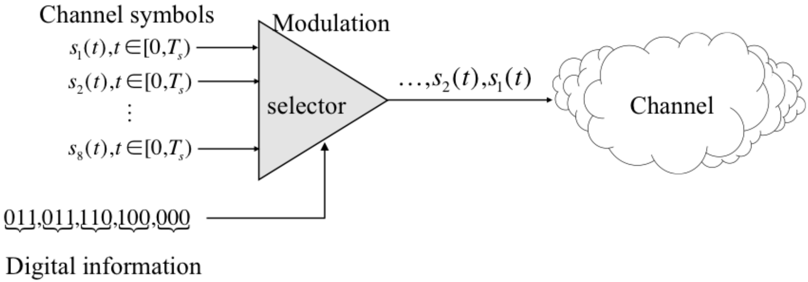
Digital Communications

Chapter 3: Digital Modulation Schemes

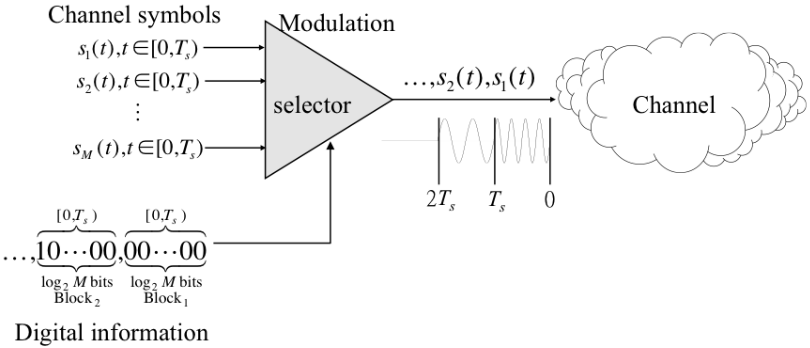
Po-Ning Chen, Professor

Institute of Communications Engineering
National Chiao-Tung University, Taiwan

3.1 Representation of digitally modulated signals



Note that the channel symbols are bandpass signals.



- Memoryless modulation: $s_{m_\ell}(t)$, $m_\ell \in \{1, 2, \ldots, M\}$,
  m_ℓ =function of Block$_\ell$

- Modulation with memory: $s_{m_\ell}(t)$,
  m_ℓ =function of (Block$_\ell$, Block$_{\ell-1}, \cdots,$ Block$_{\ell-(L-1)}$)

L = Constraint length of modulation (with memory)

Terminology

Signal $s_m(t)$, $1 \le m \le M$, $t \in [0, T_s)$

- Signaling interval: T_s (For convenience, we will sometimes use T instead.)
- Signaling rate (or symbol rate): $R_s = \frac{1}{T_s}$
- (Equivalent) Bit interval: $T_b = \frac{T_s}{\log_2 M}$
- (Eqiuvalent) Bit rate: $R_b = \frac{1}{T_b} = R_s \log_2 M$
- Average signal energy (assume equal-probable in message m)

$$\mathcal{E}_{\text{avg}} = \frac{1}{M} \sum_{m=1}^{M} \int_0^{T_s} |s_m(t)|^2 \, dt$$

- (Equivalent) Average bit energy: $\mathcal{E}_{\text{bavg}} = \frac{\mathcal{E}_{\text{avg}}}{\log_2 M}$
- Average power: $P_{\text{avg}} = \frac{\mathcal{E}_{\text{avg}}}{T_s} = R_s \mathcal{E}_{\text{avg}} = \frac{\mathcal{E}_{\text{bavg}}}{T_b} = R_b \mathcal{E}_{\text{bavg}}$

3.2 Memoryless modulation methods

Example studies of memoryless modulation

- Digital pulse amplitude modulated (PAM) signals (Amplitude-shift keying or ASK)
- Digital phase-modulated (PM) signals (Phase shift keying or PSK)
- Quadrature amplitude modulated (QAM) signals
- Multidimensional modulated signals
  - Orthogonal
  - Bi-orthogonal
- Simplex signals

M-ary pulse amplitude modulation (M-PAM)

PAM bandpass waveform

$$s_m(t) = \textbf{Re}\left\{A_m g(t) e^{\imath 2\pi f_c t}\right\} = A_m g(t)\cos(2\pi f_c t), \;\; t \in [0, T_s),$$

where $A_m = (2m - 1 - M)d$, and $m = 1, 2, \cdots, M$

Example 1 (M=4)

$$\begin{cases} s_1(t) &= -3 \cdot d \cdot g(t) \cdot \cos(2\pi f_c t) \\ s_2(t) &= -1 \cdot d \cdot g(t) \cdot \cos(2\pi f_c t) \\ s_3(t) &= +1 \cdot d \cdot g(t) \cdot \cos(2\pi f_c t) \\ s_4(t) &= +3 \cdot d \cdot g(t) \cdot \cos(2\pi f_c t) \end{cases}$$

The amplitude difference between two adjacent signals = $2d$.

$$s_m(t) = \mathbf{Re}\left\{A_m g(t) e^{\imath 2\pi f_c t}\right\} = A_m g(t)\cos(2\pi f_c t), \quad t \in [0, T_s)$$

- $g(t)$ is the pulse shaping function.
- T_s is usually assumed to be a multiple of $\frac{1}{f_c}$ in principle.

Vectorization of M-PAM signals (Gram-Schmidt)

$$\phi_1(t) = \frac{g(t)}{\|g(t)\|}\sqrt{2}\cos(2\pi f_c t) = \frac{g(t)}{\sqrt{\mathcal{E}_g}}\sqrt{2}\cos(2\pi f_c t)$$

$$\mathbf{s}_m = \left[\frac{A_m}{\sqrt{2}} \cdot \|g(t)\|\right], \text{ a one-dimensional vector}$$

By the correct Gram-Schmidt procedure,

$$\begin{aligned}\phi_1(t) &= \frac{g(t)\cos(2\pi f_c t)}{\|g(t)\cos(2\pi f_c t)\|} \\ &\neq \frac{g(t)\cos(2\pi f_c t)}{\|g(t)\| \cdot \frac{1}{\sqrt{T_s}}\|\cos(2\pi f_c t)\|} = \frac{g(t)\cos(2\pi f_c t)}{\|g(t)\|\sqrt{1/2}}\end{aligned}$$

The idea behind the above derivation is to single out "$\|g(t)\|$" in the expression! This justifies the necessity of introducing the lowpass equivalent signal where the influence of f_c has been relaxed.

For a time-limited signal, we can only claim $\mathcal{E}_{x_\ell} \approx 2\mathcal{E}_x$!

$$
\begin{aligned}
\|\phi_1(t)\|^2 &= \frac{2}{\|g(t)\|^2}\int_0^{T_s} g^2(t)\cos^2(2\pi f_c t)\,dt \\
&= \frac{2}{\|g(t)\|^2}\int_0^{T_s} g^2(t)\left[\frac{1+\cos(4\pi f_c t)}{2}\right]dt \\
&= \frac{1}{\|g(t)\|^2}\int_0^{T_s} g^2(t)\,dt \\
&\quad + \frac{1}{\|g(t)\|^2}\int_0^{T_s} g^2(t)\cos(4\pi f_c t)\,dt \\
&\approx \frac{1}{\|g(t)\|^2}\int_0^{T_s} g^2(t)\,dt = 1
\end{aligned}
$$

If $g(t)$ is constant for $t \in [0, T_s)$ and T_s is a multiple of $\frac{1}{f_c}$, then the above "$\approx$" becomes "$=$."

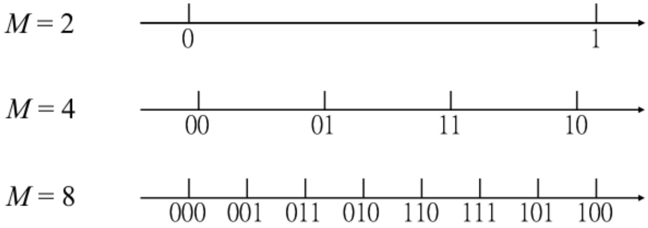
Based on the "pseudo"-vectorization,

- Transmission energy of M-PAM signals

$$\mathcal{E}_m = \int_0^{T_s} |s_m(t)|^2\, dt \approx \frac{A_m^2 \left\| g(t) \right\|^2}{2} = \frac{1}{2} A_m^2 \mathcal{E}_g$$

- Error consideration
  - The most possible error is the erroneous selection of an adjacent amplitude to the transmitted signal amplitude.
  - Therefore, the mapping (from bit pattern to channel symbol) is assigned such that the adjacent signal amplitudes differ by exactly one bit. (Gray encoding)
  - In such way, the most possible bit error pattern (caused by the noise) is a single bit error.

Gray code (Signal space diagram : one dimension)



Euclidean distance

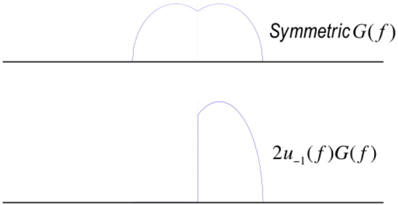
$$
\begin{aligned}
\|s_m(t) - s_n(t)\| &\approx \left| \frac{A_m \|g(t)\|}{\sqrt{2}} - \frac{A_n \|g(t)\|}{\sqrt{2}} \right| \\
&= \frac{\|g(t)\|}{\sqrt{2}} \left|(2m-1-M)d - (2n-1-M)d\right| \\
&= d\sqrt{2}\,\|g(t)\|\,|m-n|
\end{aligned}
$$

Single side band (SSB) PAM

1. $g(t)$ is real $\Leftrightarrow$ $G(f)$ is Hermitian symmetric.
2. Consequently, the previous PAM is based on the double side band (DSB) transmission which requires twice the bandwidth.
3. Recall

$$\mathcal{F}^{-1}\left\{u_{-1}(f)G(f)\right\} = \frac{1}{2}\left[g(t) + \imath\,\hat{g}(t)\right] = g_{+}(t)$$

where $\hat{g}(t)$ is the Hilbert transform of $g(t)$.



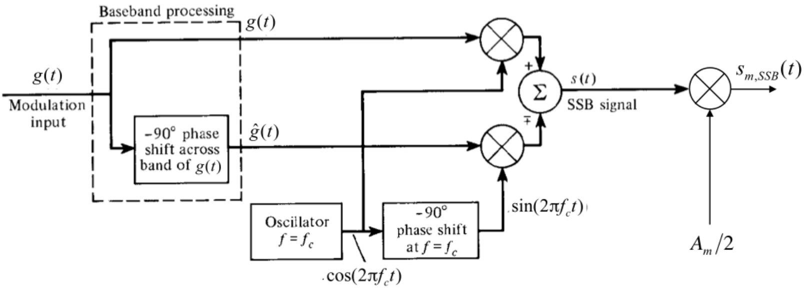
$$s_{m,SSB}(t) = \mathbf{Re}\left\{A_m g_+(t) e^{\imath 2\pi f_c t}\right\}$$

$$\phi_{1,SSB}(t) \approx \frac{\mathbf{Re}\left\{A_m g_+(t) e^{\imath 2\pi f_c t}\right\}}{\|g_+(t)\| \cdot \frac{1}{\sqrt{T_s}} \left\|\mathbf{Re}\left\{A_m e^{\imath 2\pi f_c t}\right\}\right\|} = \frac{\mathbf{Re}\left\{\sqrt{2}\, g_+(t) e^{\imath 2\pi f_c t}\right\}}{\|g_+(t)\|}$$

$$\boldsymbol{s}_{m,SSB} = \left[\frac{A_m}{\sqrt{2}} \|g_+(t)\|\right]$$

$$\begin{aligned}
&\|g_+(t)\|^2 \cdot \int_0^{T_s} \phi_{1,SSB}^2(t)\, dt \\
&= 2 \int_0^{T_s} \mathbf{Re}\left\{g_+(t) e^{\imath 2\pi f_c t}\right\}^2 dt \\
&= \frac{1}{2} \int_0^{T_s} \left[g_+(t) e^{\imath 2\pi f_c t} + g_+^*(t) e^{-\imath 2\pi f_c t}\right]^2 dt \\
&= \frac{1}{2} \int_0^{T_s} \left[|g_+(t)| e^{\imath 2\pi f_c t + \angle g_+(t)} + |g_+(t)| e^{-\imath 2\pi f_c t - \angle g_+(t)}\right]^2 dt \\
&= \int_0^{T_s} |g_+(t)|^2 dt + \int_0^{T_s} |g_+(t)|^2 \cos\left[4\pi f_c t + 2\angle g_+(t)\right] dt \\
&\approx \int_0^{T_s} |g_+(t)|^2 dt = \|g_+(t)\|^2
\end{aligned}$$

$$s_{m,SSB}(t) = \mathbf{Re}\left\{\frac{A_m}{2}\left[g(t) \pm \imath\,\hat{g}(t)\right]e^{\imath 2\pi f_c t}\right\}$$
$$= \frac{A_m}{2}g(t)\cos(2\pi f_c t) \mp \frac{A_m}{2}\hat{g}(t)\sin(2\pi f_c t)$$



$$\|g_+(t)\|^2 = \left\| \frac{1}{2} g(t) + \imath \frac{1}{2} \hat{g}(t) \right\|^2 = \frac{1}{2} \|g(t)\|^2$$

Recall from Slide 2-22, $x_+(t) = \frac{1}{2}(x(t) + \imath \hat{x}(t))$ and $\mathcal{E}_x = 2\mathcal{E}_{x_+}$.

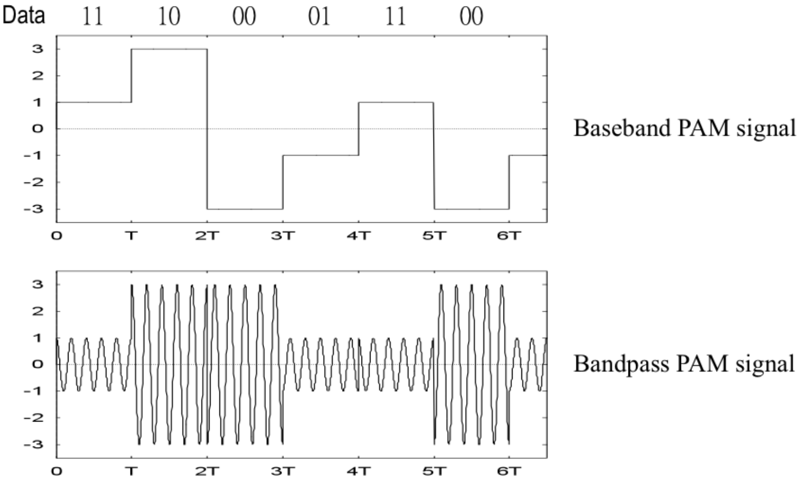
To summarize

$$\begin{cases} \phi_{1(,DSB)}(t) &= \frac{g(t)}{\|g(t)\|}\sqrt{2}\cos(2\pi f_c t) \\ \boldsymbol{s}_{m(,DSB)} &= \frac{A_m}{\sqrt{2}}\|g(t)\| \end{cases}$$

$$\begin{cases} \phi_{1,SSB}(t) &= \mathbf{Re}\left\{ \frac{g_+(t)}{\|g_+(t)\|}\sqrt{2}e^{\imath 2\pi f_c t} \right\} \\ \boldsymbol{s}_{m,SSB} &= \frac{A_m}{\sqrt{2}}\|g_+(t)\| \end{cases}$$

2-level PAM signals are particularly named *antipodal* signals. (± 1 signals)

Applications of PAM



$$T_s = T$$

Phase-modulation (PM)

Bandpass PM

$$\begin{aligned} s_m(t) &= \mathbf{Re}\left[g(t)e^{\imath 2\pi(m-1)/M} e^{\imath 2\pi f_c t}\right] \\ &= g(t)\cos(2\pi f_c t + \theta_m) \\ &= \cos(\theta_m)\underbrace{g(t)\cos(2\pi f_c t)}_{\phi_1} - \sin(\theta_m)\underbrace{g(t)\sin(2\pi f_c t)}_{\phi_2} \end{aligned}$$

where $\theta_m = 2\pi(m-1)/M,\ m = 1, 2, \cdots, M$

Example 2 (M=4)

$$\begin{cases} s_1(t) = g(t)\cos(2\pi f_c t) \\ s_2(t) = g(t)\cos(2\pi f_c t + \pi/2) \\ s_3(t) = g(t)\cos(2\pi f_c t + \pi) \\ s_4(t) = g(t)\cos(2\pi f_c t + 3\pi/2) \end{cases}$$

Signal space of PM signals

$$\begin{cases} \phi_1(t) \approx \frac{g(t)}{\|g(t)\|}\sqrt{2}\cos\left(2\pi f_c t\right) \\ \phi_2(t) \approx -\frac{g(t)}{\|g(t)\|}\sqrt{2}\sin\left(2\pi f_c t\right) \end{cases}$$

$\Longrightarrow$

$$\boldsymbol{s}_m = \left[\frac{\|g(t)\|}{\sqrt{2}}\cos(\theta_m), \frac{\|g(t)\|}{\sqrt{2}}\sin(\theta_m)\right]$$

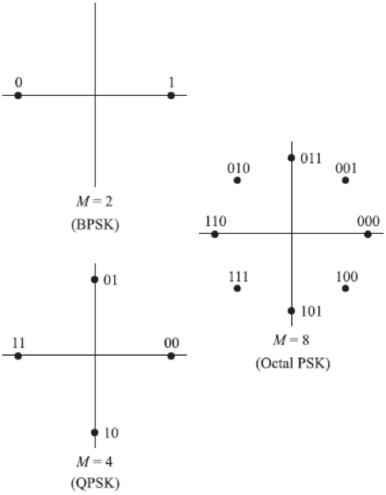
- Transmission energy of PM Signals

$$\mathcal{E}_m = \int_0^T s_m^2(t)\,dt \approx \frac{\|g(t)\|^2}{2}\left[\cos^2(\theta_m) + \sin^2(\theta_m)\right] = \frac{\mathcal{E}_g}{2}$$

  Advantages of PM signals : Equal energy for every channel symbol

- Error consideration
  - The most possible error is the erroneous selection of an adjacent phase of the transmitted signal phase.
  - Therefore, we assign the mapping from bit pattern to channel symbol as the adjacent signal phases differ only by one bit. (Gray encoding)
  - The most possible bit error pattern caused by the noise is a single-bit error.

