int_{-\infty}^{\sqrt{2k\log(2)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} dx \\
&\quad + M \int_{\sqrt{2k\log(2)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} dx \\
&= \int_{-\infty}^{\sqrt{2k\log(2)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} dx \\
&\quad + \frac{Me^{-k\gamma_b/2}}{\sqrt{2}} \int_{\sqrt{2k\log(2)}}^{\infty} \frac{1}{\sqrt{2\pi(1/2)}} e^{-\frac{(x-\sqrt{k\gamma_b/2})^2}{2(1/2)}} dx \\
&= Q\left(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)}\right) \\
&\quad + \frac{Me^{-k\gamma_b/2}}{\sqrt{2}} \left[1 - Q\left(\frac{\sqrt{k\gamma_b/2} - \sqrt{2k\log(2)}}{\sqrt{1/2}} \right) \right]
\end{aligned}
$$

$$
\boxed{\Pr\left\{ \mathcal{N}\left(m, \sigma^2\right) < r \right\} = Q\left(\frac{m-r}{\sigma} \right)}
$$

$$
\begin{aligned}
P_e & \leq Q\left(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)}\right) \\
& \quad + \frac{Me^{-k\gamma_b/2}}{\sqrt{2}} Q\left(\frac{\sqrt{2k\log(2)} - \sqrt{k\gamma_b/2}}{\sqrt{1/2}}\right) \\
& \leq \begin{cases}
\frac{1}{2}e^{-(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)})^2/2} + \frac{2^k e^{-k\gamma_b/2}}{2\sqrt{2}} e^{-(\sqrt{4k\log(2)} - \sqrt{k\gamma_b})^2/2}, \\
\qquad\qquad\qquad\qquad \text{if } \log(2) < \gamma_b < 4\log(2) \\
\frac{1}{2}e^{-(\sqrt{2k\gamma_b} - \sqrt{2k\log(2)})^2/2} + \frac{2^k e^{-k\gamma_b/2}}{\sqrt{2}} \cdot 1, \\
\qquad\qquad\qquad\qquad \text{if } \gamma_b \geq 4\log(2)
\end{cases} \\
& = \begin{cases}
\frac{1}{2}e^{-k(\sqrt{\gamma_b} - \sqrt{\log(2)})^2} + \frac{1}{2\sqrt{2}}e^{-k(\sqrt{\gamma_b} - \sqrt{\log(2)})^2}, \\
\qquad\qquad\qquad\qquad \text{if } \log(2) < \gamma_b < 4\log(2) \\
\frac{1}{2}e^{-k(\sqrt{\gamma_b} - \sqrt{\log(2)})^2} + \frac{1}{\sqrt{2}}e^{-k(\gamma_b - 2\log(2))} \\
\qquad\qquad\qquad\qquad \text{if } \gamma_b \geq 4\log(2)
\end{cases}
\end{aligned}
$$

Thus, $\gamma_b > \log(2)$ implies $\lim_{k\to\infty} P_e = 0$.

The converse to Shannon limit

> ## Shannon's channel coding theorem
>
> In 1948, Shannon proved that
> - if $R < C$, then P_e can be made arbitrarily small (by extending the code size);
> - if $R > C$, then P_e is bounded away from zero,
>
> where $C = \max_{P_X} I(X; Y)$ is the channel capacity, and R is the code rate.

- For AWGN channels,

$$C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) \quad bit/second \quad (cf. \, Eq. \, (6.5\text{–}43)).$$

Note that W is in Hz=1/second, N_0 is in Joule (so $N_0 W$ is in Joule/second=Watt), and P is in Watt.

Since P (Watt) = R (bit/second) $\times$ $\mathcal{E}_b$ (Joule/bit), we have

$$\boxed{R >} C = W \log_2 \left(1 + \frac{R\mathcal{E}_b}{N_0 W}\right) = \boxed{W \log_2 \left(1 + \frac{R}{W}\gamma_b\right)} \Leftrightarrow \boxed{\gamma_b < \frac{2^{R/W}-1}{R/W}}$$

For M-ary (orthogonal) FSK, $W = \frac{M}{2T}$ and $R = \frac{\log_2(M)}{T}$.

Hence, $R/W = \frac{2\log_2(M)}{M} = \frac{2k}{2^k}$.

This gives that

$$\text{If } \gamma_b < \lim_{k\to\infty} \frac{2^{2k/2^k} - 1}{2k/2^k} = \log(2), \text{ then } P_e \text{ is bounded away from zero.}$$

$$\frac{2^{2k/2^k}-1}{2k/2^k}$$

k
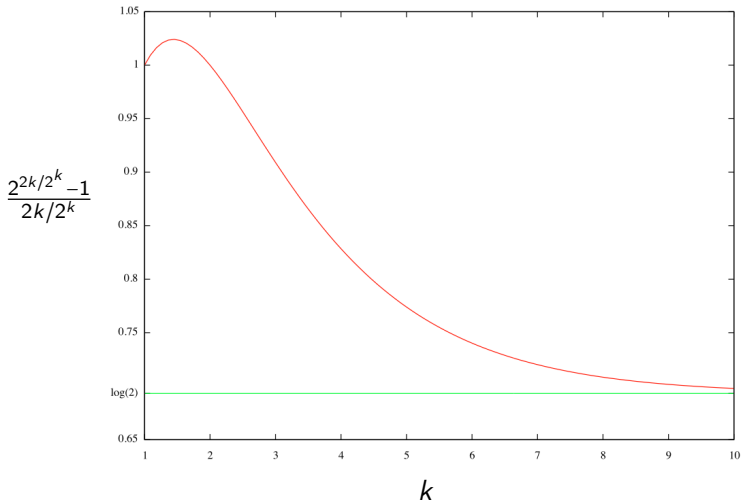
If $\gamma_b < \inf_{k \geq 1} \frac{2^{2k/2^k}-1}{2k/2^k} = \log(2)$, then P_e is bounded away from zero.

Simplex signaling

The *M*-ary simplex signaling can be obtained from *M*-ary FSK by

$$\mathcal{S}_{Simplex} = \{ \boldsymbol{s} - \mathbb{E}[\boldsymbol{s}] : \boldsymbol{s} \in \mathcal{S}_{FSK} \}$$

with the resulting energy

$$\mathcal{E}_{Simplex} = \| \boldsymbol{s} - \mathbb{E}[\boldsymbol{s}] \|^2 = \frac{M-1}{M} \mathcal{E}_{FSK}$$

Thus we can reduce the transmission power without affecting the constellation structure; hence, the performance curve will be shifted left by $10 \log_{10}[M/(M-1)]$ dB.

M	2	4	8	16	32
$10 \log_{10}[M/(M-1)]$	3.01	1.25	0.58	0.28	0.14

Biorthogonal signaling

$$(Bandpass) \quad \mathcal{S}_{BO} = \left\{ [\pm\sqrt{\mathcal{E}}, 0, \cdots, 0]^\top, \cdots, \left[0, \cdots, 0, \pm\sqrt{\mathcal{E}}\right]^\top \right\}$$

where $M = 2N$.

For convenience, we index the symbols by
$$m = -N, \ldots, -1, 1, \ldots, N,$$
where for $\boldsymbol{s}_m = [s_1, \ldots, s_N]^T$, we have

$$s_m = \text{sgn}(m) \cdot \sqrt{\mathcal{E}} \quad \text{and} \quad s_i = 0 \text{ for } i \neq m \text{ and } 1 \leq i \leq N.$$

Note that there are only N noises, i.e., $n_1, n_2, \ldots, n_N$.

Given $\boldsymbol{s}_1 = [\sqrt{\mathcal{E}}, 0, \cdots, 0]^\top$ is transmitted, correct decision calls for
$$\begin{cases} \langle \boldsymbol{r}, \boldsymbol{s}_1 \rangle = \mathcal{E} + \sqrt{\mathcal{E}}\, n_1 \geq \langle \boldsymbol{r}, \boldsymbol{s}_{-1} \rangle = -\mathcal{E} - \sqrt{\mathcal{E}}\, n_1 \\ \langle \boldsymbol{r}, \boldsymbol{s}_1 \rangle = \mathcal{E} + \sqrt{\mathcal{E}}\, n_1 \geq \langle \boldsymbol{r}, \boldsymbol{s}_m \rangle = \text{sgn}(m)\sqrt{\mathcal{E}}\, n_m = \sqrt{\mathcal{E}}|n_m|,\ 2 \leq |m| \leq N \end{cases}$$

$$
\begin{aligned}
P_c &= \Pr\left\{\sqrt{\mathcal{E}} + n_1 > 0, \sqrt{\mathcal{E}} + n_1 > |n_2|, \cdots, \sqrt{\mathcal{E}} + n_1 > |n_N|\right\} \\
&= \int_{-\sqrt{\mathcal{E}}}^{\infty} \Pr\left\{\sqrt{\mathcal{E}} + n_1 > |n_2|, \cdots, \sqrt{\mathcal{E}} + n_1 > |n_N| \,\Big|\, n_1\right\} f(n_1)\, dn_1 \\
&= \int_{-\sqrt{\mathcal{E}}}^{\infty} \left(\Pr\left\{\sqrt{\mathcal{E}} + n_1 > |n_2| \,\Big|\, n_1\right\}\right)^{N-1} f(n_1)\, dn_1 \\
&= \int_{-\sqrt{\mathcal{E}}}^{\infty} \left(1 - 2\Pr\left\{n_2 < -(\sqrt{\mathcal{E}} + n_1) \,\Big|\, n_1\right\}\right)^{N-1} f(n_1)\, dn_1 \\
&= \int_{-\sqrt{\mathcal{E}}}^{\infty} \left[1 - 2Q\left(\frac{0 + (n_1 + \sqrt{\mathcal{E}})}{\sqrt{N_0/2}}\right)\right]^{N-1} f(n_1)\, dn_1
\end{aligned}
$$

$$
\boxed{\Pr\left\{\mathcal{N}\left(m, \sigma^2\right) < r\right\} = Q\left(\frac{m-r}{\sigma}\right)}
$$

Hence

$$
\begin{aligned}
P_e &= 1 - P_c \\
&= 1 - \int_{-\sqrt{\mathcal{E}}}^{\infty} \left[1 - 2Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}} \right) \right]^{M/2-1} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_1^2}{N_0}} \, dn_1 \\
&= \frac{1}{1 - Q(\sqrt{2k\gamma_b})} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \\
&\quad - \int_0^{\infty} \left[1 - 2Q(x) \right]^{M/2-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx \\
&= \int_0^{\infty} \left(\frac{1}{1 - Q(\sqrt{2k\gamma_b})} - \left[1 - 2Q(x) \right]^{M/2-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{2k\gamma_b})^2}{2}} \, dx
\end{aligned}
$$

where $x = \frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{N_0/2}}$, $\mathcal{E} = k\mathcal{E}_b$ and $\gamma_b = \mathcal{E}_b/N_0$.

