Digital Communications

Chapter 2: Deterministic and Random Signal Analysis

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2.1 Bandpass and lowpass signal representation

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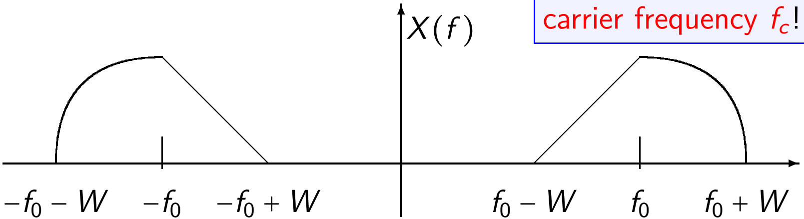
Definition (Bandpass signal)

A bandpass signal $x(t)$ is a real signal whose frequency content is located around central frequency f_0, i.e.

$$X(f) = 0 \quad \text{for all } |f \pm f_0| > W$$

f_0 may not be the carrier frequency f_c!



- The spectrum of a bandpass signal is *Hermitian symmetric*, i.e., $X(-f) = X^*(f)$. (Why? Hint: Fourier transform.)

2.1 Bandpass and lowpass signal representation

- Since the spectrum is Hermitian symmetric, we only need to retain half of the spectrum $X_+(f) = X(f)u_{-1}(f)$ (named analytic signal or pre-envelope) in order to analyze it,

where $u_{-1}(f) = \begin{cases} 1 & f > 0 \\ \frac{1}{2} & f = 0 \\ 0 & f < 0 \end{cases}$

Note: $X(f) = X_+(f) + X_+^*(-f)$

- A bandpass signal is very "real," but may contain "unnecessary" content such as the carrier frequency f_c that has nothing to do with the "digital information" transmitted.
- So, it is more convenient to remove this carrier frequency and transform $x(t)$ into its lowpass equivalent signal $x_\ell(t)$ before "analyzing" the digital content.

Definition (Baseband signal)

A lowpass or baseband (equivalent) signal $x_\ell(t)$ *is a complex signal (because it is not necessarily Hermitian symmetric!) whose spectrum is located around zero frequency, i.e.*

$$X_\ell(f) = 0 \quad \text{for all } |f| > W$$

It is generally written as

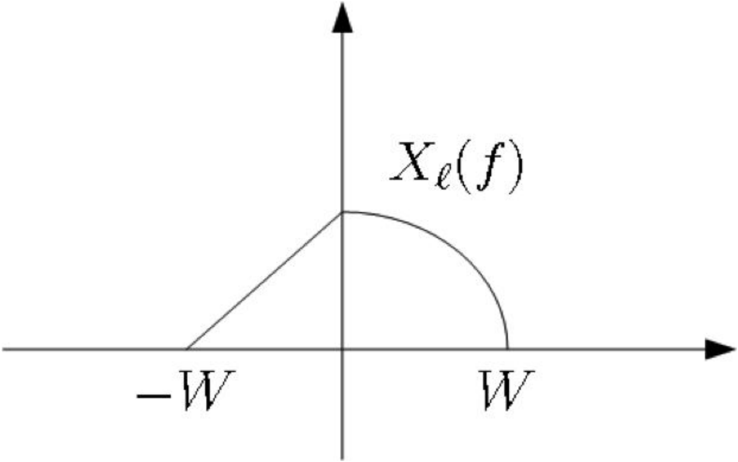
$$x_\ell(t) = x_i(t) + \imath\, x_q(t)$$

where

- $x_i(t)$ *is called the in-phase signal*
- $x_q(t)$ *is called the quadrature signal*

Baseband signal

Our goal is to relate $x_\ell(t)$ to $x(t)$ and vice versa



Bandwidths of $x_\ell(t)$ and $x(t)$

Definition of bandwidth. The bandwidth of a signal is one half of the entire range of frequencies over which the spectrum is essentially nonzero. Hence, W is the bandwidth in the lowpass signal we just defined, while $2W$ is the bandwidth of the bandpass signal by our definition.