Signal space diagram of PM with Gray code



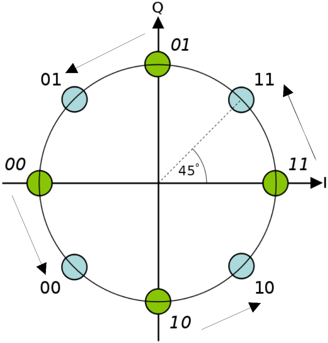
$$\boldsymbol{s}_m = \left[\frac{\|g(t)\|}{\sqrt{2}} \cos(\theta_m), \frac{\|g(t)\|}{\sqrt{2}} \sin(\theta_m) \right]$$

- Euclidean distance

$$\begin{aligned} &\|s_m(t) - s_n(t)\| \\ &= \frac{\|g(t)\|}{\sqrt{2}} \sqrt{|\cos(\theta_m) - \cos(\theta_n)|^2 + |\sin(\theta_m) - \sin(\theta_n)|^2} \\ &= \|g(t)\| \sqrt{1 - \cos(\theta_m - \theta_n)} \end{aligned}$$

$\pi/4$-QPSK

A variant of 4-phase PSK (QPSK), named $\pi/4$-QPSK, is obtained by introducing an additional $\pi/4$ phase shift in the carrier phase in each symbol interval.



Quadrature amplitude modulation (QAM)

Bandpass QAM

$$s_m(t) = x_i(t)\cos(2\pi f_c t) - x_q(t)\sin(2\pi f_c t)$$

where $x_i(t)$ and $x_q(t)$ are quadrature components. Let

$x_i(t) = A_{mi} g(t)$ and $x_q(t) = A_{mq} g(t)$; then bandpass QAM is

$$s_m(t) = A_{mi} g(t)\cos(2\pi f_c t) - A_{mq} g(t)\sin(2\pi f_c t)$$

Advantage: Transmit more digital information by using both quadrature components as information carriers. As a result, the transfer rate of digital data is doubled.

Vectorization of QAM signals

$$s_m(t) = A_{mi} \underbrace{g(t)\cos(2\pi f_c t)}_{\phi_1} - A_{mq} \underbrace{g(t)\sin(2\pi f_c t)}_{\phi_2}$$

$$\begin{cases} \phi_1(t) \approx \frac{g(t)}{\|g(t)\|}\sqrt{2}\cos(2\pi f_c t) \\ \phi_2(t) \approx -\frac{g(t)}{\|g(t)\|}\sqrt{2}\sin(2\pi f_c t) \end{cases}$$

$$\Longrightarrow \quad \boldsymbol{s}_m = \left[\frac{A_{mi}}{\sqrt{2}} \|g(t)\|, \frac{A_{mq}}{\sqrt{2}} \|g(t)\| \right]$$

$$\mathbf{s}_m = \left[\frac{A_{mi}}{\sqrt{2}} \|g(t)\|, \frac{A_{mq}}{\sqrt{2}} \|g(t)\| \right]$$

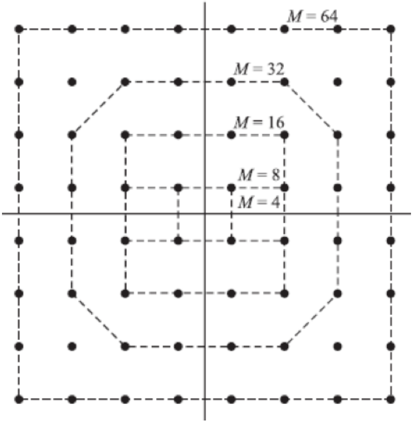
- Transmission energy of QAM signals

$$\begin{aligned} \mathcal{E}_m &= \int_0^T s_m^2(t)\,dt \\ &= \frac{1}{2}\|g(t)\|^2 A_{mi}^2 + \frac{1}{2}\|g(t)\|^2 A_{mq}^2 \\ &= \frac{1}{2}\|g(t)\|^2 \left(A_{mi}^2 + A_{mq}^2\right) \\ &= \frac{1}{2}\mathcal{E}_g \left(A_{mi}^2 + A_{mq}^2\right) \end{aligned}$$

- Euclidean Distance

$$\|s_m(t) - s_n(t)\| = \frac{\sqrt{\mathcal{E}_g}}{\sqrt{2}} \sqrt{|A_{mi} - A_{ni}|^2 + |A_{mq} - A_{nq}|^2}$$

Signal space diagram for rectangular QAM



$$\boldsymbol{s}_m = \left[\frac{A_{mi}}{\sqrt{2}} \|g(t)\|, \frac{A_{mq}}{\sqrt{2}} \|g(t)\| \right],$$

where $A_{mi}, A_{mq} \in \left\{ (2m - 1 - \sqrt{M}) : m = 1, 2, \cdots, \sqrt{M} \right\}$

- Minimum Euclidean distance (of square QAM)

$$\min_{m \neq n} \sqrt{\frac{\mathcal{E}_g}{2}} \sqrt{\underbrace{|A_{mi} - A_{ni}|^2}_{=4} + \underbrace{|A_{mq} - A_{nq}|^2}_{=0}} = \sqrt{2\mathcal{E}_g}$$

- Average symbol energy (of square QAM)

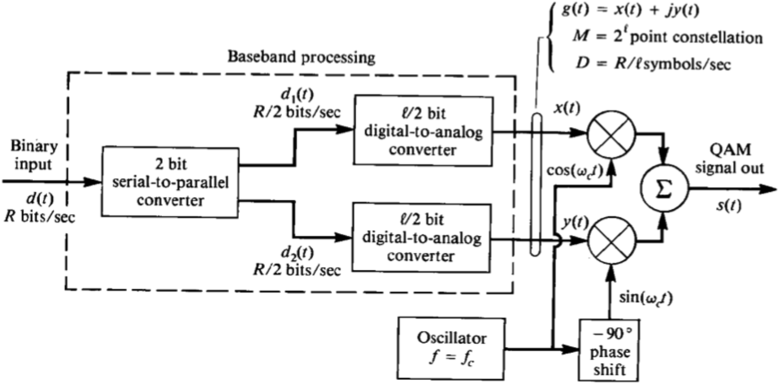
$$\mathcal{E}_{avg} = \frac{1}{M} \frac{\mathcal{E}_g}{2} \sum_{m=1}^{\sqrt{M}} \sum_{n=1}^{\sqrt{M}} \left(A_{mi}^2 + A_{nq}^2 \right) = \frac{\mathcal{E}_g}{2M} \frac{2M(M-1)}{3} = \frac{M-1}{3} \mathcal{E}_g$$

- Average bit energy (of square QAM)

$$\mathcal{E}_{bavg} = \frac{M-1}{3 \log_2 M} \mathcal{E}_g$$

Example of applications of square QAM

- CCITT V.22 modem
  - Serial binary, asynchronous or synchronous, full duplex, dial-up
  - 2400 bps or 600 baud (symbols/sec)
  - QAM, 16-point rectangular-type signal constellation



Alternative viewpoint of QAM

QAM = PM (PSK) + PAM (ASK)

- Use both amplitude and phase as digital information bearers.

$$s_m(t) = \mathbf{Re}\left[V_{m1} e^{\imath\theta_{m2}} g(t) e^{\imath 2\pi f_c t}\right] = V_{m1} g(t) \cos\left(2\pi f_c t + \theta_{m2}\right)$$

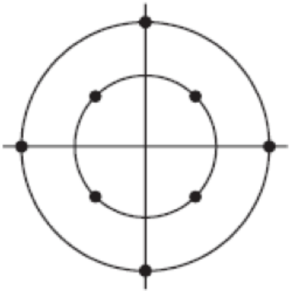
- Compare with the previous viewpoint

$$\begin{aligned} s_m(t) &= A_{mi} g(t) \cos(2\pi f_c t) - A_{mq} g(t) \sin(2\pi f_c t) \\ &= V_{m1} g(t) \cos\left(2\pi f_c t + \theta_{m2}\right) \end{aligned}$$

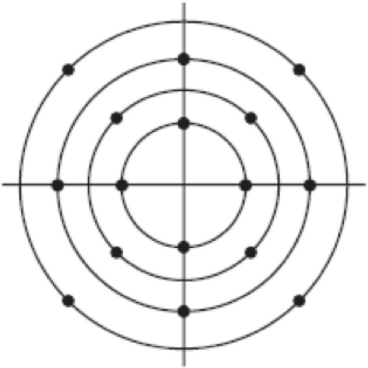
where $V_{m1} = \sqrt{A_{mi}^2 + A_{mq}^2}$ and $\theta_{m2} = \tan^{-1}(A_{mq}/A_{mi})$

- There is a one-to-one correspondence mapping from (A_{mi}, A_{mq}) domain to (V_{m1}, θ_{m2}) domain.

Signal space for non-rectangular QAM (AM-PSK)



$M = 8$



$M = 16$

Multi-dimensional signals

- PAM : one-dimensional
- PM : two-dimensional
- QAM : two-dimensional

- How to create three or higher dimensional signal?
  1. Subdivision of time
     Example. N time slots can be used to form $2N$ vector basis elements (each has two quadrature bearers.)
  2. Subdivision of frequency
     Example. N frequency subbands can be used to form $2N$ vector basis elements (each has two quadrature bearers.)
  3. Subdivision of both time and frequency

Frequency shift keying or FSK

- Subdivision of frequency

- Bandpass orthogonal multidimensional signals (Frequency shift keying or FSK)

$$\begin{aligned} s_m(t) &= \mathbf{Re}\left[\sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 2\pi (m\Delta f)t} e^{\imath 2\pi f_c t}\right] \\ &= \sqrt{\frac{2\mathcal{E}}{T}} \cos\left(2\pi f_c t + 2\pi (m\Delta f) t\right) \end{aligned}$$

- Vectorization of FSK signals under orthogonality conditions (introduced in next few slides)

$$\phi_m(t) = \frac{1}{\sqrt{\mathcal{E}}} s_m(t) \text{ and } \boldsymbol{s}_m = [0, \ldots, 0, \underbrace{\sqrt{\mathcal{E}}}_{\substack{m\text{th} \\ \text{position}}}, 0, \ldots, 0]^{\mathrm{T}}$$

Crosscorrelations of FSK signals

$$s_{m,\ell}(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 2\pi(m\Delta f)t} \quad \text{and} \quad \|s_{m,\ell}(t)\| = \sqrt{2\mathcal{E}}$$

$$\rho_{mn,\ell} = \frac{\langle s_{m,\ell}(t), s_{n,\ell}(t)\rangle}{\|s_{m,\ell}(t)\| \cdot \|s_{n,\ell}(t)\|} = \frac{1}{T}\int_0^T e^{\imath 2\pi(m-n)\Delta f \cdot t} dt$$

$$= \text{sinc}\left[T(m-n)\Delta f\right] e^{\imath \pi T(m-n)\Delta f}$$

$$\frac{\langle s_m(t), s_n(t)\rangle}{\|s_m(t)\| \, \|s_n(t)\|} = \textbf{Re}\{\rho_{mn,\ell}\} = \frac{\sin(\pi T(m-n)\Delta f)}{\pi T(m-n)\Delta f}\cos(\pi T(m-n)\Delta f)$$

$$= \text{sinc}(2T(m-n)\Delta f)$$

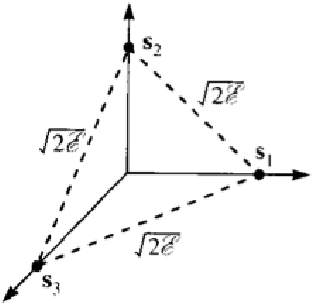
When $\Delta f = \frac{k}{2T}$, $\textbf{Re}\{\rho_{mn,\ell}\} = 0$ for $m \neq n$. In other words, the minimum frequency separation between adjacent (bandpass) signals for orthogonality is $\Delta f = \frac{1}{2T}$.

- Transmission energy of FSK signals

$$\mathcal{E}_m = \int_0^T |s_m(t)|^2 \, dt = \mathcal{E}$$

$\Longrightarrow$ Equal transmission power for each channel symbol

- Signal space diagram for FSK



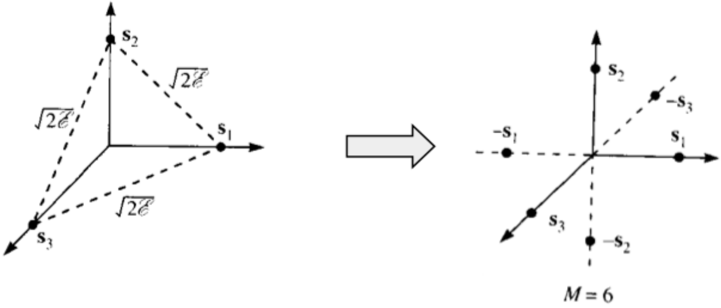
Euclidean distance between FSK signals

Equal distance between signals

$$\begin{bmatrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \cdots & \boldsymbol{s}_M \end{bmatrix} = \begin{bmatrix} \sqrt{\mathcal{E}} & 0 & \cdots & 0 \\ 0 & \sqrt{\mathcal{E}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\mathcal{E}} \end{bmatrix}$$

$$\left\| \boldsymbol{s}_m - \boldsymbol{s}_n \right\| = \sqrt{2\mathcal{E}}$$

Biorthogonal multidimensional FSK signals



- Transmission energy for biorthogonal FSK signals

$$\mathcal{E}_m = \int_0^T |s_m(t)|^2 \, dt = \mathcal{E}$$

Still, equal transmission power for each channel symbol.

- Cross-correlation of baseband biorthogonal FSK signals

$$\begin{aligned} s_{m,\ell}(t) &= \text{sgn}(m)\sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 2\pi |m| (\Delta f)\, t}, \quad m = \pm 1, \pm 2, \cdots, \pm M/2 \\ \rho_{mn,\ell} &= \begin{cases} 1, & m = n \\ -1, & m = -n \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Euclidean distance between signals

$$\begin{bmatrix} \boldsymbol{s}_{-1} & \cdots & \boldsymbol{s}_{-M/2} & \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_{M/2} \end{bmatrix}$$
$$= \begin{bmatrix} -\sqrt{\mathcal{E}} & 0 & \cdots & 0 & \sqrt{\mathcal{E}} & 0 & \cdots & 0 \\ 0 & -\sqrt{\mathcal{E}} & \cdots & 0 & 0 & \sqrt{\mathcal{E}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\sqrt{\mathcal{E}} & 0 & 0 & \cdots & \sqrt{\mathcal{E}} \end{bmatrix}$$

Hence

$$\left\| \boldsymbol{s}_m - \boldsymbol{s}_n \right\| = \begin{cases} \sqrt{2\mathcal{E}} & \text{if } m \neq -n \\ 2\sqrt{\mathcal{E}} & \text{if } m = -n \end{cases}$$

Simplex signals

Given the vector representations of orthogonal and equal-power channel symbols (such as FSK)

$$\boldsymbol{s}_m = \left[a_{m1},\ a_{m2},\ \cdots,\ a_{mk}\right]$$

for $m = 1, 2, \cdots, M$, its center (of gravity under equal prior probability assumption) is

$$\boldsymbol{c} \ = \ \left[\frac{1}{M}\sum_{m=1}^{M} a_{m1},\ \frac{1}{M}\sum_{m=1}^{M} a_{m2},\ \cdots,\ \frac{1}{M}\sum_{m=1}^{M} a_{mk}\right]$$

Define new channel symbol as

$$\boldsymbol{s}'_m \ = \ \boldsymbol{s}_m - \boldsymbol{c}$$

Then $\{\boldsymbol{s}'_1, \boldsymbol{s}'_2, \cdots, \boldsymbol{s}'_M\}$ is called the simplex signal.

Transmission energy of simplex signals

$$
\begin{aligned}
\mathcal{E}'_m &= \int_0^T |s'_m(t)|^2\, dt \\
&= \|\boldsymbol{s}_m - \boldsymbol{c}\|^2 \\
&= \|\boldsymbol{s}_m\|^2 + \|\boldsymbol{c}\|^2 - \langle \boldsymbol{s}_m, \boldsymbol{c}\rangle - \langle \boldsymbol{c}, \boldsymbol{s}_m\rangle \quad \left(\boldsymbol{c} = \frac{1}{M}\sum_{i=1}^{M} \boldsymbol{s}_i\right) \\
&= \|\boldsymbol{s}_m\|^2 + \|\boldsymbol{c}\|^2 - \frac{1}{M}\sum_{i=1}^{M} \langle \boldsymbol{s}_m, \boldsymbol{s}_i\rangle - \frac{1}{M}\sum_{i=1}^{M} \langle \boldsymbol{s}_i, \boldsymbol{s}_m\rangle \\
&= \|\boldsymbol{s}_m\|^2 + \frac{1}{M}\|\boldsymbol{s}_m\|^2 - \frac{2}{M}\|\boldsymbol{s}_m\|^2 \quad \left(\begin{array}{l}\text{by orthogonality} \\ \text{and equal-power}\end{array}\right) \\
&= \left(1 - \frac{1}{M}\right)\|\boldsymbol{s}_m\|^2
\end{aligned}
$$

- The transmission energy of a signal is reduced by "simplexing" it.

Crosscorrelation of simplex signals

$$
\begin{aligned}
\rho_{mn} = \frac{\langle \boldsymbol{s}'_m, \boldsymbol{s}'_n \rangle}{\|\boldsymbol{s}'_m\| \, \|\boldsymbol{s}'_n\|} &= \frac{\langle \boldsymbol{s}_m - \boldsymbol{c}, \boldsymbol{s}_n - \boldsymbol{c} \rangle}{\left(1 - \frac{1}{M}\right) \|\boldsymbol{s}_m\|^2} \\
&= \frac{\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle - \langle \boldsymbol{s}_m, \boldsymbol{c} \rangle - \langle \boldsymbol{c}, \boldsymbol{s}_n \rangle + \langle \boldsymbol{c}, \boldsymbol{c} \rangle}{\left(1 - \frac{1}{M}\right) \|\boldsymbol{s}_m\|^2} \\
&= \begin{cases} \frac{\|\boldsymbol{s}_m\|^2 - \frac{2}{M}\|\boldsymbol{s}_m\|^2 + \frac{1}{M}\|\boldsymbol{s}_m\|^2}{\left(1 - \frac{1}{M}\right)\|\boldsymbol{s}_m\|^2} & m = n \\ \frac{0 \quad - \frac{2}{M}\|\boldsymbol{s}_m\|^2 + \frac{1}{M}\|\boldsymbol{s}_m\|^2}{\left(1 - \frac{1}{M}\right)\|\boldsymbol{s}_m\|^2} & m \neq n \end{cases} \\
&= \begin{cases} 1 & m = n \\ -\frac{1}{M-1} & m \neq n \end{cases}
\end{aligned}
$$

Simplex signals are equally correlated !