Similar to orthogonal signals:

The larger the M is,
the better the performance
except $M = 2, 4$.

Note that P_e comparison
does not really tell the winner
in performance.

E.g., $P_e(BPSK) < P_e(QPSK)$
but $P_b(BPSK) = P_b(QPSK)$

The Shannon-limit remains
the same.
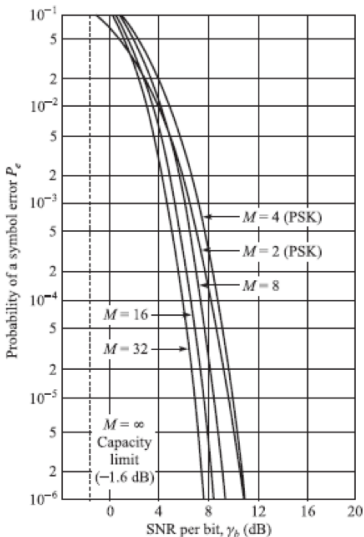
4.6 Comparison of digital signaling methods

- Signals that are both time-limited $[0, T)$ and band-limited $[-W, W]$ do not exist!

- Since the signal intended to be transmitted is always time-limited, we shall relax the strictly band-limited condition to η-band-limited defined as

$$\frac{\int_{-W}^{W} |X(f)|^2 \, df}{\int_{-\infty}^{\infty} |X(f)|^2 \, df} \geq 1 - \eta$$

for some small out-of-band ratio η.

- Such signal does exist!

Theorem 5 (Prolate spheroidal functions)

For a signal $x(t)$ with support in time $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and η-band-limited to W, there exists a set of N orthonormal signals $\{\phi_j(t), 1 \le j \le N\}$ such that

$$\frac{\int_{-\infty}^{\infty} \left| x(t) - \sum_{j=1}^{N} \langle x(t), \phi_j(t) \rangle \phi_j(t) \right|^2 dt}{\int_{-\infty}^{\infty} |X(f)|^2 df} \le 12\eta$$

where $N = \lfloor 2WT + 1 \rfloor$.

Why $N = \lfloor 2WT + 1 \rfloor$?

- For signals with bandwidth W, the Nyquist rate is $2W$ for perfect reconstruction.

- You then get $2W$ samples/second.
  $\implies 2W$ degrees of freedom (per second)

- For time duration T, you get overall $2WT$ samples.
  $\implies 2WT$ degrees of freedom (per T seconds)

Bandwidth efficiency (Simplified view)

$$N = 2WT \Rightarrow \frac{1}{W} = \frac{2T}{N}$$

Since rate $R = \frac{1}{T} \times \log_2 M$, we have for M-ary signaling

$$\frac{R}{W} = \frac{\log_2(M)}{T}\frac{2T}{N} = 2\,\frac{\log_2(M)}{N}$$

where

- $\log_2(M)$ is the number of bits transmitted at a time
- N is **usually** (see SSB PAM and DSB PAM as counterexamples) the dimensionality of the constellation

Thus $\frac{R}{W}$ can be regarded as bit/dimension (it is actually measured as bits per second per Hz).

R/W is called bandwidth efficiency.

Power-limited vs band-limited

Considering modulations satisfying $W = \frac{N}{2T}$, we have:

- for M-ary FSK, $N = M$; hence

$$\left(\frac{R}{W}\right)_{FSK} = \frac{2\log_2(M)}{M} \leq 1$$

FSK improves the performance (e.g., to reduce the required SNR for a given P_e) by increasing M; so it is **good** for channels with power constraint!

On the contrary, this improvement is achieved at a price of increasing bandwidth; so it is **bad** for channels with bandwidth constraint.

Thus, $R/W \leq 1$ is usually referred to as the power-limited region.

Power-limited vs band-limited

Considering modulations satisfying $W = \frac{N}{2T}$, we have:

- for SSB PAM, $N = 1$; hence

$$\left(\frac{R}{W}\right)_{PAM} = 2\log_2(M) > 1$$

- for PSK and QAM, $N = 2$; hence

$$\left(\frac{R}{W}\right)_{PSK} = \left(\frac{R}{W}\right)_{QAM} = \log_2(M) > 1$$

PAM/PSK/QAM worsen the performance (e.g., to increase the required SNR for a given P_e) by increasing M; so it is **bad** for channels with power constraint (because a large signal power may be necessary for performance improvement)!

Yet, such modulation schemes do not require a big bandwidth; so they are **good** for channels with bandwidth constraint.

Thus, $R/W > 1$ is usually referred to as the band-limited region.

Power-limited vs band-limited

Personal comments:

- It is better not to regard N as the **dimension** of the constellation.

- It is the ratio $N = \frac{W}{1/(2T)}$.

- Hence,

  - for SSB PAM, $N = 1$.
  - for QAM/PSK/DSB PAM as well as DPSK, $N = 2$.
  - for orthogonal signals, $N = M$.
  - for bi-orthogonal signals, $N = M/2$.

Shannon's channel coding theorem

Shannon's channel coding theorem for band-limited AWGN channels states the following:

Theorem 6

Given max power constraint P over bandwidth W, the maximal number of bits per channel use, which can be sent over the channel <u>reliably</u>, is

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P}{WN_0} \right) \quad \textit{bits/channel use}$$

Thus during a period of T, we can do $2WT$ samples (i.e., use the channel $2WT$ times).

Hence, for one "consecutive-use" of the AWGN channels,

$$\begin{aligned} C &= \frac{1}{2}\log_2\left(1+\frac{P}{WN_0}\right) \text{ bits/channel use} \\ &\quad \times\ 2WT \text{ channel uses/transmission} \\ &= WT\log_2\left(1+\frac{P}{WN_0}\right) \text{ bits/transmission} \end{aligned}$$

Considering one transmission costs T seconds, we obtain

$$\begin{aligned} C &= WT\log_2\left(1+\frac{P}{WN_0}\right) \text{ bits/transmission} \\ &\quad \times\frac{1}{T} \text{ transmission/second} \\ &= W\log_2\left(1+\frac{P}{WN_0}\right) \text{ bits/second} \end{aligned}$$

Thus, R bits/second $< C$ bits/second implies

$$\frac{R}{W} < \log_2\left(1 + \frac{P}{N_0 W}\right).$$

With

$$\mathcal{E}_b = \frac{\mathcal{E}}{\log_2 M} = \frac{PT}{\log_2(M)} = \frac{P}{R},$$

we have

$$\frac{R}{W} < \log_2\left(1 + \frac{\mathcal{E}_b}{N_0}\frac{R}{W}\right).$$

We then obtain
as previously did
$$\frac{\mathcal{E}_b}{N_0} > \frac{2^{R/W}-1}{R/W}$$

$$\frac{2^{R/W}-1}{R/W}$$

R/W
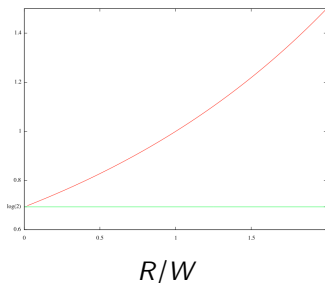
For power-limited channels, we have $0 < \frac{R}{W} \le 1$; hence

$$\frac{\mathcal{E}_b}{N_0} > \lim_{\frac{R}{W} \to 0} \frac{2^{\frac{R}{W}} - 1}{\frac{R}{W}} = \log 2 = -1.59 \text{ dB}$$

which is the Shannon Limit for (orthogonal and bi-orthogonal)
digital communications.
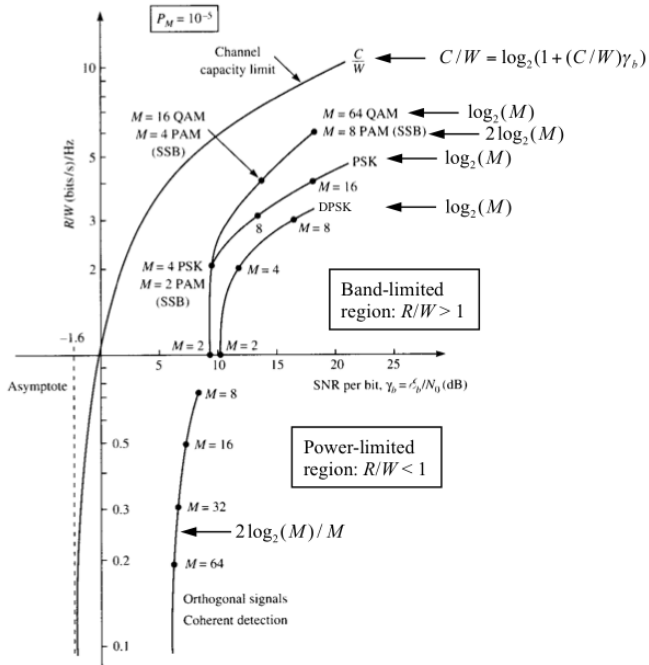
4.5 Optimal noncoherent detection

Basic theory

- Earlier we had assumed that all communication is well synchronized and $r(t) = s_m(t) + n(t)$.

- However, in practice, the signal $s_m(t)$ could be delayed and hence the receiver actually obtains $r(t) = s_m(t - t_d) + n(t)$.

- Without recognizing t_d, the receiver may perform

$$\langle r(t), \phi(t) \rangle = \int_0^T s_m(t - t_d)\phi(t)dt + \int_0^T n(t)\phi(t)dt$$

$$\neq \int_0^T s_m(t)\phi(t)dt + \int_0^T n(t)\phi(t)dt$$

- Two approaches can be used to alleviate this unsynchronization imperfection.

    - Estimate t_d and compensate it before performing demodulation.
    - Use noncoherent detection that can provide acceptable performance without the labor of estimating t_d.

- We use a parameter θ to capture the unsyn (possibly other kinds of) impairment (e.g., amplitude uncertainty) and reformulate the received signal as

$$r(t) = s_m(t; \theta) + n(t)$$

The noncoherent technique can be roughly classified into two cases:

- The distribution of θ is known (semi-blind).
- The distribution of θ is unknown (blind).