Analytic signal

- Let's start from the analytic signal $x_+(t)$.

$$
\begin{aligned}
x_+(t) &= \int_{-\infty}^{\infty} X_+(f) e^{\imath 2\pi f t} df \\
&= \int_{-\infty}^{\infty} X(f) u_{-1}(f) e^{\imath 2\pi f t} df \\
&= \mathcal{F}^{-1}\{X(f) u_{-1}(f)\} \\
&= \mathcal{F}^{-1}\{X(f)\} \star \mathcal{F}^{-1}\{u_{-1}(f)\} \\
&= x(t) \star \left(\frac{1}{2}\delta(t) + \imath \frac{1}{2\pi t}\right) \\
&= \frac{1}{2} x(t) + \imath \frac{1}{2} \hat{x}(t),
\end{aligned}
$$

$\mathcal{F}^{-1}$ Inverse Fourier transform

where $\hat{x}(t) = x(t) \star \frac{1}{\pi t} = \int_{-\infty}^{\infty} \frac{x(\tau)}{\pi(t-\tau)} d\tau$ is a real-valued signal.

Appendix: Extended Fourier transform

$$\begin{aligned}\mathcal{F}^{-1}\{2u_{-1}(f)\} &= \mathcal{F}^{-1}\{1+\text{sgn}(f)\} \\ &= \mathcal{F}^{-1}\{1\} + \mathcal{F}^{-1}\{\text{sgn}(f)\} = \delta(t) + \imath\frac{1}{\pi t}\end{aligned}$$

Since $\int_{-\infty}^{\infty}|\text{sgn}(f)| = \infty$, the inverse Fourier transform of sgn(f) does not exist in the standard sense! We therefore have to derive its inverse Fourier transform in the extended sense!

$$(\forall\ f)S(f) = \lim_{n\to\infty} S_n(f) \text{ and } (\forall\ n) \int_{-\infty}^{\infty}|S_n(f)|df < \infty$$

$$\Rightarrow \mathcal{F}^{-1}\{S(f)\} = \lim_{n\to\infty}\mathcal{F}^{-1}\{S_n(f)\}.$$

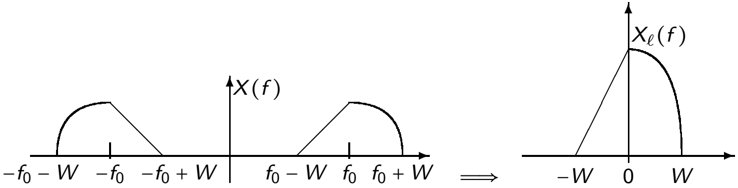
Appendix: Extended Fourier transform

Since $\lim_{a\downarrow 0} e^{-a|f|}\text{sgn}(f) = \text{sgn}(f)$,

$$\begin{aligned}
&\lim_{a\downarrow 0}\int_{-\infty}^{\infty} e^{-a|f|}\text{sgn}(f)e^{\imath 2\pi ft}df \\
&= \lim_{a\downarrow 0}\left[-\int_{-\infty}^{0} e^{f(a+\imath 2\pi t)}df + \int_{0}^{\infty} e^{f(-a+\imath 2\pi t)}df\right] \\
&= \lim_{a\downarrow 0}\left[-\frac{1}{a+\imath 2\pi t} + \frac{1}{a-\imath 2\pi t}\right] \\
&= \lim_{a\downarrow 0}\left[\frac{\imath 4\pi t}{a^2+4\pi^2 t^2}\right] = \begin{cases} 0 & t = 0 \\ \imath\frac{1}{\pi t} & t \neq 0 \end{cases}
\end{aligned}$$

Hence, $\mathcal{F}^{-1}\{2u_{-1}(f)\} = \mathcal{F}^{-1}\{1\} + \mathcal{F}^{-1}\{\text{sgn}(f)\} = \delta(t) + \imath\frac{1}{\pi t}$.