Example of simplex signals

$$\begin{bmatrix} \boldsymbol{s}_1 & \cdots & \boldsymbol{s}_M \end{bmatrix} = \begin{bmatrix} \sqrt{\mathcal{E}} & 0 & \cdots & 0 \\ 0 & \sqrt{\mathcal{E}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\mathcal{E}} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \boldsymbol{s}'_1 & \cdots & \boldsymbol{s}'_M \end{bmatrix} = \begin{bmatrix} (1-\frac{1}{M})\sqrt{\mathcal{E}} & -\frac{1}{M}\sqrt{\mathcal{E}} & \cdots & -\frac{1}{M}\sqrt{\mathcal{E}} \\ -\frac{1}{M}\sqrt{\mathcal{E}} & (1-\frac{1}{M})\sqrt{\mathcal{E}} & \cdots & -\frac{1}{M}\sqrt{\mathcal{E}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M}\sqrt{\mathcal{E}} & -\frac{1}{M}\sqrt{\mathcal{E}} & \cdots & (1-\frac{1}{M})\sqrt{\mathcal{E}} \end{bmatrix}$$

Subdivision of time: N time slots

For example: BPSK in each dimension

$$\boldsymbol{s}_m = \left[c_{m,0}, c_{m,1}, \cdots, c_{m,N-1} \right], \quad 1 \le m \le M$$

where $NT_c = T$

- "$c_{m,j} = 0$" $\equiv$ "$g_1(t)$ is transmitted at time slot j"
- "$c_{m,j} = 1$" $\equiv$ "$g_2(t)$ is transmitted at time slot j"

$$g_1(t) = +\sqrt{\frac{2\mathcal{E}_c}{T_c}} \cos(2\pi f_c t), \quad g_2(t) = -\sqrt{\frac{2\mathcal{E}}{T}} \cos(2\pi f_c t),$$

with $t \in [0, T_c)$

$$s_m(t) = \sqrt{\frac{2\mathcal{E}_c}{T_c}} \sum_{j=0}^{N-1} (-1)^{c_{m,j}} \cos\left(2\pi f_c (t - jT_c)\right) \mathbf{1}\{jT_c \le t < (j+1)T_c\}$$

- Crosscorrelation coefficient of adjacent signals (i.e., with only one distinct component)
  - For those identical components

$$\int_0^{T_c} |g_1(t)|^2\, dt = \int_0^{T_c} |g_2(t)|^2\, dt = \mathcal{E}_c$$

  - For the single distinct component

$$\int_0^{T_c} g_1(t) g_2^*(t)\, dt = \int_0^{T_c} -|g_1(t)|^2\, dt = -\mathcal{E}_c$$

  - Hence

$$\rho_{mn} = \frac{\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle}{\|\boldsymbol{s}_m\| \, \|\boldsymbol{s}_n\|} = \frac{(N-1)\mathcal{E}_c - \mathcal{E}_c}{N\mathcal{E}_c} = 1 - \frac{2}{N}$$

- Minimum Euclidean distance between adjacent codewords

$$\begin{aligned}
\min_{m \neq n} \|\boldsymbol{s}_m - \boldsymbol{s}_n\| &= \min_{m \neq n} \sqrt{\|\boldsymbol{s}_m\|^2 + \|\boldsymbol{s}_n\|^2 - \langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle - \langle \boldsymbol{s}_n, \boldsymbol{s}_m \rangle} \\
&= \sqrt{N\mathcal{E}_c + N\mathcal{E}_c - 2(N\mathcal{E}_c)\frac{N-2}{N}} = 2\sqrt{\mathcal{E}_c}
\end{aligned}$$

- Transmission energy of multidimensional BPSK signals

$$\mathcal{E}_m = \int_0^T |s_m(t)|^2 \, dt = N \|g_1(t)\|^2 = N \int_0^{T_c} |g_1(t)|^2 \, dt = N\mathcal{E}_c$$

- Largest number of channel symbols

$$M \leq 2^N$$

- Vectorization of BPSK signals

$$\boldsymbol{s}_m = \begin{bmatrix} \pm\sqrt{\mathcal{E}_c} \\ \pm\sqrt{\mathcal{E}_c} \\ \vdots \\ \pm\sqrt{\mathcal{E}_c} \end{bmatrix}_{N\times 1}$$

Can we properly choose $\{\boldsymbol{s}_m\}_{m=1}^M$ such that they are orthogonal to each other ?

Orthogonal multidimensional signals: Hadamard signals

- **Definition:** The Hadamard signals of size $M = 2^n$ can be recursively defined as

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

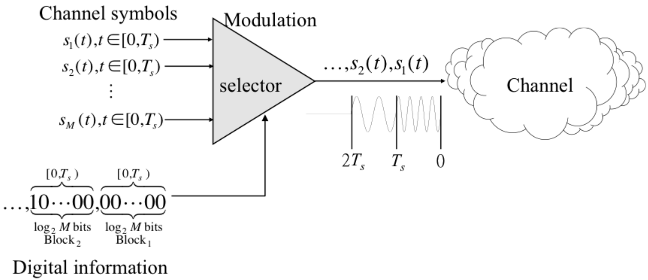
with initial value $H_0 = \begin{bmatrix} 1 \end{bmatrix}$.
For example,

$$H_1 = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & -1 \end{array}\right] \text{ and } H_2 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

Hence, when $M = 4$, the Hadamard multidimensional orthogonal (BPSK) signals are

$$\begin{bmatrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 \end{bmatrix} = \begin{bmatrix} \sqrt{\mathcal{E}_c} & \sqrt{\mathcal{E}_c} & \sqrt{\mathcal{E}_c} & \sqrt{\mathcal{E}_c} \\ \sqrt{\mathcal{E}_c} & -\sqrt{\mathcal{E}_c} & \sqrt{\mathcal{E}_c} & -\sqrt{\mathcal{E}_c} \\ \sqrt{\mathcal{E}_c} & \sqrt{\mathcal{E}_c} & -\sqrt{\mathcal{E}_c} & -\sqrt{\mathcal{E}_c} \\ \sqrt{\mathcal{E}_c} & -\sqrt{\mathcal{E}_c} & -\sqrt{\mathcal{E}_c} & \sqrt{\mathcal{E}_c} \end{bmatrix}$$

3.3 Signaling schemes with memory



- Memoryless modulation: $s_{m_i}(t)$, $m_i \in \{1, 2, \ldots, M\}$,

  m_i =function of Block$_i$

- Modulation with memory: $s_{m_i}(t)$,

  m_i =function of (Block$_i$, Block$_{i-1}, \cdots$, Block$_{i-(L-1)}$)

- Linear modulation: The modulated part of $s_{m_i}(t)$ is a linear function of the digital waveform.

  Linearity = Principle of superposition

  If $a_1 \to b_1$ and $a_2 \to b_2$, then $a_1 + a_2 \to b_1 + b_2$.

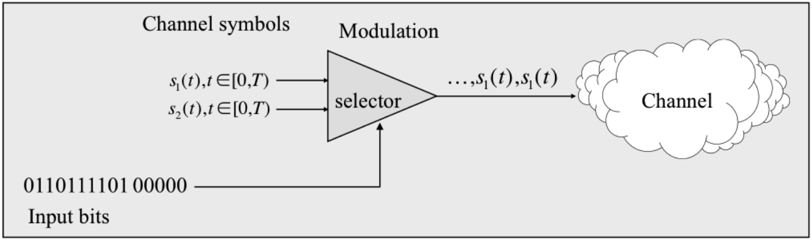
- Non-linear modulation:

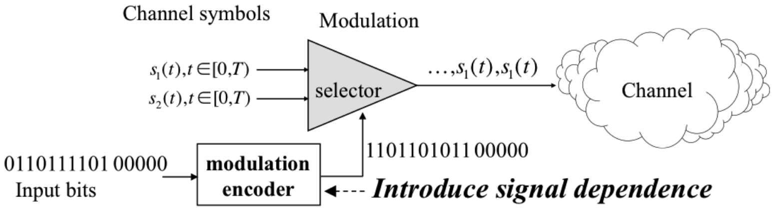
- Why introducing "memory" into signals?
  - The signal dependence is introduced for the purpose of shaping the spectrum of transmitted signal so that it matches the spectral characteristics of the channel.

- Linearity
  - For example, $s_{m_i}(t) = \textbf{Re}\left\{A_{m_i} e^{2\pi f_c t}\right\}$.

$$\begin{cases} -3 & \longrightarrow \quad \textbf{Re}\left\{-3e^{2\pi f_c t}\right\} \\ -1 & \longrightarrow \quad \textbf{Re}\left\{-1e^{2\pi f_c t}\right\} \\ +1 & \longrightarrow \quad \textbf{Re}\left\{+1e^{2\pi f_c t}\right\} \\ +3 & \longrightarrow \quad \textbf{Re}\left\{+3e^{2\pi f_c t}\right\} \end{cases}$$

  - If the modulated part of $s_{m_i}(t)$ cannot be made as a linear function of the digital waveform, the modulation is classified as nonlinear.


Channel symbols
Modulation
$s_1(t), t \in [0,T)$
$s_2(t), t \in [0,T)$
selector
$\ldots, s_1(t), s_1(t)$
Channel
0110111101 00000
Input bits


Channel symbols
Modulation
$s_1(t), t \in [0,T)$
$s_2(t), t \in [0,T)$
selector
$\ldots, s_1(t), s_1(t)$
Channel
1101101011 00000
0110111101 00000
Input bits
modulation encoder
Introduce signal dependence

Linear modulations with/without memory

- NRZ (Non-Return-to-Zero) $=$Binary PAM or binary PSK : memoryless

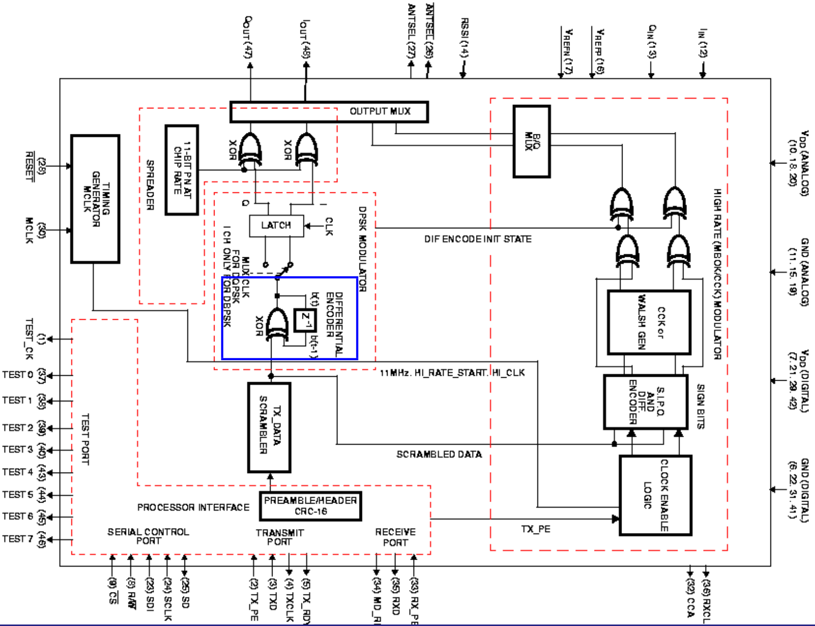
$$\text{channel code bit} = \text{input bit}$$

- NRZI (Non-Return-to-Zero, Inverted) $=$Differential encoding : with memory

$$(\text{channel code bit})_k = (\text{input bit})_k \oplus (\text{channel code bit})_{k-1}$$

$$\begin{cases} (\text{channel code bit})_k = (\text{channel code bit})_{k-1}, & \text{when } (\text{input bit})_k = 0 \\ (\text{channel code bit})_k = \overline{(\text{channel code bit})_{k-1}}, & \text{when } (\text{input bit})_k = 1 \end{cases}$$

Application: DBPSK/DQPSK in Wireless LAN



Advantage of modulation with memory

Why adding differential encoding before BPSK ?

- For PSK modulations, digital information is carried by absolute phase.
  - Synchronization is often achieved by either adding a small pilot signal or using some self-synchronization scheme.
  - The demodulator needs to detect the phase, which may have a phase ambiguity due to noise and other constraints.

- Ideal (noiseless) case

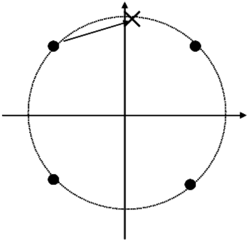
$$\begin{cases} f_{\text{transmiter}} = f_c : \text{receive } \cos(2\pi f_c t + \theta) \\ f_{\text{receiver}} = f_c : \text{estimate it based on } f_c \end{cases} \Longrightarrow \text{estimate } \hat{\theta} = \theta$$

- Ambiguous case

$$\begin{cases} f_{\text{transmiter}} = f_c : \text{receive } \cos(2\pi f_c t + \theta) \\ f_{\text{receiver}} \neq f_c : \text{estimate it based on } f_c' \end{cases}$$

$$\Longrightarrow \begin{cases} \text{receive } \cos(2\pi f_c' t + [2\pi(f_c - f_c')t] + \theta) \\ \text{estimate it based on } f_c' \end{cases}$$

$$\Longrightarrow \text{estimate } \hat{\theta} = [2\pi(f_c - f_c')t] + \theta$$



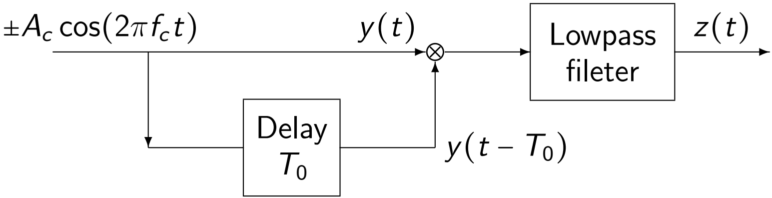
Advantage of differential encoding

$$(\text{channel code bit})_k = (\text{input bit})_k \oplus (\text{channel code bit})_{k-1}$$

- The phases or signs of the received waveforms are not important for detection.
- What is important is the change in the sign of successive pulses.
- The sign change can be detected even if the demodulating carrier has a phase ambiguity.

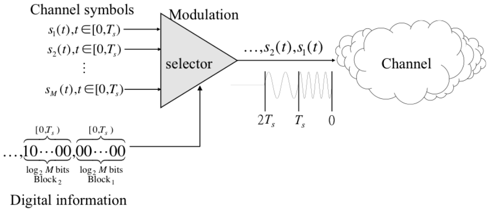
Advantage of diff encode (Noncoherent demod)

- No need to generate a local carrier at the receiver side.
- Use the received signal itself as a carrier.



$$z(t) = \begin{cases} A_c^2 \cos^2(2\pi f_c t) = \frac{A_c^2}{2} + \frac{1}{2}\cos(4\pi f_c t) \to \frac{1}{2}A_c^2, & \text{if } y(t) = y(t - T_0) \\ -A_c^2 \cos^2(2\pi f_c t) = -\frac{A_c^2}{2} - \frac{1}{2}\cos(4\pi f_c t) \to -\frac{1}{2}A_c^2, & \text{if } y(t) = -y(t - T_0) \end{cases}$$

Nonlinear modulation methods with memory



- Linear modulation: The modulated part of $s_{m_i}(t)$ is a linear function of the digital waveform.

> Linearity = Principle of superposition
> If $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$, then $a_1 + a_2 \rightarrow b_1 + b_2$.

- Nonlinear modulation: The modulated part of $s_{m_i}(t)$ cannot be made as a linear function of the digital waveform.

(Linear (from the aspect of phase)) Frequency shift keying or FSK

$$s_m(t) = \mathbf{Re}\left[\sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 2\pi(m\Delta f)t} e^{\imath 2\pi f_c t}\right]$$

where $m = \pm 1, \pm 2, \cdots, \pm(M-1)$

Motivation: Disadvantages of FSK

- Potential obstacles of multidimensional FSK with $(M-1)$ oscillators for each desired frequency
  - Abrupt switching from one oscillator to another will result in relatively large spectral side lobes outside of the main spectral band of the signal.

Continuous-Phase FSK (CPFSK)

- Alternative implementation of multidimensional FSK
- A single carrier whose frequency is changed continuously.
- This is considered as a modulated signal with memory (we will explain this point in the next few slides).

Recall

$$s(t) = \mathbf{Re}\left\{s_\ell(t)e^{\imath 2\pi f_c t}\right\}, \quad s_\ell(t) = x_i(t) + \imath\, x_q(t)$$

$s_\ell(t)$ is the baseband version of the bandpass signal $s(t)$.

For ideal FSK signals

$$s_m(t) = \mathbf{Re}\left[\sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 2\pi(m\Delta f)t} e^{\imath 2\pi f_c t}\right]$$

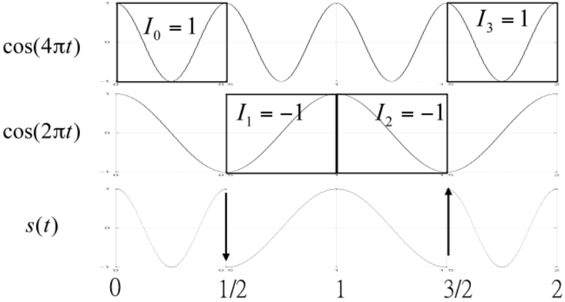
$$\Longrightarrow \quad s_{m,\ell}(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 2\pi(m\Delta f)t}$$

where $\Delta f = f_d$ and $m = \pm 1, \pm 2, \cdots, \pm(M-1)$.