- In absence of noise, the transmitter sends $\boldsymbol{s}_m$ but the receiver receives

$$\boldsymbol{s}_{m,\theta} = \begin{bmatrix} \langle s_m(t;\theta), \phi_1(t) \rangle \\ \vdots \\ \langle s_m(t;\theta), \phi_N(t) \rangle \end{bmatrix}$$

MAP: Uncertainty with known statistics

$$r = s_{m,\theta} + n$$

$$
\begin{aligned}
\hat{m} &= \arg \max_{1 \le m \le M} \Pr\{s_m | r\} = \arg \max_{1 \le m \le M} P_m f(r | s_m) \\
&= \arg \max_{1 \le m \le M} P_m \int_{\Theta} f(r | s_{m,\theta}) f_{\theta}(\theta) \, d\theta \\
&= \arg \max_{1 \le m \le M} P_m \int_{\Theta} f_n(r - s_{m,\theta}) f_{\theta}(\theta) \, d\theta
\end{aligned}
$$

The error probability is

$$
P_e = \sum_{m=1}^{M} P_m \int_{\mathcal{D}_m^c} \left(\int_{\Theta} f_n(r - s_{m,\theta}) f_{\theta}(\theta) \, d\theta \right) dr
$$

where $\mathcal{D}_m = \Bigg\{ r : P_m \int_{\Theta} f_n(r - s_{m,\theta}) f_{\theta}(\theta) \, d\theta$

$$
> P_{m'} \int_{\Theta} f_n(r - s_{m',\theta}) f_{\theta}(\theta) \, d\theta \ \text{ for all } m' \ne m \Bigg\}.
$$

Example (Channel with attenuation)

$$r(t) = \theta \cdot s_m(t) + n(t)$$

where θ is a nonnegative random variable in $\mathbb{R}$, and $s_m(t)$ is binary antipodal with $s_1(t) = s(t)$ and $s_2(t) = -s(t)$.

Rewrite the above in vector form

$$r = \theta s + n, \text{ where } s = \langle s(t), \phi(t) \rangle = \sqrt{\mathcal{E}_b}.$$

Then

$$\mathcal{D}_1 = \left\{ r : \int_0^\infty e^{-\frac{(r-\theta\sqrt{\mathcal{E}_b})^2}{N_0}} f(\theta)\, d\theta > \int_0^\infty e^{-\frac{(r+\theta\sqrt{\mathcal{E}_b})^2}{N_0}} f(\theta)\, d\theta \right\}$$

Since
$$\int_0^\infty e^{-\frac{(r-\theta\sqrt{\mathcal{E}_b})^2}{N_0}} f(\theta)\, d\theta > \int_0^\infty e^{-\frac{(r+\theta\sqrt{\mathcal{E}_b})^2}{N_0}} f(\theta)\, d\theta$$

$$\Leftrightarrow \int_0^\infty \left[e^{-\frac{(r-\theta\sqrt{\mathcal{E}_b})^2}{N_0}} - e^{-\frac{(r+\theta\sqrt{\mathcal{E}_b})^2}{N_0}} \right] f(\theta)\, d\theta > 0$$

$$\Leftrightarrow \int_0^\infty e^{-\frac{r^2+\theta^2\mathcal{E}_b}{N_0}} \underbrace{\left[e^{\frac{2\theta\sqrt{\mathcal{E}_b}}{N_0}r} - e^{-\frac{2\theta\sqrt{\mathcal{E}_b}}{N_0}r} \right]}_{>0 \text{ iff } r>0} f(\theta)\, d\theta > 0$$

$$\Leftrightarrow r > 0,$$

we have

$$\mathcal{D}_1 = \{r : r > 0\} \Rightarrow \mathcal{D}_1^c = \{r : r \leq 0\}$$

The error probability

$$P_b = \int_0^\infty \left[\int_{-\infty}^0 \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\theta\sqrt{\mathcal{E}_b})^2}{N_0}}\, dr \right] f(\theta) d\theta = \mathbb{E}\left[Q\left(\sqrt{\theta^2 \frac{2\mathcal{E}_b}{N_0}} \right) \right]$$

$$\Pr\{\mathcal{N}(m, \sigma^2) < r\} = Q\left(\frac{m-r}{\sigma}\right)$$

4.5-1 Noncoherent detection of carrier modulated signals

Noncoherent due to uncertainty of time delay

Recall the bandpass signal $s_m(t)$

$$s_m(t) = \mathbf{Re}\left\{s_{m,\ell}(t)e^{i2\pi f_c t}\right\}$$

Assume the received signal delayed by t_d

$$
\begin{aligned}
r(t) &= s_m(t - t_d) + n(t) \\
&= \mathbf{Re}\left\{s_{m,\ell}(t - t_d)e^{i2\pi f_c(t - t_d)}\right\} + n(t) \\
&= \mathbf{Re}\left\{\left[s_{m,\ell}(t - t_d)\exp\{i\underbrace{(-2\pi f_c t_d)}_{=\phi}\} + n_\ell(t)\right]e^{i2\pi f_c t}\right\}
\end{aligned}
$$

Hence, if $t_d \ll T$, then $\langle s_{m,\ell}(t - t_d), \phi_{i,\ell}(t)\rangle \approx \langle s_{m,\ell}(t), \phi_{i,\ell}(t)\rangle$.

$$r_\ell(t) = s_{m,\ell}(t - t_d)e^{i\phi} + n_\ell(t) \implies \boldsymbol{r}_\ell = e^{i\phi}\boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell$$

When f_c is large, ϕ is well modeled by uniform distribution over $[0, 2\pi)$. The MAP rule is

$$
\begin{aligned}
\hat{m} &= \arg\max_{1 \le m \le M} P_m \int_0^{2\pi} \frac{1}{2\pi} f_{\boldsymbol{n}_\ell}\left(\boldsymbol{r}_\ell - \boldsymbol{s}_{m,\ell} e^{\imath \phi}\right) d\phi \\
&= \arg\max_{1 \le m \le M} \frac{P_m}{2\pi} \frac{1}{\left(\pi(2N_0)\right)^N} \int_0^{2\pi} e^{-\frac{\left\|\boldsymbol{r}_\ell - e^{\imath\phi} \boldsymbol{s}_{m,\ell}\right\|^2}{2N_0}} d\phi
\end{aligned}
$$

(The text uses $4N_0$ instead of $2N_0$, which does not seem correct! See (4.5-18) in text and Slide 4-11.)

$$
\begin{aligned}
&= \arg\max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\mathsf{Re}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell} \cdot e^{\imath\phi}\right]}{N_0}} d\phi \\
&= \arg\max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\mathsf{Re}\left[\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right| e^{\imath\theta_m} \cdot e^{\imath\phi}\right]}{N_0}} d\phi \\
&= \arg\max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} \int_0^{2\pi} e^{\frac{\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right|}{N_0} \cos(\theta_m + \phi)} d\phi
\end{aligned}
$$

where $\theta_m = \angle\left(\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right)$ and $\mathcal{E}_m = \left\|\boldsymbol{s}_m(t)\right\|^2 = \frac{1}{2}\left\|s_{m,\ell}(t)\right\|^2 = \frac{1}{2}\left\|\boldsymbol{s}_{m,\ell}\right\|^2$.

$$\hat{m} = \arg \max_{1 \le m \le M} P_m e^{-\frac{\varepsilon_m}{N_0}} \int_0^{2\pi} e^{\frac{\left|r_\ell^\dagger s_{m,\ell}\right|}{N_0} \cos(\phi)} d\phi$$

$$= \arg \max_{1 \le m \le M} P_m e^{-\frac{\varepsilon_m}{N_0}} I_0\left(\frac{\left|r_\ell^\dagger s_{m,\ell}\right|}{N_0}\right)$$

where $I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos(\phi)} d\phi$ is the modified Bessel function of the first kind and order zero.
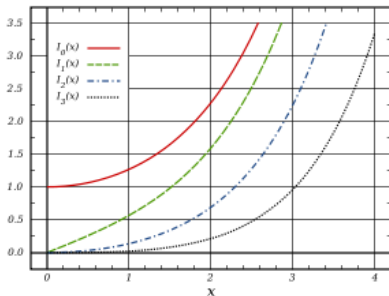
$$g_{MAP}(\boldsymbol{r}_\ell) = \arg \max_{1 \le m \le M} P_m e^{-\frac{\mathcal{E}_m}{N_0}} I_0 \left(\frac{\left| \boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell} \right|}{N_0} \right)$$

For equal-energy and equiprobable signals, the above simplifies to

$$\hat{m} = \arg \max_{1 \le m \le M} \left| \boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell} \right|$$

$$= \arg \max_{1 \le m \le M} \left| \int_0^T r_\ell^*(t) s_{m,\ell}(t) \, dt \right| \qquad (1)$$

since $I_0(x)$ is strictly increasing.

(1) is referred to as envelope detector because $|c|$ for a complex number c is called its envelope.

Compare with coherent detection

For a system modeled as

$$r_\ell(t) = s_{m,\ell}(t) + n_\ell(t) \implies r_\ell = s_{m,\ell} + n_\ell$$

The MAP rule is

$$
\begin{aligned}
\hat{m} &= \arg \max_{1 \le m \le M} P_m f_{n_\ell}\left(r_\ell - s_{m,\ell}\right) \\
&= \arg \max_{1 \le m \le M} \frac{P_m}{(4\pi N_0)^N} \exp\left(-\frac{\left\| r_\ell - s_{m,\ell} \right\|^2}{4 N_0}\right) \\
&= \arg \max_{1 \le m \le M} \mathbf{Re}\left[r_\ell^\dagger s_{m,\ell} \right]
\end{aligned}
$$

if equiprobable and equal-energy signals are assumed.

Main result

Theorem 7 (Carrier modulated signals)

For equal-probable and equal-energy carrier modulated signals with baseband equivalent received signal $\boldsymbol{r}_\ell$ over AWGN

- *Coherent MAP detection*

$$\hat{m} = \arg \max_{1 \le m \le M} \mathbf{Re}\left[\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right]$$

- *Noncohereht MAP detection*

$$\hat{m} = \arg \max_{1 \le m \le M} \left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right| \quad \text{if} \left\{ \begin{array}{l} \langle s_{m,\ell}(t-t_d), \phi_{i,\ell}(t) \rangle \approx \langle s_{m,\ell}(t), \phi_{i,\ell}(t) \rangle \\ \text{and } \phi \text{ uniform over } [0, 2\pi) \end{array} \right.$$

Note that the above two $\boldsymbol{r}_\ell$'s are different! For a coherent system, $\boldsymbol{r}_\ell = \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell$ is obtained from synchornized local carrier; however, for a noncoherent system, $\boldsymbol{r}_\ell = e^{\imath \phi} \boldsymbol{s}_{m,\ell} + \boldsymbol{n}_\ell$ is obtained from non-synchronized local carrier.

4.5-2 Optimal noncoherent detection of FSK modulated signals

Recall that baseband M-ary FSK orthogonal modulation is given by

$$s_{m,\ell}(t) \;=\; g(t)e^{i\,2\pi(m-1)\Delta f\,t}$$

for $1 \le m \le M$.