$x_\ell(t) \leftrightarrow x_+(t) \leftrightarrow x(t)$



- We then observe

$$X_\ell(f) = 2X_+(f + f_0).$$

This implies

$$\begin{aligned} x_\ell(t) &= \mathcal{F}^{-1}\{X_\ell(f)\} \\ &= \mathcal{F}^{-1}\{2X_+(f+f_0)\} \\ &= 2x_+(t)e^{-\imath 2\pi f_0 t} \\ &= (x(t) + \imath\,\hat{x}(t))e^{-\imath 2\pi f_0 t} \end{aligned}$$

As a result,

$$x(t) + \imath\,\hat{x}(t) = x_\ell(t)e^{\imath 2\pi f_0 t}$$

which gives:

$$x(t) \quad \left(= \mathbf{Re}\left\{x(t) + \imath\,\hat{x}(t)\right\}\right) \quad = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$$

By $x_\ell(t) = x_i(t) + \imath\,x_q(t)$,

$$x(t) \quad \left(= \mathbf{Re}\left\{(x_i(t) + \imath\,x_q(t))e^{\imath 2\pi f_0 t}\right\}\right)$$
$$= x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)$$

$X_\ell(f) \leftrightarrow X(f)$

- From $x(t) = \textbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\}$, we obtain

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-\imath 2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \textbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\} e^{-\imath 2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left[x_\ell(t)e^{\imath 2\pi f_0 t} + \left(x_\ell(t)e^{\imath 2\pi f_0 t}\right)^*\right] e^{-\imath 2\pi ft}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x_\ell(t)e^{-\imath 2\pi (f-f_0)t}dt \\
&\quad +\frac{1}{2}\int_{-\infty}^{\infty} x_\ell^*(t)e^{-\imath 2\pi (f+f_0)t}dt \\
&= \frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right]
\end{aligned}
$$

$$X_\ell^*(-f) = \int_{-\infty}^{\infty}\left(x_\ell(t)e^{-\imath 2\pi(-f)t}\right)^* dt = \int_{-\infty}^{\infty} x_\ell^*(f)e^{-\imath 2\pi ft}dt$$

Summary

Terminologies & relations

- Bandpass signal

$$\begin{cases} x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_0 t}\right\} \\ X(f) = \frac{1}{2}\left[X_\ell(f-f_0) + X_\ell^*(-f-f_0)\right] \end{cases}$$

- Analytic signal or pre-envelope $x_+(t)$ and $X_+(f)$
- Lowpass equivalent signal or complex envelope

$$\begin{cases} x_\ell(t) = (x(t) + \imath\hat{x}(t))e^{-\imath 2\pi f_0 t} \\ X_\ell(f) = 2X(f+f_0)u_{-1}(f+f_0) \end{cases}$$

Useful to know

Terminologies & relations

- From $x_\ell(t) = x_i(t) + \imath\, x_q(t) = (x(t) + \imath\, \hat{x}(t))e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x_i(t) = \textbf{Re}\left\{ (x(t) + \imath\, \hat{x}(t))\, e^{-\imath 2\pi f_0 t} \right\} \\ x_q(t) = \textbf{Im}\left\{ (x(t) + \imath\, \hat{x}(t))\, e^{-\imath 2\pi f_0 t} \right\} \end{cases}$$

- Also from $x_\ell(t) = (x(t) + \imath\, \hat{x}(t))e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x(t) = \textbf{Re}\left\{ x_\ell(t)\, e^{\imath 2\pi f_0 t} \right\} \\ \hat{x}(t) = \textbf{Im}\left\{ x_\ell(t)\, e^{\imath 2\pi f_0 t} \right\} \end{cases}$$

Useful to know

Terminologies & relations

- From $x_\ell(t) = x_i(t) + \imath\, x_q(t) = (x(t) + \imath\, \hat{x}(t))e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x_i(t) = \mathbf{Re}\left\{ (x(t) + \imath\, \hat{x}(t))\, e^{-\imath 2\pi f_0 t} \right\} \\ x_q(t) = \mathbf{Im}\left\{ (x(t) + \imath\, \hat{x}(t))\, e^{-\imath 2\pi f_0 t} \right\} \end{cases}$$

- Also from $x_\ell(t) = (x(t) + \imath\, \hat{x}(t))e^{-\imath 2\pi f_0 t}$,

$$\begin{cases} x(t) = \mathbf{Re}\left\{ (x_i(t) + \imath\, x_q(t))\, e^{\imath 2\pi f_0 t} \right\} \\ \hat{x}(t) = \mathbf{Im}\left\{ (x_i(t) + \imath\, x_q(t))\, e^{\imath 2\pi f_0 t} \right\} \end{cases}$$