Example of ideal (2-OSC) FSK signals

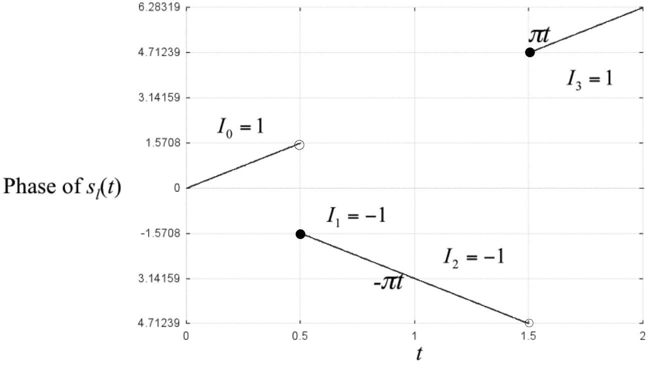
Let $T = 0.5$ sec, $\mathcal{E} = 0.25$, $f_d = 0.5$, $I_n\ (= m) \in \{1, -1\}$, and $f_c = 1.5$ Hz.

$$s(t) = \textbf{Re}\left\{s_\ell(t)e^{\imath 2\pi f_c t}\right\} = \begin{cases} \cos(4\pi t) & I_n = 1 \\ \cos(2\pi t), & I_n = -1 \end{cases}$$



Discontinuous phase of (2-OSC) FSK

$$\text{Phase of } s_\ell(t) = \begin{cases} \pi t, & I_n = 1 \\ -\pi t, & I_n = -1 \end{cases} \quad \text{for } t \in [nT, (n+1)T)$$



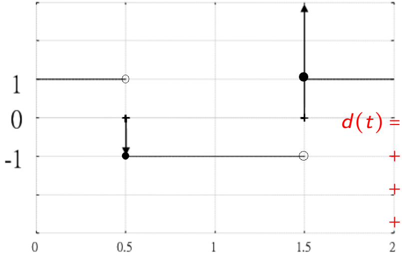
Phase change of (2-OSC) FSK

- (Normalized) phase change (for $t \in [nT, (n+1)T)$)

$$d(t) = \frac{\text{phase of } s_\ell(t)}{4\pi T f_d} = \frac{\frac{\partial}{\partial t}(2\pi I_n f_d t)}{4\pi T f_d} = \frac{I_n}{2T}$$

is the derivative of the phase!

Continue from the previous example with $T = 0.5$.



$$\begin{aligned} d(t) &= (+1)[u_{-1}(t) - u_{-1}(t-T)] - \delta(t-T) \\ &+ (-1)[u_{-1}(t-T) - u_{-1}(t-2T)] \\ &+ (-1)[u_{-1}(t-2T) - u_{-1}(t-3T)] + 3\delta(t-3T) \\ &+ (+1)[u_{-1}(t-3T) - u_{-1}(t-4T)] + \cdots \end{aligned}$$

$$
\begin{aligned}
d(t) = I_0 \left[u_{-1}(t) - u_{-1}(t-T) \right] &+ \mathbf{1}\{I_0 \neq I_1\} \cdot I_1 \cdot 1 \cdot \delta(t-T) \\
+ I_1 \left[u_{-1}(t-T) - u_{-1}(t-2T) \right] &+ \mathbf{1}\{I_1 \neq I_2\} \cdot I_2 \cdot 2 \cdot \delta(t-2T) \\
+ I_2 \left[u_{-1}(t-2T) - u_{-1}(t-3T) \right] &+ \mathbf{1}\{I_2 \neq I_3\} \cdot I_3 \cdot 3 \cdot \delta(t-3T) \\
+ I_3 \left[u_{-1}(t-3T) - u_{-1}(t-4T) \right] &+ \mathbf{1}\{I_3 \neq I_4\} \cdot I_4 \cdot 4 \cdot \delta(t-4T) \\
+ \cdots
\end{aligned}
$$

- Phase change is the derivative of the phase!
- Phase is the integration of phase change!

$$
s_\ell(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau}
$$

- Those $\delta(\cdot)$ functions result in "discontinuity" in integration! Hence, let us remove them to force "continuity" in phase.

Continuous phase FSK (CPFSK)

$$s_\ell(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{\imath 4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau}$$

where

$$d(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT) \text{ and } g(t) = \frac{1}{2T}\left[u_{-1}(t) - u_{-1}(t-T)\right].$$

- $I_n \in \{\pm 1, \pm 3, \pm 5, \cdots\}$ is the PAM information sequence.
- $g(t)$ is the "phase shaping function".
  - It is now chosen as a rectangular pulse of height $1/(2T)$ and duration $[0, T)$ (hence, the area is $1/2$.)
- T is the symbol duration.

Re-express $s_\ell(t)$ as

$$s_\ell(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{\imath \phi(t;\boldsymbol{I})}$$

where

$$\begin{aligned}
\phi(t; \boldsymbol{I}) & \\
&= 4\pi T f_d \int_{-\infty}^{t} d(\tau)\, d\tau \\
&= 4\pi T f_d \int_{-\infty}^{t} \left[\sum_{n=-\infty}^{\infty} I_n g(\tau - nT) \right] d\tau \\
&= 4\pi f_d T \left[\sum_{k=-\infty}^{n-1} I_k \left(T \times \frac{1}{2T} \right) + I_n \frac{t - nT}{2T} \right] \quad \text{for } t \in [nT, (n+1)T) \\
&= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t - nT) I_n \quad \text{for } t \in [nT, (n+1)T)
\end{aligned}$$

(Cont.) For $t \in [nT, (n+1)T)$, $s_\ell(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{\imath \phi(t;\boldsymbol{I})}$ with

$$\begin{aligned}\phi(t;\boldsymbol{I}) &= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t - nT) I_n \\ &= \theta_n + 2\pi h \cdot I_n \cdot q(t - nT),\end{aligned}$$

where

$$\begin{cases} h = 2 f_d T & \text{(modulation index)} \\ \theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k & \text{(accumulation of history/memory)} \\ q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2T} & 0 \le t < T \\ \frac{1}{2} & t \ge T \end{cases} & \text{(integration of } g(t)\text{)} \end{cases}$$

Generalization of CPFSK: CPM

We can further generalize $\phi(t;\boldsymbol{I})$ to

$$\phi(t;\boldsymbol{I}) = 2\pi \sum_{k=-\infty}^{n} h_k \cdot I_k \cdot q(t-kT)$$

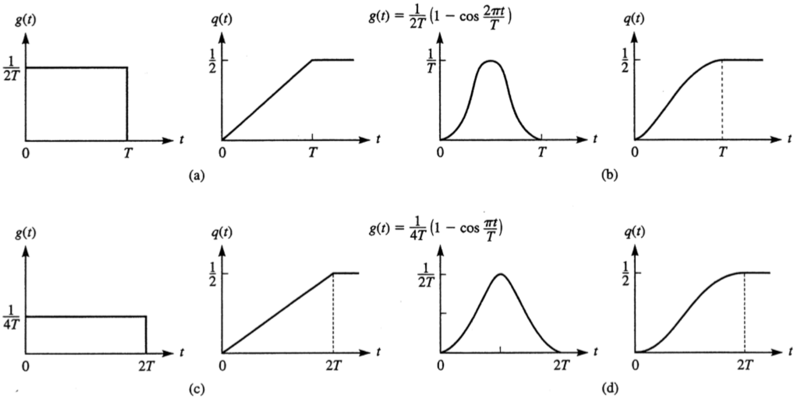
for $nT \le t < (n+1)T$

where

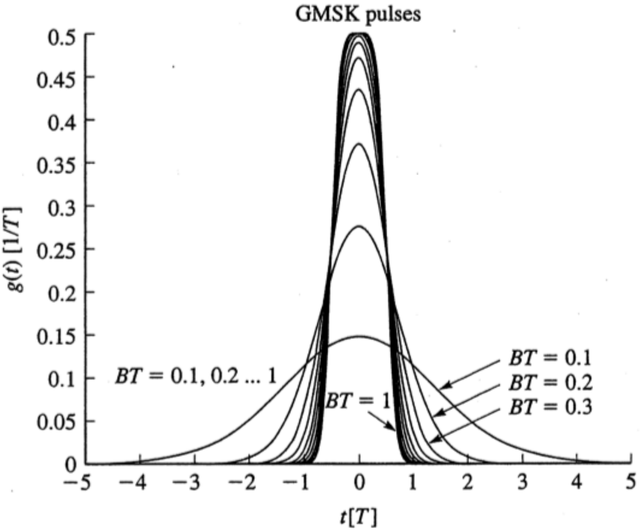
1. $\boldsymbol{I} = \{I_k\}_{k=-\infty}^{\infty}$ is the sequence of PAM symbols in $\{\pm 1, \pm 3, \ldots, \pm(M-1)\}$.
2. h_k is the modulation index.
   If h_k varies with k, it is called multi-h CPM.
3. $q(t) = \int_0^t g(\tau)\, d\tau.$

   If $g(t) = 0$ for $t \ge T$ (and $t < 0$), $s_\ell(t)$ is called full-response CPM; otherwise it is called partial-response CPM.

Examples of CPMs



Examples of CPMs



Some commonly used CPM pulse shapes

- LREC (Rectangular): LREC with $L = 1$ is CPFSK

$$g(t) = \frac{1}{2LT}\left(u_{-1}(t) - u_{-1}(t - LT)\right)$$

- LRC (Raised cosine)

$$g(t) = \frac{1}{2LT}\left(u_{-1}(t) - u_{-1}(t - LT)\right)\left(1 - \cos\left(\frac{2\pi t}{LT}\right)\right)$$

Some commonly used CPM pulse shapes

- GMSK (Gaussian minimum shift keying)

$$g(t) = Q\left(2\pi B\left(t - \frac{T}{2}\right)\Big/\sqrt{\ln 2}\right) - Q\left(2\pi B\left(t + \frac{T}{2}\right)\Big/\sqrt{\ln 2}\right)$$

where $Q(t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$, and B is 3dB Bandwidth

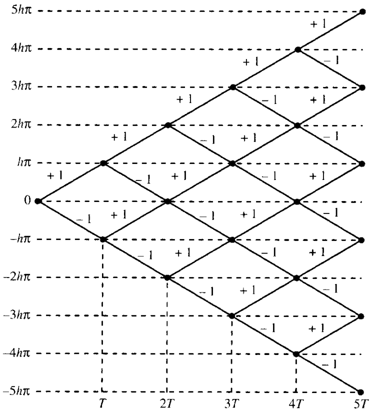
  - $g(t)$ is the response of filter $H(f) = 2^{-\frac{(f/B)^2}{2}}$ to a rectangular pulse of $u_{-1}(t + T/2) - u_{-1}(t - T/2)$.
  - GMSK with $BT = 0.3$ is used in the European digital cellular communication system, called GSM (2G).
  - At $BT = 0.3$, the GMSK pulse may be truncated at $|t| = 1.5T$ with a relatively small error incurred for $t > 1.5T$.

Representations of continuous-phase

- Phase trajectory or phase tree
- Phase trellis

Binary CPFSK (i.e., $I_n = \pm 1$ and $g(t)$ is full response rectangular function)

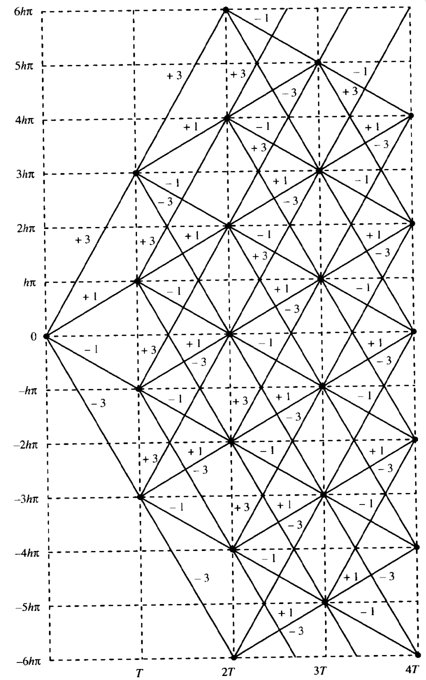
$$\phi(t; \boldsymbol{I}) = \pi h \sum_{k=-\infty}^{n-1} I_k + 2\pi h I_n \cdot q(t - nT)$$



Example 3

Quaternary CPFSK *(See the next page)* *with* $I_n \in \{-3, -1, +1, +3\}$.

- *We observe that the phase trees for CPFSK are piecewise linear as a consequence of the fact that the pulse* $g(t)$ *is rectangular.*
- *Smoother phase trajectories and phase trees are obtained by using pulses that do not contain discontinuities.*

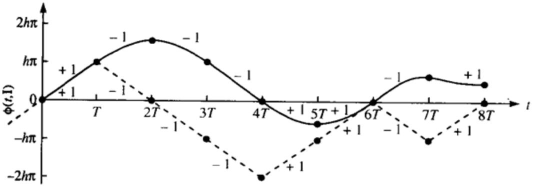

6hπ
5hπ
4hπ
3hπ
2hπ
hπ
0
–hπ
–2hπ
–3hπ
–4hπ
–5hπ
–6hπ
T
2T
3T
4T
+ 3
+ 1
– 1
– 3

If $g(t)$ is continuous (especially at boundaries), phase trajectory becomes smooth.

Example 4

$$g(t) = \frac{1}{6T}\left(1 - \cos\left(\frac{2\pi t}{3T}\right)\right) = \textit{ raised cosine of length } 3T$$

with $(I_{-2}, I_{-1}, I_0, I_1, I_2, \cdots) = (+1, +1, +1, -1, -1, -1, +1, +1, -1, +1, \cdots)$



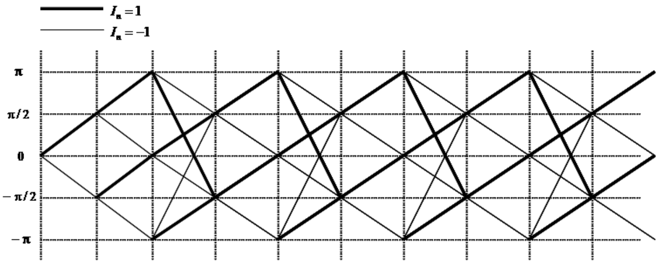
* solid line = partial response CPM based on raised cosine pulse of length $3T$.
* dashed line = binary CPFSK.

Phase trellis

Phase trellis = Phase trajectory is plotted with modulo 2π

Example 5

Binary CPFSK with $h = 1/2$ and $g(t)$ is a full response rectangular function.



Thus CPM can be decoded by Viterbi trellis decoding.

Minimum shift keying (MSK)

Recall for $nT \le t < (n+1)T$, CPM has

$$\phi(t;\boldsymbol{I}) = 2\pi \sum_{k=-\infty}^{n} h_k \cdot I_k \cdot q(t-kT).$$

CPFSK is a special case of CPM with

$$g(t) = \frac{1}{2T} \text{ for } 0 \le t < T$$

MSK is a special case of binary CPFSK with

$$h_k = \frac{1}{2},\ g(t) = \frac{1}{2T} \text{ for } 0 \le t < T \text{ and } I_n \in \{\pm 1\}$$

Thus for MSK, we have for $nT \le t < (n+1)T$,

$$\phi(t;\boldsymbol{I}) = \frac{\pi}{2}\sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t-nT) = \theta_n + \frac{1}{2}\pi I_n \left(\frac{t-nT}{T}\right)$$

$$\Phi(t;\boldsymbol{I}) = \theta_n + \frac{1}{2}\pi I_n\left(\frac{t-nT}{T}\right) = 2\pi\left(\frac{I_n}{4T}\right)t - \frac{n\pi I_n}{2} + \theta_n$$

The corresponding modulated carrier wave is

$$\begin{aligned} s_{\text{MSK}}(t) &= A\cos\left(2\pi f_c t + \Phi(t;\boldsymbol{I})\right) \\ &= A\cos\left[2\pi\left(f_c + \frac{I_n}{4T}\right)t - \frac{n\pi I_n}{2} + \theta_n\right] \end{aligned}$$

Since $I_n \in \{\pm 1\}$, $s_{MSK}(t)$ has two frequency components:

$$\begin{aligned} f_1 &= f_c - \frac{1}{4T} \\ f_2 &= f_c + \frac{1}{4T} \end{aligned}$$

Minimum shift keying (MSK)

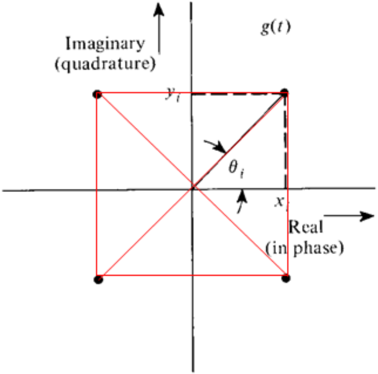
MSK is so named because $f_2 - f_1 = \frac{1}{2T}$ = the minimum (frequency) shift that makes the two frequency components orthogonal.

[See Slide 3-35] When $\Delta f = \frac{k}{2T}$, $\mathbf{Re}\{\rho_{mn,\ell}\} = 0$ for $m \neq n$. In other words, the minimum frequency separation between adjacent (passband) signals for orthogonality is $\Delta f = \frac{1}{2T}$.

MSK is sometimes regarded as a kind of OQPSK (Offset QPSK). Why?

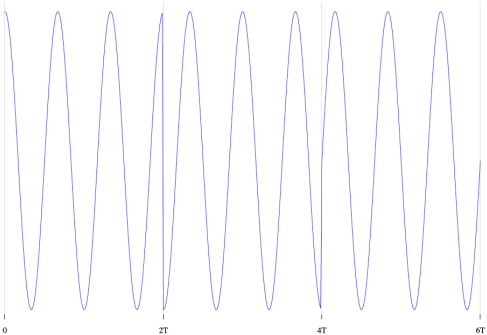
Offset QPSK

The original QPSK



There could be 180 degree of (sudden) phase change (so, not continuous phase), e.g., from $(+1,+1)$ to $(-1,-1)$.

$$
\begin{aligned}
s_{\mathrm{QPSK}}(t) &= \sum_{n=-\infty}^{\infty} I_{2n} g(t-2nT)\cos(2\pi f_c) \\
&\quad - \sum_{n=-\infty}^{\infty} I_{2n+1} g(t-2nT)\sin(2\pi f_c t)
\end{aligned}
$$

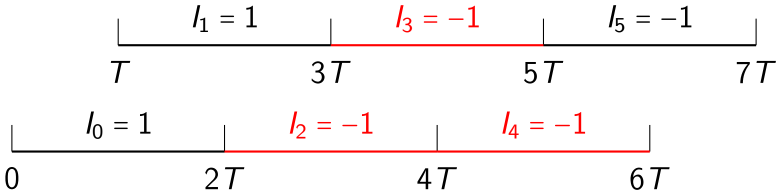


$(I_0, I_1) = (+1, +1)$, $(I_2, I_3) = (-1, -1)$ and $(I_4, I_5) = (-1, +1)$.
$g(t)$ rectangular pulse of unit height and during $2T$.