Given $s_{m,\ell}(t)$ transmitted, the received signal (for non-coherent due to uncertain of time delay) is

$$\boldsymbol{r}_\ell = \boldsymbol{s}_{m,\ell}e^{i\,\phi} + \boldsymbol{n}_\ell$$

or equivalently

$$r_\ell(t) \;=\; s_{m,\ell}(t)e^{i\,\phi} + n_\ell(t)$$

Non-coherent detection of FSK

The non-coherent ML (i.e., equal-probable) detection computes

$$
\begin{aligned}
\left| \boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m',\ell} \right| &= \left| \int_0^T r_\ell^*(t) s_{m',\ell}(t) \, dt \right| \\
&= \left| \int_0^T r_\ell(t) s_{m',\ell}^*(t) \, dt \right| \\
&= \left| \int_0^T \left(s_{m,\ell}(t) e^{\imath \phi} + n_\ell(t) \right) s_{m',\ell}^*(t) \, dt \right| \\
&= \left| e^{\imath \phi} \int_0^T s_{m,\ell}(t) s_{m',\ell}^*(t) \, dt + \int_0^T n_\ell(t) s_{m',\ell}^*(t) \, dt \right|
\end{aligned}
$$

Assuming $g(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\left[u_{-1}(t) - u_{-1}(t - T)\right]$,

$$
\begin{aligned}
&\int_0^T s_{m,\ell}(t) s_{m',\ell}^*(t)\, dt \\
&= \frac{2\mathcal{E}_s}{T} \int_0^T e^{\imath 2\pi(m-1)\Delta f\, t} e^{-\imath 2\pi(m'-1)\Delta f\, t}\, dt \\
&= \frac{2\mathcal{E}_s}{T} \int_0^T e^{\imath 2\pi(m-m')\Delta f\, t}\, dt \\
&= \frac{2\mathcal{E}_s}{T}\left(\frac{e^{\imath 2\pi(m-m')\Delta f\, T} - 1}{\imath 2\pi(m - m')\Delta f}\right) \\
&= 2\mathcal{E}_s e^{\imath \pi(m-m')\Delta f\, T}\operatorname{sinc}\left[(m - m')\Delta f\, T\right]
\end{aligned}
$$

Hence, if $\Delta f = \frac{k}{T}$,

$$
\begin{aligned}
\left|\boldsymbol{r}_\ell^\dagger \boldsymbol{s}_{m',\ell}\right| &= \left|e^{\imath \phi}\int_0^T s_{m,\ell}(t) s_{m',\ell}^*(t)\, dt + \int_0^T n_\ell(t) s_{m',\ell}^*(t)\, dt\right| \\
&= \begin{cases}
\left|e^{\imath \phi}(2\mathcal{E}_s) + \int_0^T n_\ell(t) s_{m,\ell}^*(t)\, dt\right|, & m' = m \\[2mm]
\left|\int_0^T n_\ell(t) s_{m',\ell}^*(t)\, dt\right|, & m' \neq m
\end{cases}
\end{aligned}
$$

Coherent detection of FSK

$$r_\ell(t) = s_{m,\ell}(t) + n_\ell(t)$$

$$\hat{m} = \arg \max_{1 \le m' \le M} \mathbf{Re}\left[r_\ell^\dagger s_{m',\ell}\right]$$

$$= \arg \max_{1 \le m' \le M} \mathbf{Re}\left[\int_0^T s_{m,\ell}(t)s_{m',\ell}^*(t)\,dt + \int_0^T n_\ell(t)s_{m',\ell}^*(t)\,dt\right]$$

Hence, with $g(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\left[u_{-1}(t) - u_{-1}(t - T)\right]$,

$$\mathbf{Re}\left[\int_0^T s_{m,\ell}(t)s_{m',\ell}^*(t)\,dt\right]$$

$$= 2\mathcal{E}_s \cos\left(\pi(m - m')\Delta fT\right)\operatorname{sinc}\left((m - m')\Delta fT\right)$$

$$= 2\mathcal{E}_s\operatorname{sinc}\left(2(m - m')\Delta fT\right)$$

Here we only need $\Delta f = \frac{k}{2T}$ and $\mathbb{E}\left\{\mathbf{Re}\left[r_\ell^\dagger s_{m',\ell}\right]\right\} = 0$ for $m' \ne m$ as similar to Slide 3-35.

For M-ary orthogonal signaling with symbol energy $\mathcal{E}_s$, the lowpass equivalent signal has constellation (recall that $\mathcal{E}_s$ is the transmission energy of the bandpass signal)

$$
\begin{aligned}
\boldsymbol{s}_{1,\ell} &= \begin{pmatrix} \sqrt{2\mathcal{E}_s} & 0 & \cdots & 0 \end{pmatrix}^\top \\
\boldsymbol{s}_{2,\ell} &= \begin{pmatrix} 0 & \sqrt{2\mathcal{E}_s} & \cdots & 0 \end{pmatrix}^\top \\
\vdots\; &= \quad\;\; \vdots \qquad\; \vdots \quad\; \ddots \quad\; \vdots \\
\boldsymbol{s}_{M,\ell} &= \begin{pmatrix} 0 & 0 & \cdots & \sqrt{2\mathcal{E}_s} \end{pmatrix}^\top
\end{aligned}
$$

Given $\boldsymbol{s}_{m,1}$ transmitted, the received is

$$\boldsymbol{r}_\ell = e^{\imath \phi}\boldsymbol{s}_{1,\ell} + \boldsymbol{n}_\ell.$$

The noncoherent ML computes (if $\Delta f = \frac{k}{T}$)

$$
\left| \boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell \right| =
\begin{cases}
\left| 2\mathcal{E}_s e^{\imath \phi} + \boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{n}_\ell \right|, & m = 1 \\
\left| \boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{n}_\ell \right|, & 2 \leq m \leq M
\end{cases}
$$

Recall $\boldsymbol{n}_\ell$ is a complex Gaussian random vector with

$$\mathbb{E}[\boldsymbol{n}_\ell \boldsymbol{n}_\ell^\dagger] = 2N_0.$$

Hence $\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{n}_\ell$ is a circular symmetric complex Gaussian random variable with

$$\mathbb{E}\left[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{n}_\ell \boldsymbol{n}_\ell^\dagger \boldsymbol{s}_{m,\ell}\right] = 2N_0 \cdot 2\mathcal{E}_s = 4\mathcal{E}_s N_0$$

Thus

$$\mathbf{Re}\left[\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{r}_\ell\right] \sim \mathcal{N}\left(2\mathcal{E}_s \cos\phi, 2\mathcal{E}_s N_0\right)$$

$$\mathbf{Im}\left[\boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{r}_\ell\right] \sim \mathcal{N}\left(2\mathcal{E}_s \sin\phi, 2\mathcal{E}_s N_0\right)$$

$$\mathbf{Re}\left[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell\right] \sim \mathcal{N}\left(0, 2\mathcal{E}_s N_0\right), \ m \neq 1$$

$$\mathbf{Im}\left[\boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell\right] \sim \mathcal{N}\left(0, 2\mathcal{E}_s N_0\right), \ m \neq 1$$

Define $R_1 = \left| \boldsymbol{s}_{1,\ell}^\dagger \boldsymbol{r}_\ell \right|$; then we have that R_1 is Ricean distributed with density

$$f_{R_1}(r_1) = \frac{r_1}{\sigma} I_0\left(\frac{sr_1}{\sigma^2}\right) e^{-\frac{r_1^2+s^2}{2\sigma^2}}, \qquad r_1 > 0$$

where $\sigma^2 = 2\mathcal{E}_s N_0$ and $s = 2\mathcal{E}_s$.

Define $R_m = \left| \boldsymbol{s}_{m,\ell}^\dagger \boldsymbol{r}_\ell \right|$, $m \geq 2$; then we have that R_m is Rayleigh distributed with density

$$f_{R_m}(r_m) = \frac{r_m}{\sigma^2} e^{-\frac{r_m^2}{2\sigma^2}}, \qquad r_m > 0$$

$$
\begin{aligned}
P_c &= \Pr\{R_2 < R_1, \cdots, R_M < R_1\} \\
&= \int_0^\infty \Pr\{R_2 < r_1, \cdots, R_M < r_1 | R_1 = r_1\} f(r_1)\, dr_1 \\
&= \int_0^\infty \left[\int_0^{r_1} f(r_m)\, dr_m\right]^{M-1} f(r_1)\, dr_1 \\
&= \int_0^\infty \left[1 - e^{-\frac{r_1^2}{2\sigma^2}}\right]^{M-1} f(r_1)\, dr_1 \\
&= \int_0^\infty \sum_{n=0}^{M-1} \binom{M-1}{n}(-1)^n e^{-\frac{nr_1^2}{2\sigma^2}}\; \frac{r_1}{\sigma} I_0\left(\frac{sr_1}{\sigma^2}\right) e^{-\frac{r_1^2 + s^2}{2\sigma^2}}\, dr_1 \\
&= \sum_{n=0}^{M-1} \binom{M-1}{n}(-1)^n \int_0^\infty \frac{r_1}{\sigma} I_0\left(\frac{sr_1}{\sigma^2}\right) e^{-\frac{(n+1)r_1^2 + s^2}{2\sigma^2}}\, dr_1
\end{aligned}
$$

Setting

$$s' = \frac{s}{\sqrt{n+1}} \quad \text{and} \quad r' = r_1 \sqrt{n+1}$$

gives

$$
\begin{aligned}
&\int_0^\infty \frac{r_1}{\sigma} I_0\left(\frac{sr_1}{\sigma^2}\right) e^{-\frac{(n+1)r_1^2 + s^2}{2\sigma^2}} \, dr_1 \\
&= \int_0^\infty \frac{r'}{\sigma(n+1)} I_0\left(\frac{s'r'}{\sigma^2}\right) e^{-\frac{r'^2 + (n+1)s'^2}{2\sigma^2}} \, dr' \\
&= \frac{1}{n+1} e^{-\frac{ns'^2}{2\sigma^2}} \underbrace{\int_0^\infty \frac{r'}{\sigma} I_0\left(\frac{s'r'}{\sigma^2}\right) e^{-\frac{r'^2 + s'^2}{2\sigma^2}} \, dr'}_{=1} \\
&= \frac{1}{n+1} e^{-\frac{ns'^2}{2\sigma^2}} = \frac{1}{n+1} e^{-\frac{ns^2}{2\sigma^2(n+1)}}
\end{aligned}
$$

Hence with $\sigma^2 = 2\mathcal{E}_s N_0$ and $s = 2\mathcal{E}_s$,

$$
\begin{aligned}
P_c &= \sum_{n=0}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} e^{-(\frac{n}{n+1})\frac{s^2}{2\sigma^2}} = \sum_{n=0}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} e^{-(\frac{n}{n+1})\frac{\mathcal{E}_s}{N_0}} \\
&= 1 + \sum_{n=1}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} e^{-(\frac{n}{n+1})\frac{\mathcal{E}_s}{N_0}}
\end{aligned}
$$

Thus

$$
P_e = 1 - P_c = \sum_{n=1}^{M-1} \frac{(-1)^{n+1}}{n+1} \binom{M-1}{n} e^{-(\frac{n}{n+1})\frac{\mathcal{E}_b \log_2 M}{N_0}}
$$

BFSK

For $M = 2$, the above shows

$$
P_e = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{2N_0}} > P_{e,\text{coherent}} = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)
$$

4.5-5 Differential PSK

Introduction of differential PSK

- The previous noncoherent scheme simply uses one symbol in noncoherent detection.
- The differential scheme uses two consecutive symbols to achieve the same goal but with 3 dB performance improvement.