Useful to know

Terminologies & relations

- pre-envelope $x_+(t)$
- complex envelope $x_\ell(t)$
- envelope $|x_\ell(t)| = \sqrt{x_i^2(t) + x_q^2(t)} = r_\ell(t)$

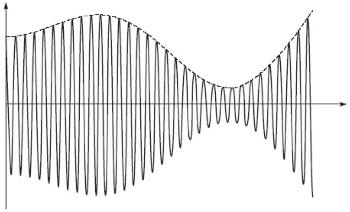


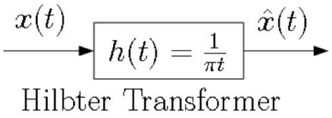
FIGURE 2.1–4
A bandpass signal. The dashed curve denotes the envelope.

- phase $\theta_\ell(t) = \arctan[x_q(t)/x_i(t)]$

Modulator/demodulator and Hilbert transformer

Usually, we will modulate and demodulate with respect to carrier frequency f_c, which may not be equal to the center frequency f_0.

- $x_\ell(t) \to x(t) = \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_c t}\right\} \Rightarrow$ modulation
- $x(t) \to x_\ell(t) = (x(t) + \imath \hat{x}(t))e^{-\imath 2\pi f_c t} \Rightarrow$ demodulation
  - The demodulation requires to generate $\hat{x}(t)$, a Hilbert transform of $x(t)$



Hilbert transform is basically a 90-degree phase shifter.

$$H(f) = \mathcal{F}\left\{\frac{1}{\pi t}\right\} = -\imath\,\mathrm{sgn}(f) = \begin{cases} -\imath, & f > 0 \\ 0, & f = 0 \\ \imath, & f < 0 \end{cases}$$

Recall that on page 10, we have shown

$$\mathcal{F}^{-1}\{\mathrm{sgn}(f)\} = \imath\frac{1}{\pi t}\mathbf{1}\{t \neq 0\};$$

hence

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = \frac{1}{\imath}\mathrm{sgn}(f) = -\imath\,\mathrm{sgn}(f).$$

Energy considerations

Definition (Energy of a signal)

The energy $\mathcal{E}_s$ of a (complex) signal $s(t)$ is

$$\mathcal{E}_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

Hence, the energies of $x(t)$, $x_+(t)$ and $x_\ell(t)$ are

$$\begin{aligned} \mathcal{E}_x &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ \mathcal{E}_{x_+} &= \int_{-\infty}^{\infty} |x_+(t)|^2 \, dt \\ \mathcal{E}_{x_\ell} &= \int_{-\infty}^{\infty} |x_\ell(t)|^2 \, dt \end{aligned}$$

We are interested in the connection among $\mathcal{E}_x$, $\mathcal{E}_{x_+}$, and $\mathcal{E}_{x_\ell}$.

- First, from Parseval's Theorem, we see

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Parseval's theorem $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$

(Rayleigh's theorem) $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

- Second

$$X(f) = \underbrace{\frac{1}{2}X_\ell(f - f_c)}_{=X_+(f)} + \underbrace{\frac{1}{2}X_\ell^*(-f - f_c)}_{=X_+^*(-f)}$$

- Third, $f_c \gg W$ and

$$X_\ell(f - f_c)X_\ell^*(-f - f_c) = 4X_+(f)X_+^*(-f) = 0 \text{ for all } f$$

It then shows

$$\begin{aligned}\mathcal{E}_x &= \int_{-\infty}^{\infty}\left|\frac{1}{2}X_\ell(f-f_c)+\frac{1}{2}X_\ell^*(-f-f_c)\right|^2 df\\ &= \frac{1}{4}\mathcal{E}_{x_\ell}+\frac{1}{4}\mathcal{E}_{x_\ell}=\frac{1}{2}\mathcal{E}_{x_\ell}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_x &= \int_{-\infty}^{\infty}\left|X_+(f)+X_+^*(-f)\right|^2 df\\ &= \mathcal{E}_{x_+}+\mathcal{E}_{x_+}=2\mathcal{E}_{x_+}\end{aligned}$$

Theorem (Energy considerations)

$$\mathcal{E}_{x_\ell} = 2\mathcal{E}_x = 4\mathcal{E}_{x_+}$$

Extension of energy considerations

Definition (Inner product)