Offset QPSK (OQPSK)

How to reduce the 180^o phase change to only 90^o?

Simple solution: Do not let the "two bits" I_{2n} and I_{2n+1} change at the same time!



$$s_{\text{OQPSK}}(t) = \sum_{n=-\infty}^{\infty} I_{2n} g(t-2nT)\cos(2\pi f_c t) - \sum_{n=-\infty}^{\infty} I_{2n+1} g(t-(2n+1)T)\sin(2\pi f_c t)$$

To synchronize with the textbook, we reverse $\{I_{2n+1}\}$ to obtain

$$\begin{aligned} s_{\text{OQPSK}}(t) &= \sum_{n=-\infty}^{\infty} I_{2n} g(t - 2nT)\cos(2\pi f_c t) \\ &\quad + \sum_{n=-\infty}^{\infty} I_{2n+1} g(t - (2n+1)T)\sin(2\pi f_c t) \end{aligned}$$

OQPSK vs. MSK

MSK can be regarded as a kind of (memoryless) OQPSK. Why?

MSK: $\phi(t;\boldsymbol{I}) = \theta_n + \frac{1}{2}\pi I_n\left(\frac{t-nT}{T}\right) = \theta_0 + \frac{\pi}{2}\sum_{k=0}^{n-1} I_k + \pi\left(\frac{I_n}{2T}\right)t - \frac{n\pi}{2} I_n$ for $nT \le t < (n+1)T$

Proof: Suppose without loss of generality,

$$\theta_0 = \frac{\pi}{2}\sum_{k=-\infty}^{-1} I_k = \frac{3\pi}{2}.$$

Then for $nT \le t < (n+1)T$ (and $n \ge 1$),

$$\begin{aligned} s_{\text{MSK},\ell}(t) &= e^{\imath\phi(t;\boldsymbol{I})} \\ &= e^{\imath\pi\left(\frac{I_n}{2T}\right)t}\cdot e^{-\imath\frac{n\pi}{2}I_n}\cdot e^{\imath\frac{\pi}{2}\sum_{k=0}^{n-1}I_k}\cdot e^{\imath\theta_0} \\ &= \left[\cos\left(\pi\frac{t}{2T}\right) + \imath I_n \sin\left(\pi\frac{t}{2T}\right)\right](-I_n\imath)^n\left(\prod_{k=0}^{n-1}(I_k\imath)\right)(-\imath) \\ &= I_n^{n+1}\left(\prod_{k=0}^{n-1}I_k\right)\sin\left(\pi\frac{t}{2T}\right) + \imath I_n^n\left(\prod_{k=0}^{n-1}I_k\right)\sin\left(\pi\frac{(t-T)}{2T}\right) \end{aligned}$$

Note $(-\imath)^n\imath^n = 1$.

n	$I_n^{n+1}\left(\prod_{k=0}^{n-1} I_k\right)$	$I_n^{n}\left(\prod_{k=0}^{n-1} I_k\right)$
0	$J_0 = I_0 = J_{2\lfloor 0/2\rfloor}$	
1	$J_0 = I_0 = J_{2\lfloor 1/2\rfloor}$	$J_1 = I_0 I_1 = J_{2\lfloor (1-1)/2\rfloor+1}$
2	$J_2 = I_0 I_1 I_2 = J_{2\lfloor 2/2\rfloor}$	$J_1 = I_0 I_1 = J_{2\lfloor (2-1)/2\rfloor+1}$
3	$J_2 = I_0 I_1 I_2 = J_{2\lfloor 3/2\rfloor}$	$J_3 = I_0 I_1 I_2 I_3 = J_{2\lfloor (3-1)/2\rfloor+1}$
4	$J_4 = I_0 I_1 I_2 I_3 I_4 = J_{2\lfloor 4/2\rfloor}$	$J_3 = I_0 I_1 I_2 I_3 = J_{2\lfloor (4-1)/2\rfloor+1}$
5	$J_4 = I_0 I_1 I_2 I_3 I_4 = J_{2\lfloor 5/2\rfloor}$	$J_5 = I_0 I_1 I_2 I_3 I_4 I_5 = J_{2\lfloor (5-1)/2\rfloor+1}$
6	$J_6 = I_0 I_1 I_2 I_3 I_4 I_5 I_6 = J_{2\lfloor 6/2\rfloor}$	$J_5 = I_0 I_1 I_2 I_3 I_4 I_5 = J_{2\lfloor (6-1)/2\rfloor+1}$

For $nT \le t < (n+1)T$,

$$s_{\text{MSK},\ell}(t) = J_{2\lfloor n/2\rfloor}(-1)^{\lfloor n/2\rfloor} \underbrace{\sin\left(\pi\frac{(t-2\lfloor n/2\rfloor T)}{2T}\right)}_{g(t-2\lfloor n/2\rfloor T)}$$

$$-\imath J_{2\lfloor (n-1)/2\rfloor+1}(-1)^{\lfloor (n-1)/2\rfloor+1} \underbrace{\sin\left(\pi\frac{(t-2\lfloor (n-1)/2\rfloor T - T)}{2T}\right)}_{g(t-2\lfloor (n-1)/2\rfloor T - T)}$$

For $2mT \le t < (2m+1)T$ (i.e., $n = 2m$),

$$s_{\text{MSK},\ell}(t) = J_{2m}(-1)^m g(t - 2mT) - \imath J_{2m-1} \underbrace{(-1)^m}_{=(-1)^{\lceil (2m-1)/2 \rceil}} g(t - (2m-1)T)$$

For $(2m+1)T \le t < (2m+2)T$ (i.e., $n = 2m+1$),

$$s_{\text{MSK},\ell}(t) = J_{2m}(-1)^m g(t - 2mT) - \imath J_{2m+1} \underbrace{(-1)^{m+1}}_{=(-1)^{\lceil (2m+1)/2 \rceil}} g(t - (2m+1)T)$$

For $(2m+2)T \le t < (2m+3)T$ (i.e., $n = 2m+2$),

$$\begin{aligned} s_{\text{MSK},\ell}(t) &= J_{2(m+1)}(-1)^{m+1} g(t - 2(m+1)T) \\ &\quad - \imath J_{2m+1} \underbrace{(-1)^{m+1}}_{=(-1)^{\lceil (2m+1)/2 \rceil}} g(t - (2m+1)T) \end{aligned}$$

with $g(t) = \sin\left(\pi \frac{t}{2T}\right) [u_{-1}(t) - u_{-1}(t - 2T)]$.

MSK can be regarded as a memoryless OQPSK by setting

$$\begin{aligned} s_{\text{MSK}}(t) &= \left[\sum_{n=-\infty}^{\infty} \tilde{I}_{2n} g(t-2nT)\right]\cos(2\pi f_c t) \\ &\quad + \left[\sum_{n=-\infty}^{\infty} \tilde{I}_{2n+1} g(t-(2n+1)T)\right]\sin(2\pi f_c t) \end{aligned}$$

with

$$\tilde{I}_n = (-1)^{\lceil n/2 \rceil} J_n = (-1)^{\lceil n/2 \rceil} \prod_{k=0}^{n} I_k.$$

- MSK can be "composed" using "memoryless" circuits with "with-memory" information sequence $\tilde{\boldsymbol{I}}$.
- Please be noted that the textbook abuses the notation by using $g(t)$ to denote both amplitude and phase pulse shaping functions for CPM signals!

A linear representation of CPM

The **key** of OQPSK representation of MSK is that phase can be "pulled down" as a multiplicative adjustment in amplitude when $I_n \in \{-1, +1\}$!

For example, $$e^{\imath 2\pi\left(\frac{I_n}{4T}\right)t} = \cos\left(\pi\frac{t}{2T}\right) + \imath\, I_n \sin\left(\pi\frac{t}{2T}\right).$$

(1986 Laurent)

- CPM can also be represented as a linear superposition of AM signal waveforms (if $I_n \in \{\pm 1\}$).
- Such a representation provides an alternative method for synthesizing CPM signal at the transmitter and for demodulating the signal at the receiver.

An important and useful fact

For $I \in \{-1, +1\}$,

$$e^{\imath A \cdot I} = \frac{\sin(B - A)}{\sin(B)} + e^{\imath B \cdot I} \frac{\sin(A)}{\sin(B)}.$$

Proof:

$$\begin{aligned}
&\sin(B) e^{\imath A \cdot I} \\
&= \sin(B)[\cos(A) + \imath I \sin(A)] \\
&= \sin(B)\cos(A) + \imath \sin(B \cdot I)\sin(A) \\
&= \sin(B - A) + \cos(B)\sin(A) + \imath \sin(B \cdot I)\sin(A) \\
&= \sin(B - A) + \sin(A)[\cos(B \cdot I) + \imath \sin(B \cdot I)] \\
&= \sin(B - A) + \sin(A) e^{\imath B \cdot I}
\end{aligned}$$

□

For general h and $g(\cdot)$ function of duration L and of integral $1/2$ (but each $I_n \in \{\pm 1\}$), we have for $nT \le t < (n+1)T$ (for a binary CPM signal),

$$\begin{aligned}
s_{\text{b-CPM},\ell}(t) &= e^{\imath\phi(t;\boldsymbol{I})} \\
&= e^{\imath\left(\pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^{n} I_k q(t-kT)\right)} \\
&= e^{\imath\pi h \sum_{k=-\infty}^{n-L} I_k} \prod_{k'=0}^{L-1} e^{\imath 2\pi h I_{n-k'} q(t-(n-k')T)} \qquad (n-k'=k) \\
&= e^{\imath\pi h \sum_{k=-\infty}^{n-L} I_k} \prod_{k'=0}^{L-1} \left(\frac{\sin(B - 2\pi h\, q(t-(n-k')T))}{\sin(B)} \right. \\
&\qquad \left. + e^{\imath B \cdot I_{n-k'}} \frac{\sin(2\pi h\, q(t-(n-k')T))}{\sin(B)} \right),
\end{aligned}$$

where $B = \pi h$.

Define

$$s_0(t) = \begin{cases} \frac{\sin(2\pi h\, q(t))}{\sin(B)} & 0 \le t < LT \\ \frac{\sin(B - 2\pi h\, q(t-LT))}{\sin(B)} & LT \le t < 2LT \\ 0 & \text{otherwise} \end{cases}$$

Since $q(0) = 0$ and $q(LT) = 1/2$, $s_0(t)$ is continuous for $t \in \mathbb{R}$.

Continue the derivation:

$$\begin{aligned} s_{\text{b-CPM},\ell}(t) &= e^{\imath\pi h\sum_{k=-\infty}^{n-L} I_k} \prod_{k'=0}^{L-1}\left(\frac{\sin(B-2\pi hq(t-(n-k')T+LT-LT))}{\sin(B)}\right. \\ &\qquad\qquad \left. +e^{\imath B\cdot I_{n-k'}}\frac{\sin(2\pi hq(t-(n-k')T))}{\sin(B)}\right) \\ &= e^{\imath\pi h\sum_{k=-\infty}^{n-L} I_k} \prod_{k'=0}^{L-1}\left(s_0(t-(n-k')T+LT)\right. \\ &\qquad\qquad \left. +e^{\imath B\cdot I_{n-k'}}s_0(t-(n-k')T)\right) \end{aligned}$$

$nT \le t < (n+1)T$ and $0 \le k' \le L-1$ imply that

$$0 \le t-(n-k')T < LT \text{ and } LT \le t-(n-k')T+LT < 2LT.$$

$$\prod_{k'=0}^{L-1}\left(s_0(t-(n-k')T+LT)+e^{\imath B\cdot I_{n-k'}}s_0(t-(n-k')T)\right)$$

$$= \left(\underbrace{s_0(t-nT+0\cdot T+LT)}_{a_{i,0}=1\ (k'=0)}+e^{\imath B\cdot I_{n-0}}\underbrace{s_0(t-nT+0\cdot T)}_{a_{i,0}=0\ (k'=0)}\right)$$

$$\times \left(\underbrace{s_0(t-nT+1\cdot T+LT)}_{a_{i,1}=1\ (k'=1)}+e^{\imath B\cdot I_{n-1}}\underbrace{s_0(t-nT+1\cdot T)}_{a_{i,1}=0\ (k'=1)}\right)$$

$$\vdots$$

$$\times \left(\underbrace{s_0(t-nT+(L-1)\cdot T+LT)}_{a_{i,L-1}=1\ (k'=L-1)}+e^{\imath B\cdot I_{n-(L-1)}}\underbrace{s_0(t-nT+(L-1)\cdot T)}_{a_{i,L-1}=0\ (k'=L-1)}\right)$$

$$= \sum_{i=0}^{2^L-1} e^{\imath B\sum_{k'=0}^{L-1}(1-a_{i,k'})I_{n-k'}}\prod_{k'=0}^{L-1}s_0(t-nT+k'T+a_{i,k'}LT)$$

where $(a_{i,0}, a_{i,1}, \ldots, a_{i,L-1})$ is the binary representation of i with $a_{i,0}$ being the most significant bit.

Continue the derivation:

$$
\begin{aligned}
s_{\text{b-CPM},\ell}(t) \\
&= e^{\imath B \sum_{k=-\infty}^{n-L} I_k} \sum_{i=0}^{2^L-1} e^{\imath B \sum_{k'=0}^{L-1}(1-a_{i,k'})I_{n-k'}} \prod_{k'=0}^{L-1} s_0(t - nT + k'T + a_{i,k'}LT) \\
&= \sum_{i=0}^{2^L-1} \underbrace{e^{\imath \pi h A_{i,n}}}_{\substack{\text{complex} \\ \text{amplitude}}} \underbrace{c_i(t-nT)}_{\substack{\text{pulse shaping} \\ \text{function}}}
\end{aligned}
$$

where

$$
A_{i,n} = \sum_{k=-\infty}^{n} I_k - \sum_{k'=0}^{L-1} a_{i,k'} I_{n-k'} \quad \text{and} \quad c_i(t) = \prod_{k'=0}^{L-1} s_0(t+k'T+a_{i,k'}LT).
$$

Binary CPM can be expressed as a weighted sum of 2^L real-valued pulses $\{c_i(t)\}$ where the complex amplitudes depends on the information sequence. This is useful, especially when L is small!

Property of $c_i(t)$

- Duration: $c_i(t) = 0$ if any of $s_0(t + k'T + a_{i,k'}LT) = 0$. Hence, $c_i(t) \neq 0$ only possible in

$$\max_{0\le k'<L}\left(-k'T - a_{i,k'}LT\right) \le t < \min_{0\le k'<L}\left[\left(-k'T - a_{i,k'}LT\right) + 2LT\right]$$

$$\Leftrightarrow \quad -\Bigg(\underbrace{\min_{\substack{0\le k'\le L\\ \text{and } a_{i,k'}=0}} k'}_{\substack{\text{“}\le L\text{” for the case}\\ \text{of } a_{i,k'}=1\ \forall 0\le k'<L}}\Bigg)T \le t < LT - \Bigg(\underbrace{\max_{\substack{-1\le k'<L\\ \text{and } a_{i,k'}=1}} k'}_{\substack{\text{“}-1\le\text{” for the case}\\ \text{of } a_{i,k'}=0\ \forall 0\le k'<L}}\Bigg)T$$

where we define $a_{i,L} = 0$ and $a_{i,-1} = 1$. So, the duration is equal to:

$$\Bigg(L - \underbrace{\Bigg(\max_{-1\le k'<L \text{ and } a_{i,k'}=1} k'\Bigg)}_{k_{\max_1}} + \underbrace{\Bigg(\min_{0\le k'\le L \text{ and } a_{i,k'}=0} k'\Bigg)}_{k_{\min_0}}\Bigg)T.$$

$L = 3$

i	$a_{i,0}a_{i,1}a_{i,2}$	$-k_{\min_0}$	$L - k_{\max_1}$	$(L - k_{\max_1}) - (-k_{\min_0})$
0	000	0	4	4
1	001	0	1	1
2	010	0	2	2
3	011	0	1	1
4	100	−1	3	4
5	101	−1	1	2
6	110	−2	2	4
7	111	−3	1	4

It can be shown that $L - k_{\max_1} + k_{\min_0} \leq L + 1$, and the upper bound can always be achieved by $i = 0$.