Advantage of differential PSK

- Phase ambiguity (due to frequency shift) of M-ary PSK (under noiseless transmission)
  - Receive $\cos(2\pi f_c t + \theta)$ but estimate θ in terms of f_c'
    $\implies$ Receive $\cos(2\pi f_c t + 2\pi(f_c - f_c')t + \theta)$ but estimate θ in terms of f_c'
    $\implies \hat{\theta} = 2\pi(f_c - f_c')t + \theta = \phi + \theta.$
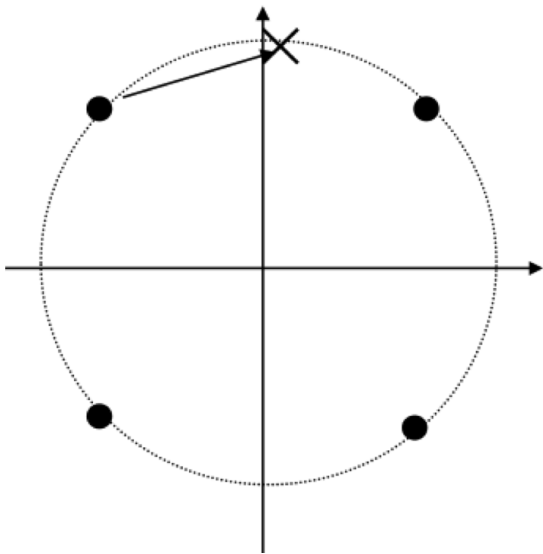
Differential encoding

- BDPSK
  - Shift the phase of the previous symbol by 0 degree, if input $= 0$
  - Shift the phase of the previous symbol by 180 degree, if input $= 1$
- QDPSK
  - Shift the phase of the previous symbol by 0 degree, if input $= 00$
  - Shift the phase of the previous symbol by 90 degree, if input $= 01$
  - Shift the phase of the previous symbol by 180 degree, if input $= 11$
  - Shift the phase of the previous symbol by 270 degree, if input $= 10$
- ... (further extensions)

The two consecutive lowpass equivalent signals are

$$s_\ell^{(k-1)} = \sqrt{2\mathcal{E}_s}\, e^{\imath\,\phi_0} \quad \text{and} \quad s_{m,\ell}^{(k)} = \sqrt{2\mathcal{E}_s}\, e^{\imath\,(\theta_m + \phi_0)}.$$

Note: We denote the $(k-1)th$ symbol by $s_\ell^{(k-1)}$ instead of $s_{m',\ell}^{(k-1)}$ because m' is not the digital information to be detected now, and hence is not important! $s_\ell^{(k-1)}$ is simply the base to help detecting m.

The received signals given $s_\ell^{(k-1)}$ and $s_{m,\ell}^{(k)}$ are

$$\vec{r}_\ell = \begin{bmatrix} r_\ell^{(k-1)} \\ r_\ell^{(k)} \end{bmatrix} = e^{\imath\,\phi}\begin{bmatrix} s_\ell^{(k-1)} \\ s_{m,\ell}^{(k)} \end{bmatrix} + \begin{bmatrix} n_\ell^{(k-1)} \\ n_\ell^{(k)} \end{bmatrix} = e^{\imath\,\phi}\vec{s}_{m,\ell} + \vec{n}_\ell$$

$$\Rightarrow \vec{s}_{m,\ell}^{\dagger}\vec{r}_\ell = \begin{bmatrix} \sqrt{2\mathcal{E}_s}\,e^{-\imath\,\phi_0} & \sqrt{2\mathcal{E}_s}\,e^{-\imath\,(\theta_m+\phi_0)} \end{bmatrix}\begin{bmatrix} r_\ell^{(k-1)} \\ r_\ell^{(k)} \end{bmatrix}$$

$$= \sqrt{2\mathcal{E}_s}\,e^{-\imath\,\phi_0}\Big(r_\ell^{(k-1)} + r_\ell^{(k)}\,e^{-\imath\,\theta_m} \Big)$$

$$
\begin{aligned}
\hat{m} &= \arg\max_{1 \le m \le M} \left| \vec{s}_{m,\ell}^{\dagger} \vec{r}_{\ell} \right| = \arg\max_{1 \le m \le M} \left| \sqrt{2\mathcal{E}_s} e^{-\imath \phi_0} \right| \left| r_{\ell}^{(k-1)} + r_{\ell}^{(k)} e^{-\imath \theta_m} \right| \\
&= \arg\max_{1 \le m \le M} \left| r_{\ell}^{(k-1)} + r_{\ell}^{(k)} e^{-\imath \theta_m} \right|^2 \\
&= \arg\max_{1 \le m \le M} \mathbf{Re} \left\{ \left(r_{\ell}^{(k-1)} \right)^* r_{\ell}^{(k)} e^{-\imath \theta_m} \right\} \\
&= \arg\max_{1 \le m \le M} \cos\left(\angle r_{\ell}^{(k)} - \angle r_{\ell}^{(k-1)} - \theta_m \right) \\
&= \arg\min_{1 \le m \le M} \left| \angle r_{\ell}^{(k)} - \angle r_{\ell}^{(k-1)} - \theta_m \right|
\end{aligned}
$$

The error probability of M-ary differential PSK can generally be obtained from $\Pr[D < 0]$, where the random variable of the general quadratic form is given by

$$
D = \sum_{k=1}^{L} \left(A|X_k|^2 + B|Y_k|^2 + C X_k Y_k^* + C^* X_k^* Y_k \right)
$$

and $\{X_k, Y_k\}_{k=1}^{L}$ is independent complex Gaussain with common covariance matrix. (See Slide 4-181.)

Error probability for binary DPSK

In a special case, where $M = 2$, the error of differential PSK can be derived without using the quadratic form.

Specifically, with $M = 2$,

$$\vec{s}_{1,\ell} = \sqrt{2\mathcal{E}_s}\, e^{\imath\phi_0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{s}_{2,\ell} = \sqrt{2\mathcal{E}_s}\, e^{\imath\phi_0} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can perform 45-degree counterclockwise rotation on the received signals given $s_\ell^{(k-1)}$ and $s_{m,\ell}^{(k)}$:

$$\mathbb{R}\vec{r}_\ell = \mathbb{R}\begin{bmatrix} r_\ell^{(k-1)} \\ r_\ell^{(k)} \end{bmatrix} = e^{\imath\phi}\mathbb{R}\begin{bmatrix} s_\ell^{(k-1)} \\ s_{m,\ell}^{(k)} \end{bmatrix} + \mathbb{R}\begin{bmatrix} n_\ell^{(k-1)} \\ n_\ell^{(k)} \end{bmatrix} = e^{\imath\phi}\mathbb{R}\vec{s}_{m,\ell} + \mathbb{R}\vec{n}_\ell$$

where $\mathbb{R} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

This gives

$$\mathbb{R}\vec{s}_{1,\ell} = \begin{bmatrix} 0 \\ \sqrt{2(2\mathcal{E}_s)} \end{bmatrix} \quad \text{and} \quad \mathbb{R}\vec{s}_{2,\ell} = \begin{bmatrix} \sqrt{2(2\mathcal{E}_s)} \\ 0 \end{bmatrix} \quad \text{and set } \mathcal{E}_s' = 2\mathcal{E}_s.$$

Notably, the distribution of "rotated" additive noises remains the same.

The decision rule on Slide 4-176 becomes

$$\hat{m} = \arg \max_{1 \le m \le 2} \left| \vec{s}_{m,\ell}^{\dagger} \vec{r}_\ell \right| = \arg \max_{1 \le m \le 2} \left| (\mathbb{R}\vec{s}_{m,\ell})^{\dagger} \mathbb{R}\vec{r}_\ell \right|$$

The same analysis as non-coherent orthogonal signals (cf. Slide 4-170) can thus be used for BDPSK:

$$P_{e,BDPSK} = \frac{1}{2} e^{-\frac{\mathcal{E}_s'}{2N_0}} = \frac{1}{2} e^{-\frac{(2\mathcal{E}_s)}{2N_0}} = \frac{1}{2} e^{-\frac{\mathcal{E}_s}{N_0}} = \frac{1}{2} e^{-\frac{\mathcal{E}_b}{N_0}}$$

For coherent detection of BPSK, we have

$$P_{e,BPSK} = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) < \frac{1}{2} e^{-\frac{\mathcal{E}_b}{N_0}}.$$

The bit error rate (not symbol error rate) for QDPSK under Gray mapping can only be derived using the quadratic form formula (cf. Slide 4-181) and is given by

$$P_{b,\text{QDPSK}} = Q_1(a, b) - \frac{1}{2} I_0(ab) e^{-(a^2+b^2)/2}$$

where $Q_1(a, b)$ is the Marcum Q function,

$$a = \sqrt{2\gamma_b \left(1 - \frac{1}{\sqrt{2}}\right)} \quad \text{and} \quad b = \sqrt{2\gamma_b \left(1 + \frac{1}{\sqrt{2}}\right)}.$$

BDPSK is in general 1 dB inferior than BPSK/QPSK.

QDPSK is in general 2.3 dB inferior than BPSK/QPSK.