We define the inner product of two (complex) signals $x(t)$ and $y(t)$ as

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt.$$

- Parseval's relation immediately gives

$$\langle x(t), y(t)\rangle = \langle X(f), Y(f)\rangle.$$

- $\mathcal{E}_x = \langle x(t), x(t)\rangle = \langle X(f), X(f)\rangle$
- $\mathcal{E}_{x_\ell} = \langle x_\ell(t), x_\ell(t)\rangle = \langle X_\ell(f), X_\ell(f)\rangle$

We can similarly prove that

$$
\begin{aligned}
&\langle x(t), y(t)\rangle \\
&= \langle X(f), Y(f)\rangle \\
&= \left\langle \frac{1}{2}X_\ell(f-f_c)+\frac{1}{2}X_\ell^*(-f-f_c), \frac{1}{2}Y_\ell(f-f_c)+\frac{1}{2}Y_\ell^*(-f-f_c)\right\rangle \\
&= \frac{1}{4}\langle X_\ell(f-f_c), Y_\ell(f-f_c)\rangle + \frac{1}{4}\underbrace{\langle X_\ell(f-f_c), Y_\ell^*(-f-f_c)\rangle}_{=0} \\
&\quad + \frac{1}{4}\underbrace{\langle X_\ell^*(-f-f_c), Y_\ell(f-f_c)\rangle}_{=0} + \frac{1}{4}\langle X_\ell^*(-f-f_c), Y_\ell^*(-f-f_c)\rangle \\
&= \frac{1}{4}\langle x_\ell(t), y_\ell(t)\rangle + \frac{1}{4}\left(\langle x_\ell(t), y_\ell(t)\rangle\right)^* = \frac{1}{2}\mathbf{Re}\left\{\langle x_\ell(t), y_\ell(t)\rangle\right\}.
\end{aligned}
$$

Corss-correlation of two signals

Definition (Cross-correlation)

The cross-correlation of two signals $x(t)$ and $y(t)$ is defined as

$$\rho_{x,y} = \frac{\langle x(t), y(t)\rangle}{\sqrt{\langle x(t), x(t)\rangle}\sqrt{\langle y(t), y(t)\rangle}} = \frac{\langle x(t), y(t)\rangle}{\sqrt{\mathcal{E}_x \mathcal{E}_y}}.$$

Definition (Orthogonality)

Two signals $x(t)$ and $y(t)$ are said to be orthogonal if $\rho_{x,y} = 0$.

- The previous slide then shows $\rho_{x,y} = \mathbf{Re}\left\{\rho_{x_\ell,y_\ell}\right\}$.
- $\rho_{x_\ell,y_\ell} = 0 \Rightarrow \rho_{x,y} = 0$ but $\rho_{x,y} = 0 \not\Rightarrow \rho_{x_\ell,y_\ell} = 0$

2.1-4 Lowpass equivalence of a bandpass system

Definition (Bandpass system)

A bandpass **system** *is an LTI system with real impulse response $h(t)$ whose transfer function is located around a frequency f_c.*

- Using a similar concept, we set the lowpass equivalent impulse response as

$$h(t) = \mathbf{Re}\left\{h_\ell(t)e^{\imath 2\pi f_c t}\right\}$$

and

$$H(f) = \frac{1}{2}\left[H_\ell(f - f_c) + H_\ell^*(-f - f_c)\right]$$

Baseband input-output relation

- Let $x(t)$ be a bandpass input signal and let

$$y(t) = h(t) \star x(t) \quad \text{or equivalently} \quad Y(f) = H(f)X(f)$$

- Then, we know

$$\begin{aligned} x(t) &= \mathbf{Re}\left\{x_\ell(t)e^{\imath 2\pi f_c t}\right\} \\ h(t) &= \mathbf{Re}\left\{h_\ell(t)e^{\imath 2\pi f_c t}\right\} \\ y(t) &= \mathbf{Re}\left\{y_\ell(t)e^{\imath 2\pi f_c t}\right\} \end{aligned}$$

and

Theorem (Baseband input-output relation)

$$y(t) = h(t) \star x(t) \quad \Longleftrightarrow \quad y_\ell(t) = \frac{1}{2} h_\ell(t) \star x_\ell(t)$$