Example. $h = 1/2$ and $q(t) = \begin{cases} 0 & t < 0 \\ t/(6T) & 0 \le t < 3T \\ 1/2 & \text{otherwise} \end{cases}$. Then

$$s_0(t) = \begin{cases} \sin\left(\frac{\pi}{6T} t\right) & 0 \le t < 6T \\ 0 & \text{otherwise} \end{cases}$$

$$A_{i,n} = \sum_{k=-\infty}^{n} I_k - \sum_{k'=0}^{2} a_{i,k'} I_{n-k'} \quad \text{and} \quad c_i(t) = \prod_{k'=0}^{2} s_0(t + k'T + a_{i,k'} LT).$$

$a_{i,0}a_{i,1}a_{i,2}$	duration	$c_i(t)$	$e^{\imath\pi hA_{i,n}}$
0≡000	[0,4T)	$s_0(t)s_0(t+T)s_0(t+2T)$	$e^{\imath\theta_{n+1}}$
1≡001	[0,T)	$s_0(t)s_0(t+T)s_0(t+5T)$	$e^{\imath(\theta_{n-2}+\pi hI_n+\pi hI_{n-1})}$
2≡010	[0,2T)	$s_0(t)s_0(t+4T)s_0(t+2T)$	$e^{\imath(\theta_{n-1}+\pi hI_n)}$
3≡011	[0,T)	$s_0(t)s_0(t+4T)s_0(t+5T)$	$e^{\imath(\theta_{n-2}+\pi hI_n)}$
4≡100	[-T,3T)	$s_0(t+3T)s_0(t+T)s_0(t+2T)$	$e^{\imath\theta_n}$
5≡101	[-T,T)	$s_0(t+3T)s_0(t+T)s_0(t+5T)$	$e^{\imath(\theta_{n-2}+\pi hI_{n-1})}$
6≡110	[-2T,2T)	$s_0(t+3T)s_0(t+4T)s_0(t+2T)$	$e^{\imath\theta_{n-1}}$
7≡111	[-3T,T)	$s_0(t+3T)s_0(t+4T)s_0(t+5T)$	$e^{\imath\theta_{n-2}}$

Note that

$$\begin{cases} c_4(t) = c_0(t+T) \\ e^{\imath\pi hA_{4,n}} = e^{\imath\pi hA_{0,n-1}} \end{cases} \quad \begin{cases} c_6(t) = c_0(t+2T) \\ e^{\imath\pi hA_{6,n}} = e^{\imath\pi hA_{0,n-2}} \end{cases} \quad \begin{cases} c_7(t) = c_0(t+3T) \\ e^{\imath\pi hA_{7,n}} = e^{\imath\pi hA_{0,n-3}} \end{cases}$$

and

$$\begin{cases} c_5(t) = c_2(t+T) \\ e^{\imath\pi hA_{5,n}} = e^{\imath\pi hA_{2,n-1}} \end{cases}$$

For $nT \le t < (n+1)T$,

$$s_{\text{b-CPM},\ell}(t) = e^{\imath\phi(t;\boldsymbol{I})} = \sum_{i=0}^{7} e^{\imath\pi h A_{i,n}} c_i(t-nT)$$

$$\begin{aligned}
&= e^{\imath\pi h A_{0,n}} c_0(t-nT) + e^{\imath\pi h A_{1,n}} c_1(t-nT) + e^{\imath\pi h A_{2,n}} c_2(t-nT) \\
&+ e^{\imath\pi h A_{3,n}} c_3(t-nT) + e^{\imath\pi h A_{4,n}} c_4(t-nT) + e^{\imath\pi h A_{5,n}} c_5(t-nT) \\
&+ e^{\imath\pi h A_{6,n}} c_6(t-nT) + e^{\imath\pi h A_{7,n}} c_7(t-nT)
\end{aligned}$$

$$\begin{aligned}
&= e^{\imath\pi h A_{0,n}} c_0(t-nT) + e^{\imath\pi h A_{1,n}} c_1(t-nT) + e^{\imath\pi h A_{2,n}} c_2(t-nT) \\
&+ e^{\imath\pi h A_{3,n}} c_3(t-nT) + e^{\imath\pi h A_{0,n-1}} c_0(t-(n-1)T) \\
&+ e^{\imath\pi h A_{2,n-1}} c_2(t-(n-1)T) + e^{\imath\pi h A_{0,n-2}} c_0(t-(n-2)T) \\
&+ e^{\imath\pi h A_{0,n-3}} c_0(t-(n-3)T)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty} \Big[\underbrace{e^{\imath\pi h A_{0,m}} c_0(t-mT)}_{\text{(for } m=n-3,\,n-2,\,n-1,\,n)} + e^{\imath\pi h A_{1,m}} c_1(t-mT) \\
&\qquad + e^{\imath\pi h A_{2,m}} c_2(t-mT) + e^{\imath\pi h A_{3,m}} c_3(t-mT) \Big] \\
&= \sum_{m=-\infty}^{\infty} \left[\sum_{i=0}^{2^{3-1}-1} e^{\imath\pi h A_{i,m}} c_i(t-mT) \right]
\end{aligned}$$

So, we notice that when $a_{i,0} = 1$, $c_i(t)$ is always a shift-version of some $c_j(t)$ with $0 \le j \le 2^{L-1} - 1$.

This concludes to that:

Theorem 1 (Laurent '86)

For $nT \le t < (n+1)T$,

$$s_{b\text{-}CPM,\ell}(t) = \sum_{m=-\infty}^{\infty} \left[\sum_{i=0}^{2^{L-1}-1} e^{\imath \pi h A_{i,m}} c_i(t - mT) \right]$$

where

$$A_{i,n} = \sum_{k=-\infty}^{n} I_k - \sum_{k'=1}^{L-1} a_{i,k'} I_{n-k'}$$

and

$$c_i(t) = s_0(t) \prod_{k'=1}^{L-1} s_0(t + k'T + a_{i,k'} LT)$$

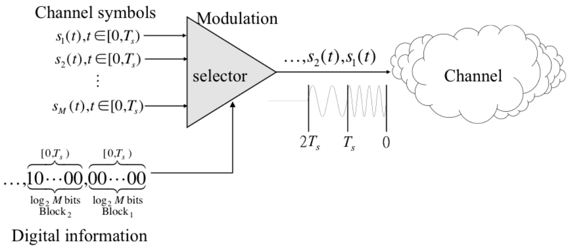
with duration $0 \le t < (L - k_{\max_1})T$.

3.4 Power spectrum of digital modulated signals

- Why studying spectral characteristics?
  - Bandwidth limitation in a real channel.

- Random process $\Longrightarrow$ Power spectral density
  - PAM
  - CPM

Power spectra of modulated signals



- The modulated waveform $s(t)$ is deterministic given the information sequence $\boldsymbol{I}$, so only the information sequence $\boldsymbol{I} = (\ldots, I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots)$ is random!
- For convenience, we denote the waveform at $nT \le t < (n+1)T$ as $s(t - nT; I_n)$ if the modulation is memoryless, and as $s(t - nT; \boldsymbol{I}_n)$ if the modulation is with memory, where $\boldsymbol{I}_n = (\ldots, I_{n-2}, I_{n-1}, I_n)$.

Hence, the modulated lowpass equivalent signal can be expressed as

$$\boldsymbol{v}_\ell(t) = \sum_{n=-\infty}^{\infty} s(t - nT; \boldsymbol{I}_n).$$

Note that $\boldsymbol{v}_\ell(t)$ is usually not a (wide-sense) stationary process but a cyclostationary process.

Its spectral characteristics is then determined by the time-averaged autocorrelation function rather than the usual authocorrelation function for a WSS proess.

2.7.2 Cyclostationary processes

- How to model a waveform source that carries digital information?

- For example,

$$\boldsymbol{X}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{a}_n \cdot g(t - nT)$$

where $\{\boldsymbol{a}_n\}_{n=-\infty}^{\infty}$ is a discrete-time random sequence, and $g(t)$ is a deterministic pulse shaping function.

Cyclostationary processes

Given that $\{a_n\}_{n=-\infty}^{\infty}$ is WSS, what is the statistical property of $X(t)$?

- $X(t)$ is not necessarily (strictly) stationary. Its mean becomes periodic with period T:

$$\mathbb{E}[X(t)] = \mathbb{E}\left[\sum_{n=-\infty}^{\infty} a_n g(t-nT)\right] = \mu_a \sum_{n=-\infty}^{\infty} g(t-nT) = E[X(t+KT)]$$

- Autocorrelation function becomes periodic with period T

$$\begin{aligned}
R_X(t_1,t_2) &= \mathbb{E}[X(t_1)X^*(t_2)] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathbb{E}[a_n a_m^*] g(t_1-nT)g(t_2-mT) \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_a(n-m) g(t_1-nT)g(t_2-mT) \\
&= R_X(t_1+KT, t_2+KT)
\end{aligned}$$

Definition 1 (Cyclostationary process)

A random process is said to be cyclostationary or periodically stationary in the wide sense if its mean and autocorrelation function are both periodic.

- Time-average autocorrelation function

$$\overline{R}_{\boldsymbol{X}}(\tau) = \frac{1}{T}\int_0^T R_{\boldsymbol{X}}(t+\tau, t)\, dt$$

- Average power spectral density

$$\overline{S}_{\boldsymbol{X}}(f) = \mathcal{F}\left\{\overline{R}_{\boldsymbol{X}}(\tau)\right\}$$

3.4-1 Power spectral density of a digitally modulated signal with memory

$$\mathbb{E}[\boldsymbol{v}_\ell(t)] = \sum_{n=-\infty}^{\infty} \mathbb{E}[I_n] g(t - nT) = \mu_{\boldsymbol{I}} \sum_{n=-\infty}^{\infty} g(t - nT) = \mathbb{E}[\boldsymbol{v}_\ell(t + T)]$$

and

$$\begin{aligned} R_{\boldsymbol{v}_\ell}(t_1, t_2) &= \mathbb{E}[\boldsymbol{v}_\ell(t_1)\boldsymbol{v}_\ell^*(t_2)] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbb{E}[I_n I_m^*] g(t_1 - nT) g^*(t_2 - mT) = R_{\boldsymbol{v}_\ell}(t_1 + T, t_2 + T) \end{aligned}$$

implies that $\boldsymbol{v}_\ell(t)$ is cyclostationary.

$$\begin{aligned} &\overline{R}_{\boldsymbol{v}_\ell}(\tau) \\ &= \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{\boldsymbol{I}}(n - m) g(t + \tau - nT) g^*(t - mT) dt \\ &= \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{\boldsymbol{I}}(k) g(t + \tau - kT - mT) g^*(t - mT) dt \\ &\quad (k = n - m) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_{\mathbf{I}}(k) \sum_{m=-\infty}^{\infty} \int_0^T g(t+\tau-kT-mT)g^*(t-mT)dt \\
&\stackrel{u=t-mT}{=} \frac{1}{T}\sum_{k=-\infty}^{\infty} R_{\mathbf{I}}(k) \sum_{m=-\infty}^{\infty} \int_{-mT}^{-(m-1)T} g(u+\tau-kT)g^*(u)du \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_{\mathbf{I}}(k) \int_{-\infty}^{\infty} g(u+\tau-kT)g^*(u)du \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} g_k(\tau-kT)
\end{aligned}
$$

where

$$
g_m(\tau) = R_{\mathbf{I}}(m)\int_{-\infty}^{\infty} g(u+\tau)g^*(u)du.
$$

$$
\begin{aligned}
G_m(f) &= \int_{-\infty}^{\infty} g_m(\tau) e^{-\imath 2\pi f \tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(R_{\boldsymbol{I}}(m) \int_{-\infty}^{\infty} g(u+\tau) g^*(u) du \right) e^{-\imath 2\pi f \tau} d\tau \\
&= R_{\boldsymbol{I}}(m) \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u+\tau) e^{-\imath 2\pi f \tau} d\tau \right) g^*(u) du \\
&\overset{v=u+\tau}{=} R_{\boldsymbol{I}}(m) \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(v) e^{-\imath 2\pi f (v-u)} dv \right) g^*(u) du \\
&= R_{\boldsymbol{I}}(m) \left(\int_{-\infty}^{\infty} g(v) e^{-\imath 2\pi f v} dv \right) \left(\int_{-\infty}^{\infty} g^*(u) e^{\imath 2\pi f u} du \right) \\
&= R_{\boldsymbol{I}}(m) |G(f)|^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \overline{S}_{\boldsymbol{v}_\ell}(f) &= \mathcal{F}\left\{ \overline{R}_{\boldsymbol{v}_\ell}(\tau) \right\} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathcal{F}\{ g_k(\tau - kT) \} \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} R_{\boldsymbol{I}}(k) |G(f)|^2 e^{-\imath 2\pi k f T} \\
&= \frac{1}{T} S_{\boldsymbol{I}}(f) |G(f)|^2 \quad \text{where } S_{\boldsymbol{I}}(f) = \sum_{k=-\infty}^{\infty} R_{\boldsymbol{I}}(k) e^{-\imath 2\pi k f T}.
\end{aligned}
$$

Theorem 2

$$\overline{S}_{\boldsymbol{v}_\ell}(f) = \frac{1}{T} S_{\boldsymbol{I}}(f) \left|G(f)\right|^2$$

The average power spectrum density of PAM signals is determined by the pulse shape, as well as the input information.

Example

Input information is real and mutually uncorrelated

$$R_{\boldsymbol{I}}(k) = \begin{cases} \sigma_{\boldsymbol{I}}^2 + \mu_{\boldsymbol{I}}^2, & k = 0 \\ \mu_{\boldsymbol{I}}^2, & k \neq 0 \end{cases}$$

Hence

$$S_{\boldsymbol{I}}(f) = \sigma_{\boldsymbol{I}}^2 + \mu_{\boldsymbol{I}}^2 \sum_{k=-\infty}^{\infty} e^{-\imath 2\pi f k T} = \sigma_{\boldsymbol{I}}^2 + \frac{\mu_{\boldsymbol{I}}^2}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

and

$$\overline{S}_{\boldsymbol{v}_\ell}(f) = \frac{\sigma_{\boldsymbol{I}}^2}{T} |G(f)|^2 + \frac{\mu_{\boldsymbol{I}}^2}{T^2} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) |G(f)|^2$$

$$\overline{S}_{\boldsymbol{v}_\ell}(f) = \underbrace{\frac{\sigma_{\boldsymbol{I}}^2}{T}|G(f)|^2}_{continuous} + \underbrace{\frac{\mu_{\boldsymbol{I}}^2}{T^2}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)|G(f)|^2}_{discrete}$$

- Observation 1: Discrete spectrum vanishes when the input information has zero mean, which is often desirable for digital modulation techniques.

- Observation 2: With a zero-mean input information, the average power spectrum density is determined by $G(f)$.

Example 6

The average power spectrum density for rectangular pulses

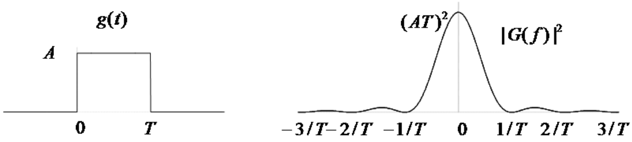
$$g(t) = A[u_{-1}(t) - u_{-1}(t-T)]$$

It shows

$$G(f) = AT\text{sinc}(fT)e^{-\imath\pi fT} \Rightarrow |G(f)|^2 = A^2T^2\text{sinc}^2(fT).$$

Hence

$$\begin{aligned}\overline{S}_{\mathbf{v}_\ell}(f) &= \frac{\sigma_{\mathbf{I}}^2}{T}|G(f)|^2 + \frac{\mu_{\mathbf{I}}^2}{T^2}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)|G(f)|^2 \\ &= \sigma_{\mathbf{I}}^2A^2T\text{sinc}^2(fT) + \mu_{\mathbf{I}}^2A^2\delta(f).\end{aligned}$$



Example 7

The average power spectrum density for raised cosine pulse

$$g(t) = \frac{A}{2}\left[1 + \cos\left(\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right)\right](u_{-1}(t) - u_{-1}(t - T)).$$

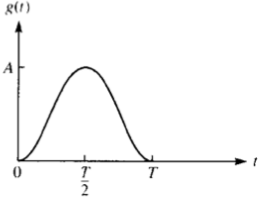
It gives

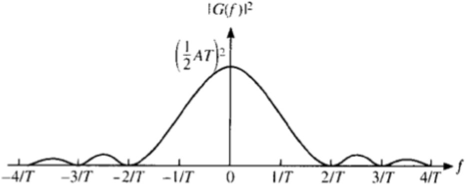
$$G(f) = \frac{AT}{2}\text{sinc}(fT)\frac{1}{1 - f^2T^2}e^{-\imath\pi fT}.$$

Hence

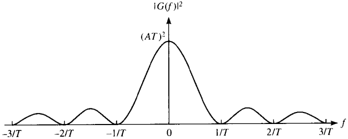
$$\begin{aligned}\overline{S}_{\boldsymbol{v}_\ell}(f) &= \frac{\sigma_{\boldsymbol{I}}^2}{T}|G(f)|^2 + \frac{\mu_{\boldsymbol{I}}^2}{T^2}\sum_{k=-\infty}^{\infty}\delta\left(f - \frac{k}{T}\right)|G(f)|^2 \\ &= \frac{\sigma_{\boldsymbol{I}}^2 A^2 T\text{sinc}^2(fT)}{4(1 - f^2T^2)^2} + \frac{\mu_{\boldsymbol{I}}^2 A^2}{4}\delta(f) + \frac{\mu_{\boldsymbol{I}}^2 A^2}{16}\delta\left(f - \frac{1}{T}\right) + \frac{\mu_{\boldsymbol{I}}^2 A^2}{16}\delta\left(f + \frac{1}{T}\right).\end{aligned}$$

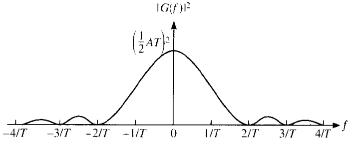
Note: $\lim_{x\to\pm 1}\frac{\text{sinc}^2(x)}{(1-x^2)^2} = \frac{1}{4}$


g(t)
A
0
T/2
T
t


|G(f)|²
(½AT)²
−4/T
−3/T
−2/T
−1/T
0
1/T
2/T
3/T
4/T
f

Comparison of the previous two examples

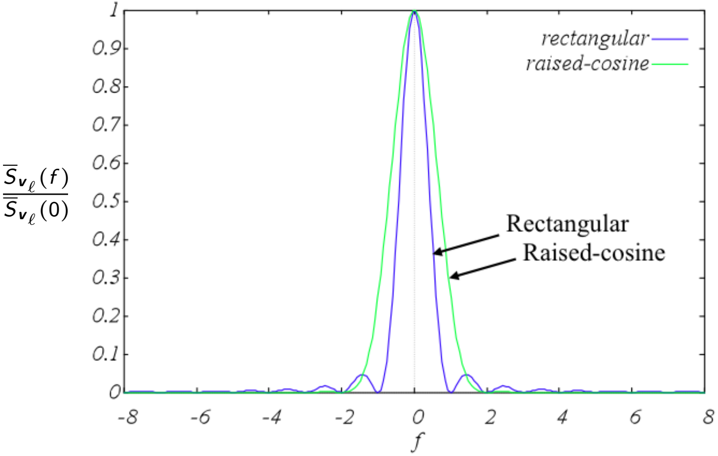




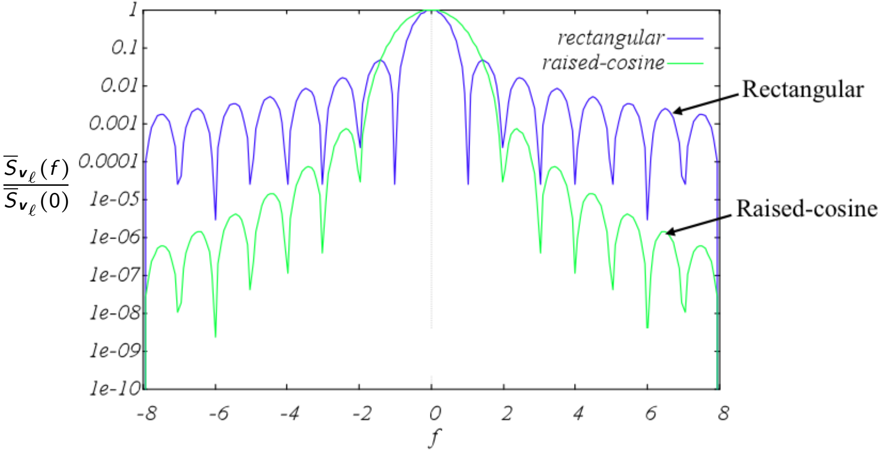
Broader side lobe

Faster decay in tail ($f^{-6} < f^{-2}$)

Assume $A = T = \sigma_I^2 = 1$ and $\mu_I = 0$



Assume $A = T = \sigma_I^2 = 1$ and $\mu_I = 0$



Assume $A = T = \sigma_I^2 = 1$ and $\mu_I = 0$

- The smoother (meaning, continuity of derivatives) the pulse shape, the greater the bandwidth efficiency (lower bandwidth occupancy).