Appendix B
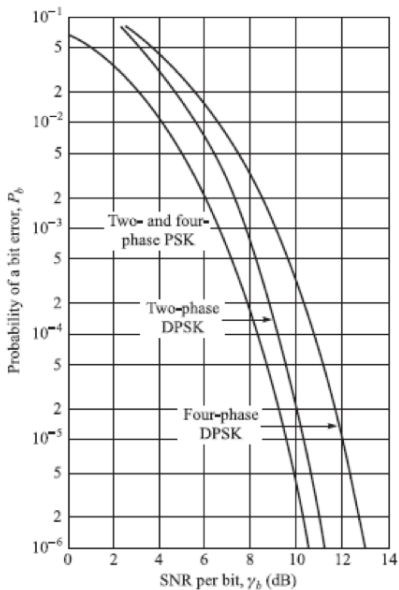
The ML decision is

$$\hat{m} = \arg \max_{1 \le m \le M} \mathbf{Re} \left\{ \left(r_\ell^{(k-1)} \right)^* r_\ell^{(k)} e^{-\imath \theta_m} \right\}$$

where in absence of noise,

$$\begin{bmatrix} r_\ell^{(k-1)} \\ r_\ell^{(k)} \end{bmatrix} = e^{\imath \phi} \begin{bmatrix} s_\ell^{(k-1)} \\ s_{m,\ell}^{(k)} \end{bmatrix} = \sqrt{2\mathcal{E}_s} e^{\imath (\phi + \phi_0)} \begin{bmatrix} 1 \\ e^{\imath \theta_m} \end{bmatrix}$$

$$\implies \left(r_\ell^{(k-1)} \right)^* r_\ell^{(k)} = 2\mathcal{E}_s e^{\imath \theta_m} = \begin{cases} 2\mathcal{E}_s & 00 \\ 2\mathcal{E}_s \imath & 01 \\ -2\mathcal{E}_s & 11 \\ -2\mathcal{E}_s \imath & 10 \end{cases}$$
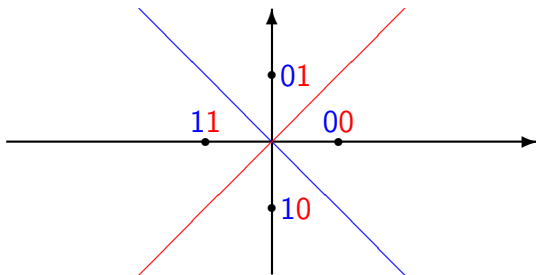
As the phase noise is unimodal, the optimal decision should be

$$\mathbf{Re}\left\{\left(r_\ell^{(k-1)}\right)^* r_\ell^{(k)}\right\} + \mathbf{Im}\left\{\left(r_\ell^{(k-1)}\right)^* r_\ell^{(k)}\right\} \begin{cases} > 0 & \text{the 1st bit} = 0 \\ < 0 & \text{the 1st bit} = 1 \end{cases}$$

$$\mathbf{Re}\left\{\left(r_\ell^{(k-1)}\right)^* r_\ell^{(k)}\right\} - \mathbf{Im}\left\{\left(r_\ell^{(k-1)}\right)^* r_\ell^{(k)}\right\} \begin{cases} > 0 & \text{the 2nd bit} = 0 \\ < 0 & \text{the 2nd bit} = 1 \end{cases}$$

The bit error rate for the 1st/2nd bit is given by

$$\Pr[D < 0],$$

where

$$D = A|X|^2 + B|Y|^2 + CXY^* + C^*X^*Y = A|X|^2 + B|Y|^2 + 2\mathbf{Re}\{CXY^*\}$$

and

$$\begin{cases} A = B = 0 \\[2mm] 2C = \begin{cases} 1 - \imath & \text{for the 1st bit} \\ 1 + \imath & \text{for the 2nd bit} \end{cases} \\[4mm] X = \boldsymbol{r}_\ell^{(k)} = \sqrt{2\mathcal{E}_s}e^{\imath(\phi+\phi_0)}e^{\imath\theta_m} + \boldsymbol{n}_\ell^{(k)} \\[2mm] Y = \boldsymbol{r}_\ell^{(k-1)} = \sqrt{2\mathcal{E}_s}e^{\imath(\phi+\phi_0)} + \boldsymbol{n}_\ell^{(k-1)} \end{cases}$$

4.8-1 Maximum likelihood sequence detector

- Optimal detector for signals with memory (not channel with memory or noise with memory. Still, the noise is AWGN)

  - It is implicitly assumed that the order of the signal memory is known.

Example. NRZI signal of (signal) memory order $L = 1$
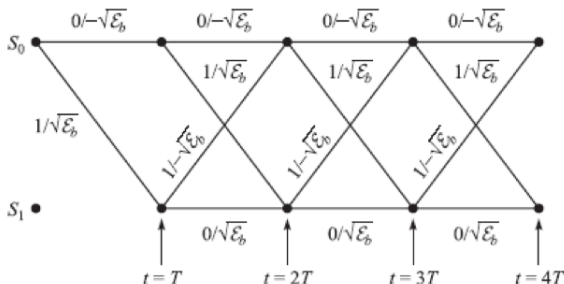
The channel gives

$$r_k = s_k + n_k = \pm\sqrt{\mathcal{E}_b} + n_k, \quad k = 1, \ldots, K.$$

The pdf of a sequence of demodulation outputs

$$f(r_1, \ldots, r_K | s_1, \ldots, s_K) = \frac{1}{(\pi N_0)^{K/2}} \exp\left\{-\frac{1}{N_0} \sum_{k=1}^{K} (r_k - s_k)^2\right\}$$

Note again that $s_1, \ldots, s_K$ has memory!

The ML decision is therefore

$$\arg \min_{(s_1, \ldots, s_K) \in \{\pm\sqrt{\mathcal{E}_b}\}^K} \sum_{k=1}^{K} (r_k - s_k)^2$$

Viterbi algorithm

- Since $s_1, \ldots, s_K$ has memory and

$$\min_{(s_1,\ldots,s_K) \in \{\pm\sqrt{\mathcal{E}_b}\}^K} \sum_{k=1}^{K} (r_k - s_k)^2 \neq \sum_{k=1}^{K} \min_{s_k \in \{\pm\sqrt{\mathcal{E}_b}\}} (r_k - s_k)^2,$$

the ML decision cannot be obtained based on individual decisions.

- Viterbi (demodulation) Algorithm: A sequential trellis search algorithm that performs ML sequence detection

- It transforms a search over 2^K vector points into a sequential search over a trellis.

Explaining the Viterbi algorithm

There are two paths entering each node at $t = 2T$ (In the sequence, we denote $s(t) = A = \sqrt{\mathcal{E}_b}$.)

path $(I_1, I_2) = (0,0)$ or $(1,1)$
$\rightarrow$ node S_0 at $t = 2T$,
    denoted by $S_0(2T)$.
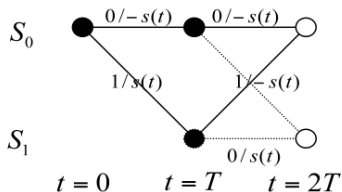
path $(I_1, I_2) = (0,1)$ or $(1,0)$
$\rightarrow$ node S_1 at $t = 2T$,
    denoted by $S_1(2T)$.

S_0 0/−s(t)   0/−s(t)

1/s(t)   1/−s(t)

S_1

0/s(t)

$t = 0$ $t = T$ $t = 2T$
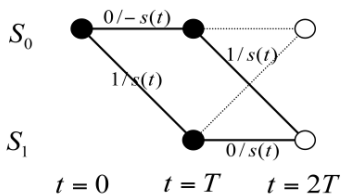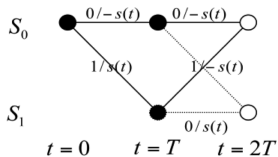
- Euclidean distance for path $(0, 0)$ entering node $S_0(2T)$:

$$D_0(0, 0) = (r_1 - (-\sqrt{\mathcal{E}_b}))^2 + (r_2 - (-\sqrt{\mathcal{E}_b}))^2$$

- Euclidean distance for path $(1, 1)$ entering node $S_0(2T)$:

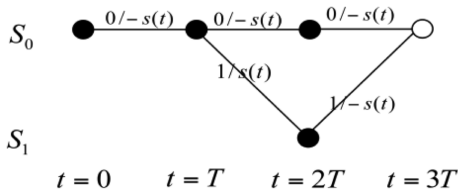$$D_0(1, 1) = (r_1 - \sqrt{\mathcal{E}_b})^2 + (r_2 - (-\sqrt{\mathcal{E}_b}))^2$$

Viterbi algorithm

- Of the above two paths, discard the one with larger Euclidean distance.

- The remaining path is called survivor at $t = 2T$.

S_0 $0/-s(t)$ $1/s(t)$

$1/s(t)$

S_1 $0/s(t)$

$t = 0$ $t = T$ $t = 2T$
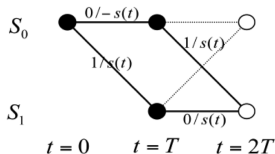
- Euclidean distance for path $(0, 1)$ entering node $S_1(2T)$:

$$D_1(0, 1) = (r_1 - (-\sqrt{\mathcal{E}_b}))^2 + (r_2 - \sqrt{\mathcal{E}_b})^2$$

- Euclidean distance for path $(1, 0)$ entering node $S_1(2T)$:

$$D_1(1, 0) = (r_1 - \sqrt{\mathcal{E}_b})^2 + (r_2 - \sqrt{\mathcal{E}_b})^2$$

Viterbi algorithm

- Of the above two paths, discard the one with larger Euclidean distance.

- The remaining path is called survivor at $t = 2T$.

We therefore have two survivor paths after observing r_2.

Suppose the two survivor paths are $(0,0)$ and $(0,1)$.
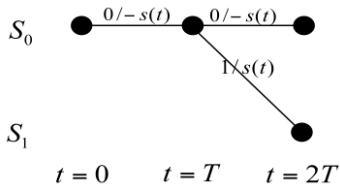
Then, there are two possible paths entering S_0 at $t = 3T$, i.e., $(0,0,0)$ and $(0,1,1)$.

Euclidean distance for each path:

$$D_0(0,0,0) = D_0(0,0) + (r_3 - (-\sqrt{\mathcal{E}_b}))^2$$
$$D_0(0,1,1) = D_1(0,1) + (r_3 - (-\sqrt{\mathcal{E}_b}))^2$$

Viterbi algorithm

- Of the above two paths, discard the one with larger Euclidean distance.

- The remaining path is called survivor at $t = 3T$.
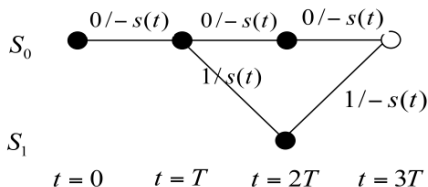
Euclidean distance for each path:

$$D_1(0,0,1) = D_0(0,0) + (r_3 - \sqrt{\mathcal{E}_b})^2$$
$$D_1(0,1,0) = D_1(0,1) + (r_3 - \sqrt{\mathcal{E}_b})^2$$

Viterbi algorithm.

- Of the above two paths, discard the one with larger Euclidean distance.

- The remaining path is called survivor at $t = 3T$.
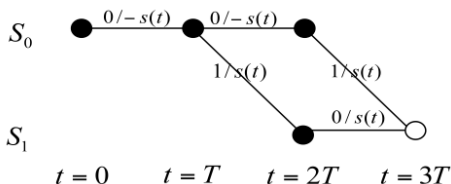
Viterbi algorithm

- Compute two metrics for the two signal paths entering a node at each stage of the trellis search.
- Remove the one with larger Euclidean distance.
- The survivor path for each node is then extended to the next state.

The elimination of one of the two paths is done without compromising the optimality of the trellis search because any extension of the path with larger distance will always have a larger metric than the survivor that is extended along the same path (as long as the path metric is non-decreasing along each path).