Proof:

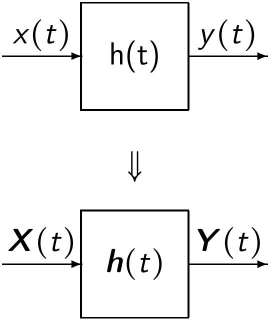
For $f \neq -f_c$ (or specifically, for $u_{-1}(f+f_c) = u_{-1}^2(f+f_c)$),

Note $\frac{1}{2} = u_{-1}(0) \neq u_{-1}^2(0) = \frac{1}{4}$.

$$\begin{aligned} Y_\ell(f) &= 2Y(f+f_c)u_{-1}(f+f_c) \\ &= 2H(f+f_c)X(f+f_c)u_{-1}(f+f_c) \\ &= \frac{1}{2}\left[2H(f+f_c)u_{-1}(f+f_c)\right]\cdot\left[2X(f+f_c)u_{-1}(f+f_c)\right] \\ &= \frac{1}{2}H_\ell(f)\cdot X_\ell(f) \end{aligned}$$

The case for $f = -f_c$ is valid since $Y_\ell(-f_c) = X_\ell(-f_c) = 0$. □

- The above theorem applies to a deterministic system. How about a stochastic system?



The text abuses the notation by using $X(f)$ as the spectrum of $x(t)$ but using $X(t)$ as the stochastic counterpart of $x(t)$.

2.7 Random processes

Random Process

Definition

A random process is a set of indexed random variables $\{\boldsymbol{X}(t), t \in \mathcal{T}\}$, where $\mathcal{T}$ is often called the index set.

Classification

1. If $\mathcal{T}$ is a finite set $\Rightarrow$ Random Vector
2. If $\mathcal{T} = \mathbb{Z}$ or $\mathbb{Z}^+$ $\Rightarrow$ Discrete Random Process
3. If $\mathcal{T} = \mathbb{R}$ or $\mathbb{R}^+$ $\Rightarrow$ Continuous Random Process
4. If $\mathcal{T} = \mathbb{R}^2, \mathbb{Z}^2, \cdots, \mathbb{R}^n, \mathbb{Z}^n$ $\Rightarrow$ Random Field

Examples of random process

Example

Let $\boldsymbol{U}$ *be a random variable uniformly distributed over* $[-\pi, \pi)$*. Then*

$$\boldsymbol{X}(t) = \cos\left(2\pi f_c t + \boldsymbol{U}\right)$$

is a continuous random process.

Example

Let $\boldsymbol{B}$ *be a random variable taking values in* $\{-1, 1\}$*. Then*

$$\boldsymbol{X}(t) = \begin{cases} \cos(2\pi f_c t) & \text{if } \boldsymbol{B} = -1 \\ \sin(2\pi f_c t) & \text{if } \boldsymbol{B} = +1 \end{cases} = \cos\left(2\pi f_c t - \frac{\pi}{4}(\boldsymbol{B}+1)\right)$$

is a continuous random process.

Statistical properties of random process

For any integer $k > 0$ and any $t_1, t_2, \cdots, t_k \in \mathcal{T}$, the finite-dimensional cumulative distribution function (cdf) for $\boldsymbol{X}(t)$ is given by:

$$F_{\boldsymbol{X}}(t_1, \cdots, t_k; x_1, \cdots, x_k) = \Pr\{\boldsymbol{X}(t_1) \le x_1, \cdots, \boldsymbol{X}(t_k) \le x_k\}$$

As event $[\boldsymbol{X}(t) < \infty]$ (resp. $[\boldsymbol{X}(t) \le -\infty]$) is always regarded

as true (resp. false),

$$\begin{aligned} &\lim_{x_s \to \infty} F_{\boldsymbol{X}}(t_1, \cdots, t_k; x_1, \cdots, x_k) \\ &\quad = F_{\boldsymbol{X}}(t_1, \cdots, t_{s-1}, t_{s+1}, t_k; x_1, \cdots, x_{s-1}, x_{s+1}, \cdots, x_k) \end{aligned}$$

and

$$\lim_{x_s \to -\infty} F_{\boldsymbol{X}}(t_1, \cdots, t_k; x_1, \cdots, x_k) = 0$$