- The raised cosine pulse shape will result in higher bandwidth efficiency than the rectangular pulse shape.

What if $\boldsymbol{I}$ correlated?

Example 8

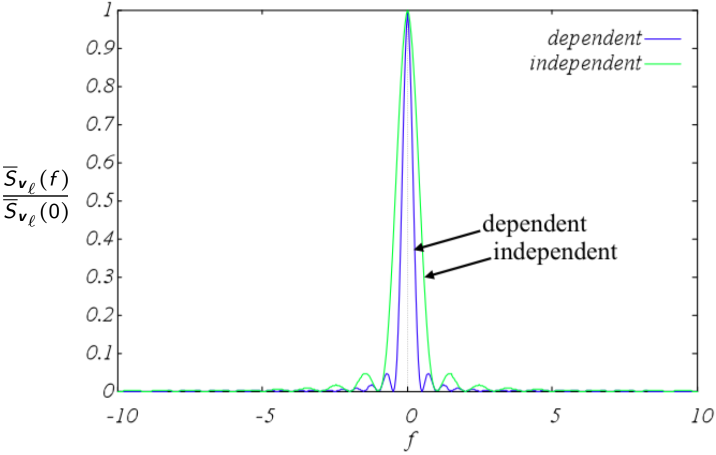
$$I_n = b_n + b_{n-1}$$

where $\{b_n\}$ mutually uncorrelated with zero mean and unit variance.

Then,

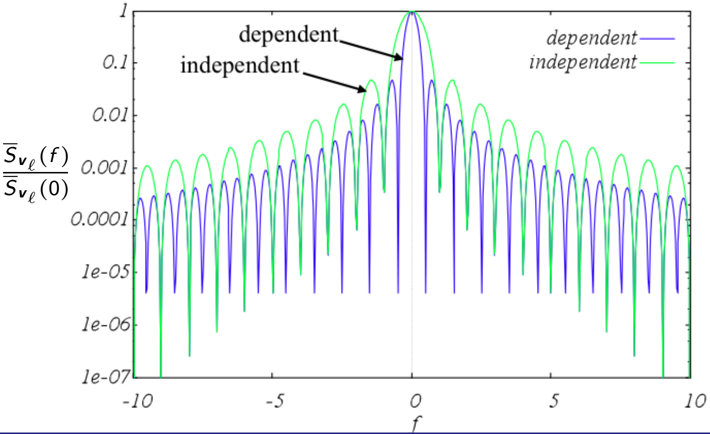
$$
\begin{aligned}
R_{\boldsymbol{I}}(k) &= \begin{cases} 2 & k = 0 \\ 1 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases} \\
S_{\boldsymbol{I}}(f) &= 2 + e^{\imath 2\pi f T} + e^{-\imath 2\pi f T} = 2\left(1 + \cos(2\pi f T)\right) = 4\cos^2(\pi f T) \\
\overline{S}_{\boldsymbol{v}_\ell}(f) &= \frac{1}{T}\left|G(f)\right|^2 S_{\boldsymbol{I}}(f) = \frac{4}{T}\left|G(f)\right|^2 \cos^2(\pi f T)
\end{aligned}
$$

Rectangular pulse shape with $A = T = 1$



Rectangular pulse shape with $A = T = 1$

Dependence in transmitted information (not the original information) can improve the bandwidth efficiency.



CPM: Assume $\boldsymbol{I}$ i.i.d.

$$\boldsymbol{v}_\ell(t) \;=\; e^{\imath\phi(t;\boldsymbol{I})}$$

where

$$\phi(t;\boldsymbol{I}) = 2\pi h \sum_{k=-\infty}^{\infty} I_k q(t-kT)$$

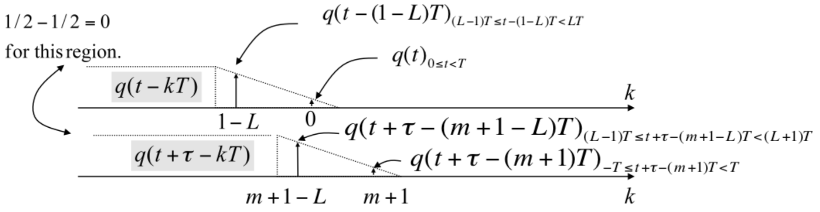
$$\begin{aligned}
&R_{\boldsymbol{v}_\ell}(t_1,t_2)\\
&\quad= \mathbb{E}\left[\boldsymbol{v}_\ell(t_1)\boldsymbol{v}_\ell^*(t_2)\right]\\
&\quad= \mathbb{E}\left[e^{\imath\phi(t_1,\boldsymbol{I})}e^{-\imath\phi(t_2,\boldsymbol{I})}\right]\\
&\quad= \mathbb{E}\left[\exp\left(\imath 2\pi h \sum_{k=-\infty}^{\infty} I_k\left[q(t_1-kT)-q(t_2-kT)\right]\right)\right]
\end{aligned}$$

$$
\begin{aligned}
R_{\mathbf{v}_\ell}(t_1, t_2) \\
&= \mathbb{E}\left[\prod_{k=-\infty}^{\infty} \exp\left(\imath 2\pi h I_k \left[q(t_1 - kT) - q(t_2 - kT)\right]\right)\right] \\
&= \prod_{k=-\infty}^{\infty} \mathbb{E}\left[\exp\left(\imath 2\pi h I_k \left[q(t_1 - kT) - q(t_2 - kT)\right]\right)\right] \\
&= \prod_{k=-\infty}^{\infty} \left[\sum_{n\in\mathcal{S}} P_n \exp\left(\imath 2\pi h n \left[q(t_1 - kT) - q(t_2 - kT)\right]\right)\right],
\end{aligned}
$$

where $I_k = n \in \mathcal{S}$ and $P_n \triangleq \Pr[I_k = n]$.

$$
\begin{aligned}
\bar{R}_{\mathbf{v}_\ell}(\tau) &= \frac{1}{T}\int_0^T R_{\mathbf{v}_\ell}(t+\tau, t)\, dt \\
&= \frac{1}{T}\int_0^T \prod_{k=-\infty}^{\infty} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n [q(t+\tau-kT) - q(t-kT)]}\right] dt.
\end{aligned}
$$

Assume $\tau \geq 0$. For $mT \leq \tau = \xi + mT < (m+1)T$ and $0 \leq t < T$ (i.e., the range of integration)



$t+\tau-(m+1)T = t+\xi-T$ and $t+\tau-(m+1-L)T = t+\xi-(1-L)T$.

$$\overline{R}_{\mathbf{v}_\ell}(\tau)$$

$$= \frac{1}{T}\int_0^T \prod_{k=\min\{m+1-L,1-L\}}^{\max\{m+1,0\}} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi hn[q(t+\tau-kT)-q(t-kT)]}\right] dt$$

$$\stackrel{m\geq 0}{=} \frac{1}{T}\int_0^T \prod_{k=1-L}^{m+1} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi hn[q(t+\tau-kT)-q(t-kT)]}\right] dt.$$

Hermitian symmetry of $\overline{R}_{\mathbf{v}_\ell}(\tau)$

It suffices to derive $\overline{R}_{\mathbf{v}_\ell}(\tau)$ for $\tau \geq 0$ because $\overline{R}_{\mathbf{v}_\ell}(-\tau) = \overline{R}^*_{\mathbf{v}_\ell}(\tau)$.

Proof:

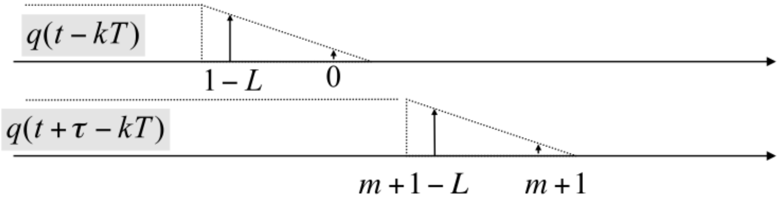
$$\begin{aligned}
\overline{R}^*_{\mathbf{v}_\ell}(\tau) &= \frac{1}{T}\int_0^T \mathbb{E}\left[e^{-\imath 2\pi h \sum_{k=-\infty}^{\infty} I_k[q(t+\tau-kT)-q(t-kT)]}\right] dt \\
&= \frac{1}{T}\int_0^T \mathbb{E}\left[e^{\imath 2\pi h \sum_{k=-\infty}^{\infty} I_k[q(t-kT)-q(t+\tau-kT)]}\right] dt \\
&= \frac{1}{T}\int_\tau^{T+\tau} \mathbb{E}\left[e^{\imath 2\pi h \sum_{k=-\infty}^{\infty} I_k[q(v-\tau-kT)-q(v-kT)]}\right] dv \\
&\quad (v = t+\tau) \\
&= \frac{1}{T}\int_0^T \mathbb{E}\left[e^{\imath 2\pi h \sum_{k=-\infty}^{\infty} I_k[q(v-\tau-kT)-q(v-kT)]}\right] dv \\
&= \overline{R}_{\mathbf{v}_\ell}(-\tau).
\end{aligned}$$

Average PSD of CPM

$$
\begin{aligned}
\overline{S}_{\mathbf{v}_\ell}(f) &= \int_{-\infty}^{\infty} \overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \int_{-\infty}^{0} \overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau + \int_{0}^{\infty} \overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \int_{0}^{\infty} \overline{R}_{\mathbf{v}_\ell}(-\tau) e^{\imath 2\pi f \tau}\, d\tau + \int_{0}^{\infty} \overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \int_{0}^{\infty} \left[\overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\right]^{*} d\tau + \int_{0}^{\infty} \overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= 2\mathbf{Re}\left[\int_{0}^{\infty} \overline{R}_{\mathbf{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau\right].
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty \overline{R}_{\mathbf{v}_\ell}(\tau)e^{-\imath 2\pi f\tau}\,d\tau \\
&= \int_0^{LT} \overline{R}_{\mathbf{v}_\ell}(\tau)e^{-\imath 2\pi f\tau}\,d\tau + \int_{LT}^{\infty} \overline{R}_{\mathbf{v}_\ell}(\tau)e^{-\imath 2\pi f\tau}\,d\tau \\
&= \int_0^{LT} \overline{R}_{\mathbf{v}_\ell}(\tau)e^{-\imath 2\pi f\tau}\,d\tau + \sum_{m=L}^{\infty}\int_{mT}^{(m+1)T} \overline{R}_{\mathbf{v}_\ell}(\tau)e^{-\imath 2\pi f\tau}\,d\tau.
\end{aligned}
$$

For $m \geq L$, the two "regions" below are non-overlapping!



$$
\begin{aligned}
\overline{R}_{\boldsymbol{v}_\ell}(\tau) &\overset{m\geq L}{=} \frac{1}{T}\int_0^T \prod_{k=1-L}^{m+1}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[q(t+\tau-kT)-q(t-kT)]}\right] dt \\
&= \frac{1}{T}\int_0^T \left(\prod_{k=1-L}^{0}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[q(t+\tau-kT)-q(t-kT)]}\right]\right. \\
&\qquad \prod_{k=1}^{m-L}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[q(t+\tau-kT)-q(t-kT)]}\right] \\
&\qquad \left.\prod_{k=m+1-L}^{m+1}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[q(t+\tau-kT)-q(t-kT)]}\right]\right) dt \\
&= \frac{1}{T}\int_0^T \left(\prod_{k=1-L}^{0}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[1/2-q(t-kT)]}\right]\right. \\
&\qquad \prod_{k=1}^{m-L}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[1/2-0]}\right] \\
&\qquad \left.\prod_{k=m+1-L}^{m+1}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[q(t+\tau-kT)-0]}\right]\right) dt
\end{aligned}
$$

$$
\begin{aligned}
\overline{R}_{\boldsymbol{v}_\ell}(\tau) &\overset{m\geq L}{=} \frac{1}{T}\int_0^T \left(\prod_{k=1-L}^{0} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi hn[1/2-q(t-kT)]} \right] [\Phi_{\boldsymbol{I}}(h)]^{m-L} \right. \\
&\qquad \left. \prod_{k'=1-L}^{1} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi hn[q(t+\tau-k'T-mT)]} \right] \right) dt \quad (k' = k - m) \\
&= [\Phi_{\boldsymbol{I}}(h)]^{m-L} \lambda(\tau - mT),
\end{aligned}
$$

where $\Phi_{\boldsymbol{I}}(h) = \sum_{n\in\mathcal{S}} P_n e^{\imath\pi hn}$ and

$$
\begin{aligned}
\lambda(\xi) &= \frac{1}{T}\int_0^T \left(\prod_{k=1-L}^{0} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi hn[1/2-q(t-kT)]} \right] \right. \\
&\qquad \left. \prod_{k'=1-L}^{1} \left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi hn[q(t+\xi-k'T)]} \right] \right) dt.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{m=L}^{\infty} \int_{mT}^{(m+1)T} \overline{R}_{\boldsymbol{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \sum_{m=L}^{\infty} \int_{mT}^{(m+1)T} [\Phi_{\boldsymbol{I}}(h)]^{m-L} \lambda(\tau - mT) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \sum_{m=L}^{\infty} \int_{0}^{T} [\Phi_{\boldsymbol{I}}(h)]^{m-L} \lambda(\xi) e^{-\imath 2\pi f (\xi + mT)}\, d\xi \quad (\xi = \tau - mT) \\
&= \left(\sum_{m=L}^{\infty} [\Phi_{\boldsymbol{I}}(h)]^{m-L} e^{-\imath 2\pi f m T} \right) \left(\int_0^T \lambda(\xi) e^{-\imath 2\pi f \xi} d\xi \right) \\
&= \begin{cases} \left(\frac{e^{-\imath 2\pi f L T}}{1 - \Phi_{\boldsymbol{I}}(h) e^{-\imath 2\pi f T}} \right) \left(\int_0^T \lambda(\xi) e^{-\imath 2\pi f \xi} d\xi \right) & \text{if } |\Phi_{\boldsymbol{I}}(h)| < 1 \\ \left(e^{-\imath 2\pi f L T} \sum_{m'=0}^{\infty} e^{-\imath 2\pi T (f - \nu/T) m'} \right) \left(\int_0^T \lambda(\xi) e^{-\imath 2\pi f \xi} d\xi \right) & \\ \qquad \text{if } |\Phi_{\boldsymbol{I}}(h)| = |e^{\imath 2\pi \nu}| = 1 & \end{cases} \\
&= \begin{cases} \left(\frac{e^{-\imath 2\pi f L T}}{1 - \Phi_{\boldsymbol{I}}(h) e^{-\imath 2\pi f T}} \right) \left(\int_0^T \lambda(\xi) e^{-\imath 2\pi f \xi} d\xi \right) & \text{if } |\Phi_{\boldsymbol{I}}(h)| < 1 \\ e^{-\imath 2\pi f L T} \left(\frac{1}{2} + \frac{1}{2T} \sum_{m'=-\infty}^{\infty} \left(\delta\left(f - \frac{\nu + m'}{T} \right) - \imath \frac{1}{\pi (f - (\nu + m')/T)} \right) \right) & \\ \left(\int_0^T \lambda(\xi) e^{-\imath 2\pi f \xi} d\xi \right) \qquad \text{if } |\Phi_{\boldsymbol{I}}(h)| = |e^{\imath 2\pi \nu}| = 1 & \end{cases}
\end{aligned}
$$

$$g(t) \leftrightarrow G(f) \Rightarrow \begin{cases} g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \\ G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)e^{-\imath 2\pi n T_s f} = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G\left(f - \frac{n}{T_s}\right) \end{cases}$$

Slide 2-9: $u_{-1}(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \leftrightarrow U_{-1}(f) = \frac{1}{2}\left(\delta(f) - \imath \frac{1}{\pi f}\right)$

$$\Rightarrow U_{-1,\delta}(f) = \sum_{n=-\infty}^{\infty} u_{-1}(nT_s)e^{-\imath 2\pi n T_s f} = -\frac{1}{2} + \sum_{n=0}^{\infty} e^{-\imath 2\pi n T_s f}$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} U_{-1}\left(f - \frac{n}{T_s}\right) = \frac{1}{2T_s}\sum_{n=-\infty}^{\infty}\left(\delta\left(f - \frac{n}{T_s}\right) - \imath \frac{1}{\pi\left(f - \frac{n}{T_s}\right)}\right)$$

$$\sum_{m'=0}^{\infty} e^{-\imath 2\pi T(f-\nu/T)m'} = \frac{1}{2} + \frac{1}{2T}\sum_{m'=-\infty}^{\infty}\left(\delta\left(f - \frac{\nu + m'}{T}\right) - \imath \frac{1}{\pi\left(f - \frac{\nu+m'}{T}\right)}\right)$$