Unfolding the trellis to a tree structure, the number of paths searched is reduced by a factor of two at each stage.

survivor paths = (0,0) and (0,1)

These dotted paths are removed.
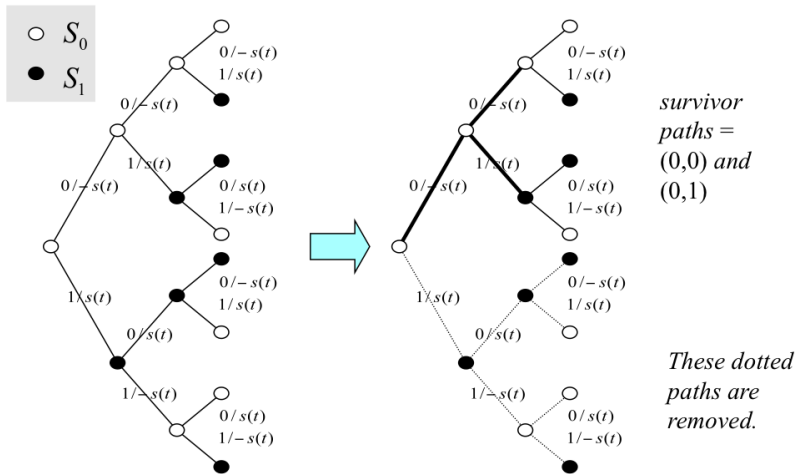
Apply the Viterbi algorithm to delay modulation

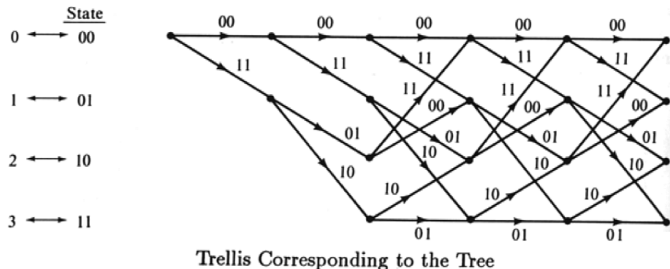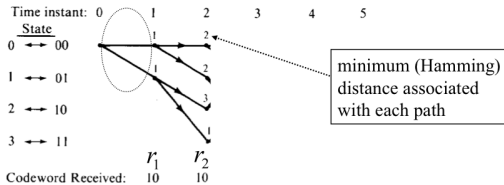- 2 entering paths for each node
- 4 survivor paths at each stage
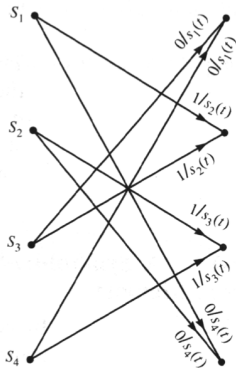
Decision delay of the Viterbi algorithm

- The final decision of the Viterbi algorithm shall wait until it traverses to the end of the trellis, where $\hat{s}_1, \ldots, \hat{s}_K$ correspond to the survivor path with the smallest metric.

- When K is large, the decision delay will be large!
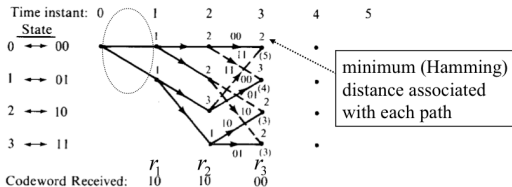
- Can we make an early decision?

  Let's borrow an example from Example 14.3-1 of *Digital and Analog Communications* by J. D. Gibson. (A code with $L = 2$)

  – Assume the received codeword is $(10, 10, 00, 00, 00, \ldots)$

Trellis Corresponding to the Tree

Time instant: 0, 1, 2, 3, 4, 5
State
0 ↔ 00
1 ↔ 01
2 ↔ 10
3 ↔ 11

minimum (Hamming) distance associated with each path
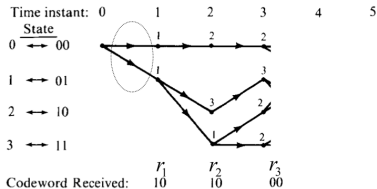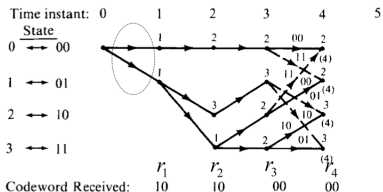
Codeword Received: r_1 = 10, r_2 = 10

At time instant 2, one does not know what the first two transmitted bits are. There are two possibilities for time period 1; hence, the decision delay $> T$.


Time instant: 0, 1, 2, 3, 4, 5
State
0 ↔ 00
1 ↔ 01
2 ↔ 10
3 ↔ 11

minimum (Hamming) distance associated with each path

Codeword Received: r_1 = 10, r_2 = 10, r_3 = 00

We then get r_3 and compute the accumulated metrics for each path.

Time instant: 0, 1, 2, 3, 4, 5
State
0 ↔ 00
1 ↔ 01
2 ↔ 10
3 ↔ 11

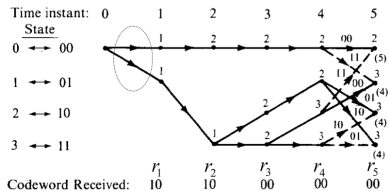Codeword Received: r_1 = 10, r_2 = 10, r_3 = 00

At time instant 3, one does not know what the first two transmitted bits are. Still, there are two possibilities for time period 1; hence, the decision delay $> 2T$.

We then get r_4 and compute the accumulated metrics for each path.
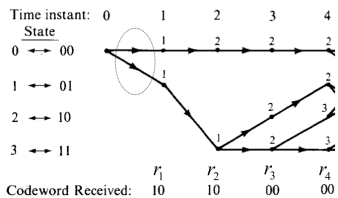
At time instant 4, one does not know what the first two transmitted bits are. Still, there are two possibilities for time period 1; hence, the decision delay $> 3T$.
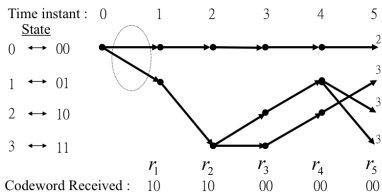
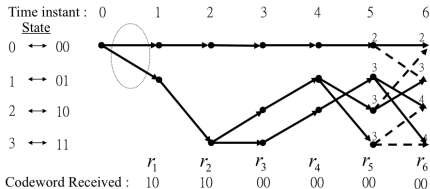We then get r_5 and compute the accumulated metrics for each path.

At time instant 5, one does not know what the first two transmitted bits are. Still, there are two possibilities for time period 1; hence, the decision delay $> 4T$.

We then get r_6 and compute the accumulated metrics for each path.

At time instant 6, one does not know what the first two transmitted bits are. Still, there are two possibilities for time period 1; hence, the decision delay $> 5T$.
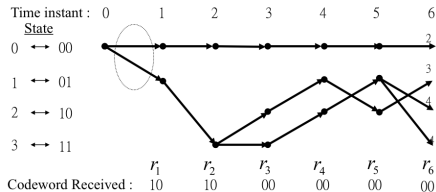
Time instant : 0   1   2   3   4   5   6   7

State

$0 \leftrightarrow 00$

$1 \leftrightarrow 01$

$2 \leftrightarrow 10$

$3 \leftrightarrow 11$

Codeword Received : r_1 10 r_2 10 r_3 00 r_4 00 r_5 00 r_6 00 r_7 00

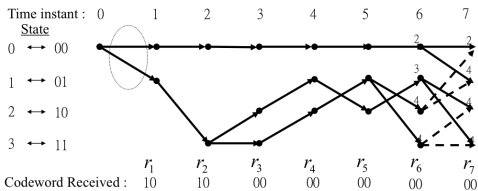We then get r_7 and compute the accumulated metrics for each path.

At time instant 7, one does not know what the first two transmitted bits are. Still, there are two possibilities for time period 1; hence, the decision delay $> 6T$.
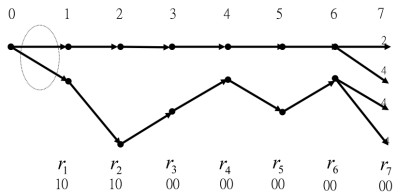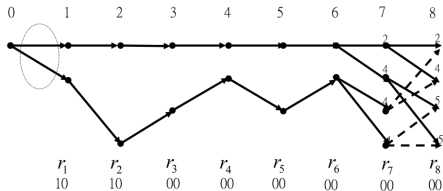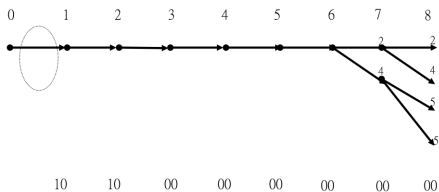
We then get r_8 and compute the accumulated metrics for each path.

At time instant 8, one is finally certain what the first two transmitted bits are, which is 00. Hence, the decision delay for the first two bits are $7T$.

Suboptimal Viterbi algorithm

- If there are more than one survivor paths remaining for time period $i - \Delta$ at time instance i, just select the one with smaller metric.

  Example (NRZI). Suppose the two metrics of the two survivor paths at time $\Delta + 1$ are

  $$D_0(0, b_2, b_3, \ldots, b_{\Delta+1}) < D_1(1, \tilde{b}_2, \tilde{b}_3, \ldots, \tilde{b}_{\Delta+1}).$$

  Then, adjust them to

  $$D_0(b_2, b_3, \ldots, b_{\Delta+1}) \text{ and } D_1(\tilde{b}_2, \tilde{b}_3, \ldots, \tilde{b}_{\Delta+1})$$

  and output the first bit 0.

- Forney (1974) proved theoretically that as long as $\Delta > 5.8L$, the suboptimal Viterbi algorithm achieves near optimal performance.

We may extend the use of the Viterbi algorithm to the MAP problem as long as the metric can be computed recursively:

$$\arg \max_{(s_1,\ldots,s_K)} f\left(r_1,\ldots,r_K | s_1,\ldots,s_K\right) \Pr\left\{s_1,\ldots,s_K\right\}$$

Further assumptions

- We may assume the channel is memoryless

$$f\left(r_1, \ldots, r_K \middle| s_1, \ldots, s_K\right) = \prod_{k=1}^{K} f\left(r_k \middle| s_k\right)$$

- $S^{(0)}, S^{(1)}, \ldots, S^{(K)}$ can be formulated as the output of a **first-order finite-state Markov chain**:
  1. A state space $\mathcal{S} = \{S_0, S_1, \ldots, S_{N-1}\}$
  2. An output function $\mathcal{O}(S^{(k-1)}, S^{(k)}) = s$, where $S^{(k)} \in \mathcal{S}$ is the state at time k.
  3. Notably, $s_1, s_2, \ldots, s_K$ and $S^{(0)}, S^{(1)}, \ldots, S^{(K)}$ are 1-1 correspondence.

Example (NRZI).

1. A state space $\mathcal{S} = \{S_0, S_1\}$

2. An output function

$$\begin{cases} \mathcal{O}(S_0, S_0) = \mathcal{O}(S_1, S_0) = -\sqrt{\mathcal{E}_b} \\ \mathcal{O}(S_0, S_1) = \mathcal{O}(S_1, S_1) = \sqrt{\mathcal{E}_b} \end{cases}$$

3. $s_1, s_2, \ldots, s_K$ and $S^{(0)}, S^{(1)}, \ldots, S^{(K)}$ are 1-1 correspondence.

E.g., $(S_0, S_1, S_0, S_0, S_0) \leftrightarrow (\sqrt{\mathcal{E}_b}, -\sqrt{\mathcal{E}_b}, -\sqrt{\mathcal{E}_b}, -\sqrt{\mathcal{E}_b})$
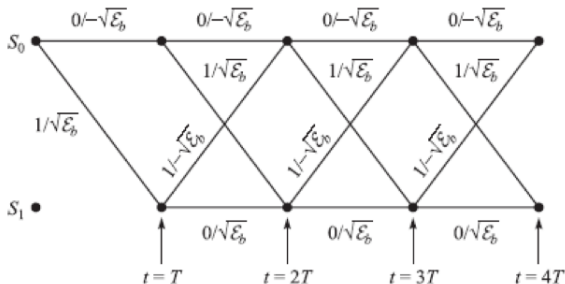
Then, we can rewrite the original MAP problem as

$$\arg \max_{S^{(0)},\cdots,S^{(K)}} \prod_{k=1}^{K} \left[f\left(r_k \middle| s_k = \mathcal{O}(S^{(k-1)}, S^{(k)}) \right) \Pr\left\{ S^{(k)} \middle| S^{(k-1)} \right\} \right]$$

Example (NRZI).

$$\begin{cases} \Pr\left\{ S^{(k)} = S_0 \middle| S^{(k-1)} = S_0 \right\} = \Pr\left\{ S^{(k)} = S_1 \middle| S^{(k-1)} = S_1 \right\} = \Pr\left\{ I_k = 0 \right\} \\ \Pr\left\{ S^{(k)} = S_0 \middle| S^{(k-1)} = S_1 \right\} = \Pr\left\{ S^{(k)} = S_1 \middle| S^{(k-1)} = S_0 \right\} = \Pr\left\{ I_k = 1 \right\} \end{cases}$$

Dynamic programming

Rewrite the above as

$$
\begin{aligned}
\max_{S^{(0)},\ldots,S^{(K)}} \quad & \prod_{k=1}^{K}\left[f\left(r_k\big|s_k=\mathcal{O}\left(S^{(k-1)},S^{(k)}\right)\right)\Pr\left\{S^{(k)}\big|S^{(k-1)}\right\}\right] \\
=\max_{S^{(K)}} \quad & \max_{S^{(K-1)}} f\left(r_K\big|\mathcal{O}\left(S^{(K-1)},S^{(K)}\right)\right)\Pr\left\{S^{(K)}\big|S^{(K-1)}\right\} \\
& \times \max_{S^{(K-2)}} f\left(r_{K-1}\big|\mathcal{O}\left(S^{(K-2)},S^{(K-1)}\right)\right)\Pr\left\{S^{(K-1)}\big|S^{(K-2)}\right\} \\
& \times \quad \cdots \\
& \times \max_{S^{(1)}} f\left(r_2\big|\mathcal{O}\left(S^{(1)},S^{(2)}\right)\right)\Pr\left\{S^{(2)}\big|S^{(1)}\right\} \\
& \quad \times \quad f\left(r_1\big|\mathcal{O}\left(S^{(0)},S^{(1)}\right)\right)\Pr\left\{S^{(1)}\big|S^{(0)}\right\}
\end{aligned}
$$

$$\max_{S^{(1)}} f\left(r_2 | \mathcal{O}\left(S^{(1)}, S^{(2)}\right)\right) \Pr\left\{S^{(2)} | S^{(1)}\right\} f\left(r_1 | \mathcal{O}\left(S^{(0)}, S^{(1)}\right)\right) \Pr\left\{S^{(1)} | S^{(0)}\right\}$$

Note

- This is a function of $S^{(2)}$ only.
- I.e., given any $S^{(2)}$, there is at least one state $\hat{S}^{(1)}$ such that it maximizes the objective function.
- If there exist more than one choices of $\hat{S}^{(1)}$ such that the object function is maximized, just pick arbitrary one.

Hence we define for (previous state) S and (current state) $\tilde{S}$ $\in \mathcal{S}^2$

① the branch metric function

$$B(S, \tilde{S}|r) = f\left(r \middle| \mathcal{O}\left(S, \tilde{S}\right)\right) \Pr\left\{\tilde{S}|S\right\}$$

② the state metric function

$$\varphi_1(\tilde{S}) = \max_{S = S_0} B(S, \tilde{S}|r_1)$$
$$\varphi_k(\tilde{S}) = \max_{S \in \mathcal{S}} B(S, \tilde{S}|r_k)\varphi_{k-1}(S) \quad k = 2, 3, \ldots$$

③ the survival path function

$$P_k(\tilde{S}) = \arg\max_{S \in \mathcal{S}} B(S, \tilde{S}|r_k)\varphi_{k-1}(S) \quad k = 2, 3, \ldots$$

Dynamic programming

We can then rewrite the decision criterion in a recursive form as

$$\max_{S^{(K)}} \max_{S^{(K-1)}} B\left(S^{(K-1)}, S^{(K)}|r_K\right) \max_{S^{(K-2)}} B\left(S^{(K-2)}, S^{(K-1)}|r_{K-1}\right)$$

$$\cdots \max_{S^{(1)}} B\left(S^{(1)}, S^{(2)}|r_2\right) \max_{S^{(0)}=S_0} B\left(S^{(0)}, S^{(1)}|r_1\right)$$

$$= \max_{S^{(K)}} \max_{S^{(K-1)}} B\left(S^{(K-1)}, S^{(K)}|r_K\right) \max_{S^{(K-2)}} B\left(S^{(K-2)}, S^{(K-1)}|r_{K-1}\right)$$

$$\cdots \max_{S^{(1)}} B\left(S^{(1)}, S^{(2)}|r_2\right) \varphi_1(S^{(1)})$$

$$= \max_{S^{(K)}} \max_{S^{(K-1)}} B\left(S^{(K-1)}, S^{(K)}|r_K\right) \max_{S^{(K-2)}} B\left(S^{(K-2)}, S^{(K-1)}|r_{K-1}\right)$$

$$\cdots \varphi_2(S^{(2)})$$

$$= \max_{S^{(K)}} \max_{S^{(K-1)}} B\left(S^{(K-1)}, S^{(K)}|r_K\right) \varphi_{K-1}(S^{(K-1)})$$

$$= \max_{S^{(K)}} \varphi_K(S^{(K)}).$$

Input: Channel observations $r_1, \ldots, r_K$

Output: MAP estimates $\hat{s}_1, \cdots, \hat{s}_K$

Initializing

1: **for all** $S^{(1)} \in \mathcal{S}$ **do**

2:     Compute $\varphi_1(S^{(1)})$ based on $B(S^{(0)} = S_0, S^{(1)}|r_1)$ for each $S^{(1)} \in \mathcal{S}$ (There are $|\mathcal{S}|$ survivor path metrics)

3:     Record $P_1(S^{(1)})$ for each $S^{(1)} \in \mathcal{S}$ (There are $|\mathcal{S}|$ survivor paths)

4: **end for**

Viterbi algorithm: Recursive stage

1: **for** $k = 2$ to K **do**
2:      **for all** $S^{(k)} \in \mathcal{S}$ (i.e., for each $S^{(k)} \in \mathcal{S}$) **do**
3:          **for all** $S^{(k-1)} \in \mathcal{S}$ **do**
4:              Compute $B(S^{(k-1)}, S^{(k)}|r_k)$.
5:          **end for**
6:          Compute $\varphi_k(S^{(k)})$ based on $\begin{cases} \varphi_{k-1}(S^{(k-1)}) \\ B(S^{(k-1)}, S^{(k)}|r_k) \end{cases}$
7:          Record $P_k(S^{(k)})$
8:      **end for**
9: **end for**

1: $\hat{S}_K = \arg\max_{S^{(K)}} \varphi_K(S^{(K)})$
2: **for** $k = K$ downto 1 **do**
3:     $\hat{S}_{k-1} = P_k\left(\hat{S}_k\right)$
4:     $\hat{s}_k = \mathcal{O}\left(\hat{S}_{k-1}, \hat{S}_k\right)$
5: **end for**

Advantage of Viterbi algorithm

- Intuitive exhaustive checking for

$$\arg \max_{S^{(1)}, \cdots, S^{(K)}}$$

  has exponential complexity $O\left(|\mathcal{S}|^K\right)$

- The Viterbi algorithm has linear complexity $O\left(K |\mathcal{S}|^2\right)$.

- Many communication problems can be formulated as a 1st-order finite-state Markov chain. To name a few:
  1. Demodulation of CPM
  2. Demodulation of differential encoding
  3. Decoding of convolutional codes
  4. Estimation of correlated channels

- It is easy to generate to high-order Markov chains.

Baum-Welch or BCJR algorithm (1974)

- The Viterbi algorithm provides the best estimate of a sequence $\hat{s}_1, \cdots, \hat{s}_K$ (equivalently, the information sequence $I_0, I_1, I_2, \dots$)

- How about the best estimate of a single-branch information bit $\hat{I}_i$?

- The best MAP estimate of a single-branch information bit $\hat{I}_i$ is the following:

$$\hat{I}_i = \arg\max_{I_i} \sum_{\substack{S^{(1)}, \cdots, S^{(K)} \\ \mathcal{I}(S^{(i-1)}, S^{(i)}) = I_i}} \prod_{k=1}^{K} \left[f\left(r_k | \mathcal{O}\left(S^{(k-1)}, S^{(k)} \right) \right) \Pr\left\{ S^{(k)} | S^{(k-1)} \right\} \right]$$

where $\mathcal{I}(S^{(i-1)}, S^{(i)})$ reports the information bit corresponding to branch from state $S^{(i-1)}$ to state $S^{(i)}$.

- This can be solved by another dynamic programming, known as **Baum-Welch** (or BCJR) algorithm.

Applications of Baum-Welch algorithm

The Baum-Welch algorithm has been applied to

1. situation when a soft-output is needed such as turbo codes
2. image pattern recognitions
3. bio-DNA sequence detection

What you learn from Chapter 4

- Analysis of error rate based on signal space vector points
  - (Important) Optimal MAP/ML decision rule
  - (Important) Binary antipodal signal & binary orthogonal signal
  - (Important) Union bounds and lower bound on error rate
  - (Advanced) M-ary PAM (exact), M-ary QAM (exact), M-ary biorthogonal signals (exact), M-ary PSK (approximate)
  - (Advanced) Optimal non-coherent receivers for carrier modulated signals and orthogonal signals as well as differential PSK

- (Important) Matched filter that maximizes the output SNR
- (Important) Maximum-likelihood sequence detector (Viterbi algorithm)
    - General dynamic programming & BCJR algorithm (outside the scope of exam)
- Shannon limit
    - A refined union bound
- (Good to know) Power-limited versus band-limited modulations