We finally obtain a numerically computable/plotable formula for the average PSD of CPM. For example, if $|\Phi_{\boldsymbol{I}}(h)| < 1$,

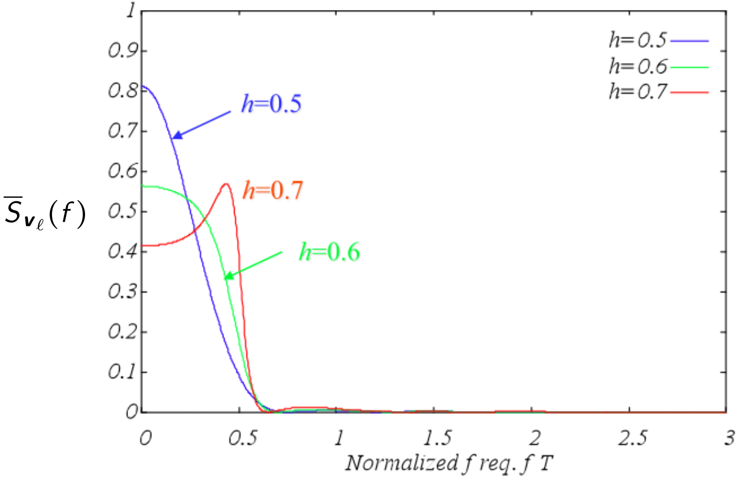
$$\begin{aligned}\overline{S}_{\boldsymbol{v}_\ell}(f) &= 2\,\mathbf{Re}\Bigg[\int_0^{LT} \overline{R}_{\boldsymbol{v}_\ell}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\ &\quad + \left(\frac{1}{1-\Phi_{\boldsymbol{I}}(h) e^{-\imath 2\pi f T}}\right)\left(\int_0^T \lambda(\xi) e^{-\imath 2\pi f(\xi+LT)} d\xi\right)\Bigg]\end{aligned}$$

where for $0 \le \tau = \xi + mT < LT$,

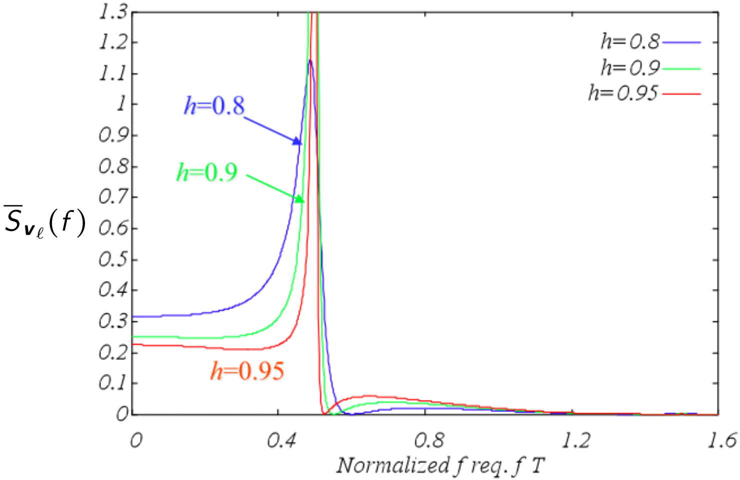
$$\overline{R}_{\boldsymbol{v}_\ell}(\tau) \overset{m\ge 0}{=} \frac{1}{T}\int_0^T \prod_{k=1-L}^{m+1}\left[\sum_{n\in\mathcal{S}} P_n e^{\imath 2\pi h n[q(t+\tau-kT)-q(t-kT)]}\right] dt.$$

However, if $|\Phi_{\boldsymbol{I}}(h)| = |e^{\imath 2\pi\nu}| = 1$, where $0 \le \nu < 1$, the average PSD of CPM signals has impulses at $f_{m'} = \frac{\nu+m'}{T}$ for integer m'.

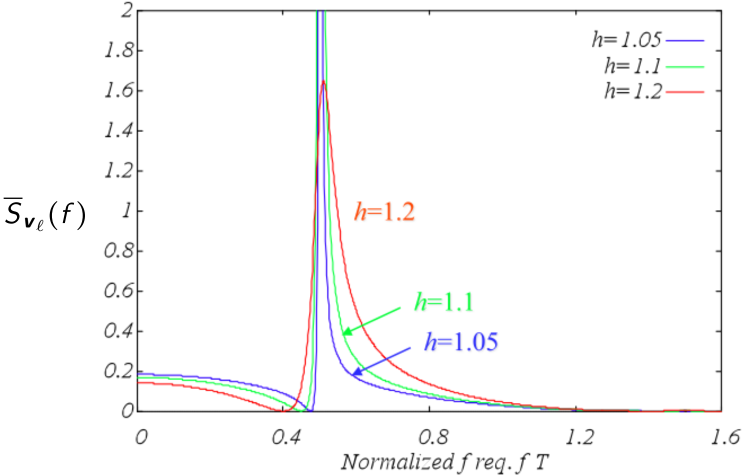
Numerically plotted average PSD of the equivalent lowpass CPFSK signal ($M = 2$, $T = 0.5$, P_n uniform over $\mathcal{S} = \{\pm 1\}$ and $\Phi_{\boldsymbol{I}}(h) = \frac{1}{2}(e^{\imath\pi h} + e^{-\imath\pi h}) = \cos(\pi h)$)



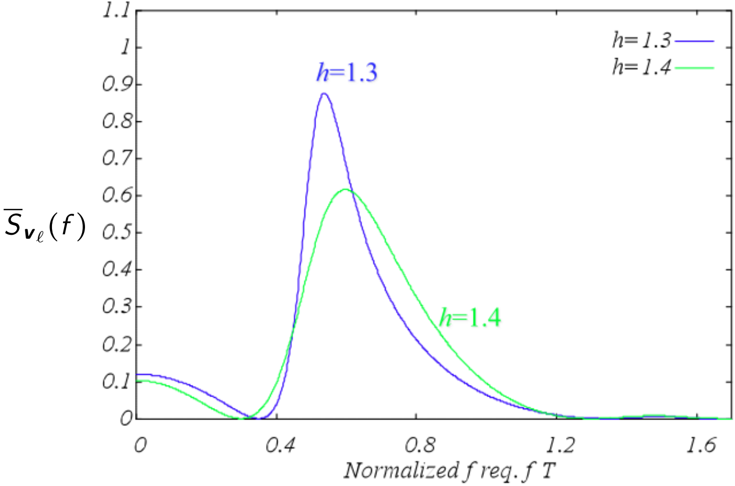
Numerically plotted average PSD of the equivalent lowpass CPFSK signal ($M = 2$, $T = 0.5$ and P_n uniform over $\mathcal{S} = \{\pm 1\}$)



Numerically plotted average PSD of the equivalent lowpass CPFSK signal ($M = 2$, $T = 0.5$ and P_n uniform over $\mathcal{S} = \{\pm 1\}$)



Numerically plotted average PSD of the equivalent lowpass CPFSK signal ($M = 2$, $T = 0.5$ and P_n uniform over $\mathcal{S} = \{\pm 1\}$)



Observation 1

For $h < 1$

- Its average PSD is relatively smooth and well confined.
- Almost all power is confined within

$$fT < 0.6 \text{ or } f < \frac{0.6}{T}$$

where T is the width of the channel symbols.

Observation 2

For $h > 1$

- Its average PSD becomes broader and hence the bandwidth is approximately

$$fT < 1.2 \text{ or } f < \frac{1.2}{T}$$

- This is the main reason why in communication systems, where CPFSK is used, the modulation index h is usually taken to be < 1.

Example: Bluetooth RF specification (Version 1.0)

- GFSK (Gaussian FSK) with $BT = 0.5$
  - B = Bandwidth (for baseband symbol) $= 0.5$ MHz, $T = 1\mu$ sec
  - $1 =$ positive frequency deviation
  - $0 =$ negative frequency deviation

- Modulation index $0.28 \sim 0.35$
  - Modulation index $= 2f_d T$, where f_d is the peak frequency deviation.
  - $0.28 < h = 2f_d T < 0.35 \implies 140KHz < f_d < 175KHz$

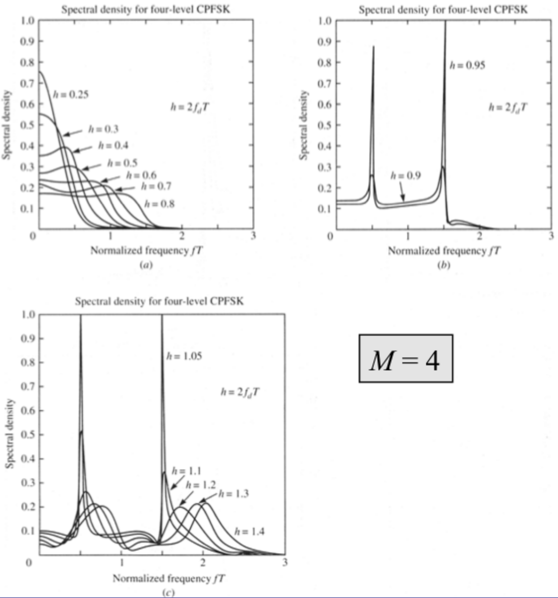
Observation3

By letting $h \to 1$

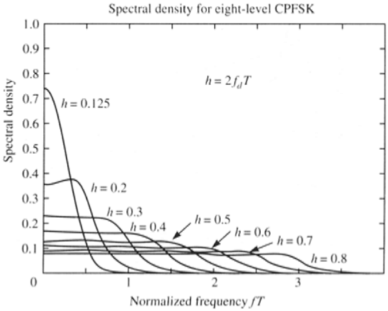
- we can observe M impulses in the average PSD of the equivalent lowpass CPFSK signal.

Numerically plotted average PSD of the equivalent lowpass CPFSK signal ($M = 4$, P_n uniform over $\mathcal{S} = \{\pm 1, \pm 3\}$ and $\Phi_I(h) = \frac{1}{2}(\cos(\pi h) + \cos(3\pi h))$)

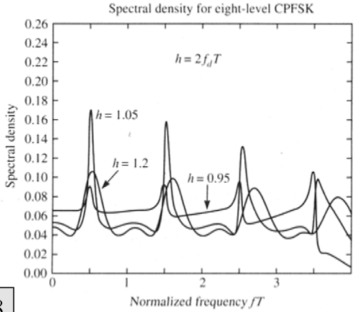
- Approximately 4 impulses appear when $h \approx 1$.
- The bandwidth becomes broader than almost twice of that of $M = 2$.



- Approximately 8 impulses are observed when $h \approx 1$.
- Bandwidth becomes broader than almost four times of that of $M = 2$.

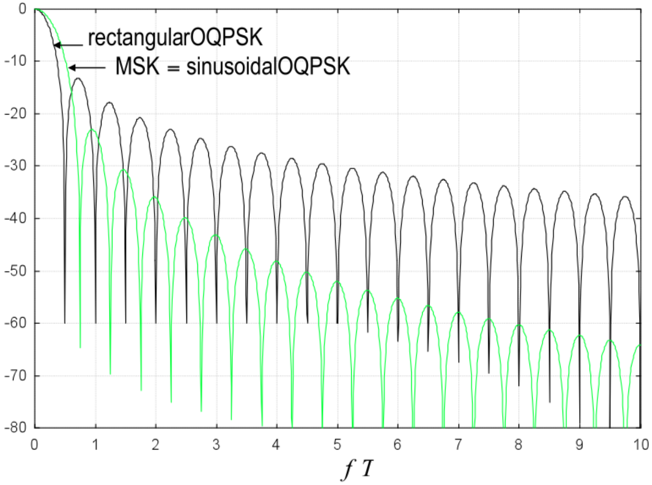


M = 8



Re-visit MSK versus OQPSK

$\frac{\overline{S}_{\mathbf{v}_\ell}(f)}{\overline{S}_{\mathbf{v}_\ell}(0)}$ (dB)



Observations

- Main Lobe: MSK is 50% wider than rectangular OQPSK, i.e., MSK $= 1.5\times$ rectangular OQPSK.

- Side Lobe:
  - Compare the bandwidth that contains 99% of the total power: MSK $= 1.2/T$ and rectangular OQPSK $= 8.0/T$.
  - MSK decreases much faster than OQPSK.
  - MSK is significantly more bandwidth efficient than rectangular OQPSK.
  - By further decreasing the modulation index h (i.e., making $h < 1/2$), the bandwidth efficiency of MSKs can be increased. However, in such case, MSK signals are no longer orthogonal. $f_d = 1/(4T) \Leftrightarrow h = 2f_d T = 1/2$

Appendix: Fractional out-of-band power

- Fractional in-band power

$$\Delta P_{\text{In-band}}(W) = \frac{1}{P_{\text{Total}}} \int_{-W}^{W} \overline{S}_{\boldsymbol{v}_\ell}(f)\, df,$$

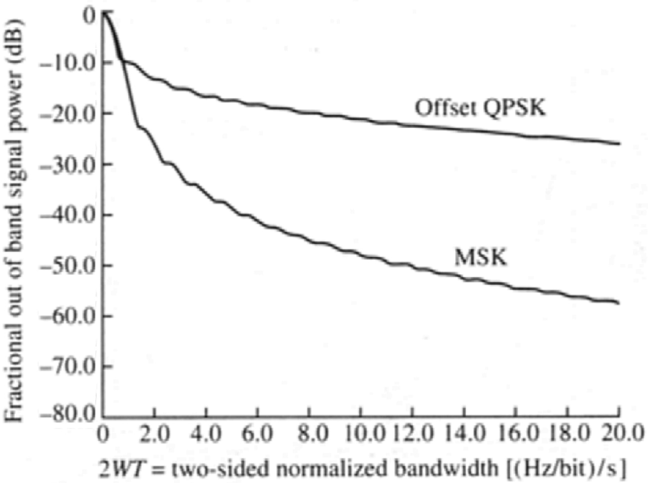
  where

$$P_{\text{Total}} = \int_{-\infty}^{\infty} \overline{S}_{\boldsymbol{v}_\ell}(f)\, df.$$

- Fractional out-of-band power

$$\Delta P_{\text{Out-of-band}}(W) = 1 - \Delta P_{\text{In-band}}(W)$$

  - This quantity is often used to measure the bandwidth efficiency of a modulation scheme. For example, finding the bandwidth W under some acceptable condition, say fractional-out-of-band power is no greater than 0.01.


Fractional out of band signal power (dB)
0
-10.0
-20.0
-30.0
-40.0
-50.0
-60.0
-70.0
-80.0
Offset QPSK
MSK
0
2.0
4.0
6.0
8.0
10.0
12.0
14.0
16.0
18.0
20.0
2WT = two-sided normalized bandwidth [(Hz/bit)/s]

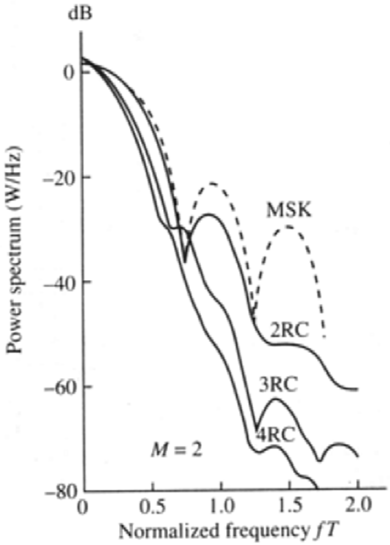
Summary of spectral characteristics of CPFSK signals

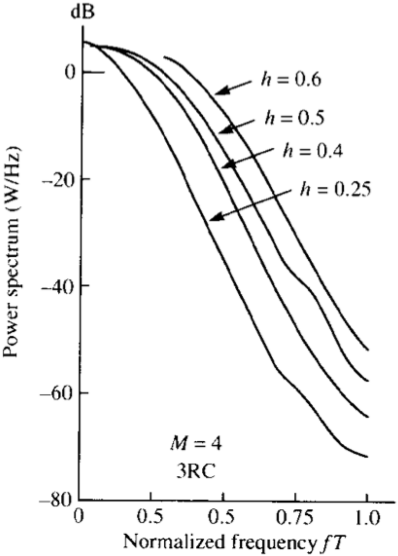
Modulation Index h

- In general, the lower the modulation index h, the higher the bandwidth efficiency.

Pulse shape $g(t)$

- The smoother (meaning, e.g., continuity of the derivatives) the $g(t)$, the greater the bandwidth efficiency.
- For example, the raised cosine $g(t)$ will result in higher bandwidth efficiency than the rectangular $g(t)$.
- For example, LRC (raised cosine $g(t)$ with duration LT) with larger L (i.e., smoother) will result in greater bandwidth efficiency.


dB
0
-20
-40
-60
-80
Power spectrum (W/Hz)
MSK
2RC
3RC
4RC
M = 2
0
0.5
1.0
1.5
2.0
Normalized frequency fT


dB
0
-20
-40
-60
-80
Power spectrum (W/Hz)
h = 0.6
h = 0.5
h = 0.4
h = 0.25
M = 4
3RC
0
0.5
0.5
0.75
1.0
Normalized frequency fT

What you learn from Chapter 3

- (Pseudo-)Vectorization of standard ASK, PSK and QAM signals
    - Computation of average energy based on signal space vector points
    - Euclidean distance based on signal space vector points
    - Gray code mapping from binary pattern to the signal space vector points (in terms of their Euclidean distances)

- (Good to know) QPSK versus $\pi/4$-QPSK

- Vectorization of standard orthogonal (FSK or multi-dimensional) and bi-orthogonal signals
  - Computation of average energy based on signal space vector points
  - Euclidean distance based on signal space vector points
  - (Important) Cross-correlation of FSK bandpass and lowpass signals (Minimum shift keying)

- (Good to know) Simplex signals (from orthogonal signals)

- (Important) Why cyclo-stationarity for digitally modulated signals and its power spectrum

- Modulation with memory – CPM signals
  - Its basic formation

$$\phi(t; \boldsymbol{I}) = 4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau$$

    based on phase change $d(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$
  - (Good to know) Full response and partial response
  - MSK versus OQPSK
  - Linear representation of CPM
  - (Important) Time-average autocorrelation and power spectrum (of cyclostationary PAM and MSK)