Definition

Let $\boldsymbol{X}(t)$ be a random process; then the mean function is

$$m_{\boldsymbol{X}}(t) = \mathbb{E}[\boldsymbol{X}(t)],$$

the (auto)correlation function is

$$R_{\boldsymbol{X}}(t_1, t_2) = \mathbb{E}\left[\boldsymbol{X}(t_1)\boldsymbol{X}^*(t_2)\right],$$

and the (auto)covariance function is

$$K_{\boldsymbol{X}}(t_1, t_2) = \mathbb{E}\Big[\, (\boldsymbol{X}(t_1) - m_{\boldsymbol{X}}(t_1))\, (\boldsymbol{X}(t_2) - m_{\boldsymbol{X}}(t_2))^* \,\Big]$$

Definition

Let $\boldsymbol{X}(t)$ *and* $\boldsymbol{Y}(t)$ *be two random processes; then the* *cross-correlation function is*

$$R_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) = \mathbb{E}[\boldsymbol{X}(t_1)\boldsymbol{Y}^*(t_2)],$$

and cross-covariance function is

$$K_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) = \mathbb{E}\Big[(\boldsymbol{X}(t_1) - m_{\boldsymbol{X}}(t_1)) \ (\boldsymbol{Y}(t_2) - m_{\boldsymbol{Y}}(t_2))^* \Big]$$

Proposition

$$R_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) = K_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) + m_{\boldsymbol{X}}(t_1) m^*_{\boldsymbol{Y}}(t_2)$$

$$R_{\boldsymbol{Y},\boldsymbol{X}}(t_2, t_1) = R^*_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) \qquad R_{\boldsymbol{X}}(t_2, t_1) = R^*_{\boldsymbol{X}}(t_1, t_2)$$

$$K_{\boldsymbol{Y},\boldsymbol{X}}(t_2, t_1) = K^*_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) \qquad K_{\boldsymbol{X}}(t_2, t_1) = K^*_{\boldsymbol{X}}(t_1, t_2)$$

Stationary random processes

Definition

A random process $\mathbf{X}(t)$ is said to be strictly or strict-sense stationary (SSS) if its finite-dimensional joint distribution function is shift-invariant, i.e. for any integer $k > 0$, any $t_1, \cdots, t_k \in \mathcal{T}$ and any τ,

$$F_{\mathbf{X}}(t_1 - \tau, \cdots, t_k - \tau; x_1, \cdots, x_k) = F_{\mathbf{X}}(t_1, \cdots, t_k; x_1, \cdots, x_k)$$

Definition

A random process $\mathbf{X}(t)$ is said to be weakly or wide-sense stationary (WSS) if its mean function and (auto)correlation function are shift-invariant, i.e. for any $t_1, t_2 \in \mathcal{T}$ and any τ,

$$m_{\mathbf{X}}(t - \tau) = m_{\mathbf{X}}(t) \text{ and } R_{\mathbf{X}}(t_1 - \tau, t_2 - \tau) = R_{\mathbf{X}}(t_1, t_2).$$

The above condition is equivalent to

$$m_{\mathbf{X}}(t) = constant \text{ and } R_{\mathbf{X}}(t_1, t_2) = R_{\mathbf{X}}(t_1 - t_2).$$

Wide-sense stationary random processes

Definition

Two random processes $\boldsymbol{X}(t)$ *and* $\boldsymbol{Y}(t)$ *are said to be jointly wide-sense stationary if*

- *Both* $\boldsymbol{X}(t)$ *and* $\boldsymbol{Y}(t)$ *are WSS;*
- $R_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) = R_{\boldsymbol{X},\boldsymbol{Y}}(t_1 - t_2)$.

Proposition

For jointly WSS $\boldsymbol{X}(t)$ *and* $\boldsymbol{Y}(t)$*,*

$$\begin{aligned} R_{\boldsymbol{Y},\boldsymbol{X}}(t_2, t_1) = R^*_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) &\implies R_{\boldsymbol{X},\boldsymbol{Y}}(\tau) = R^*_{\boldsymbol{Y},\boldsymbol{X}}(-\tau) \\ K_{\boldsymbol{Y},\boldsymbol{X}}(t_2, t_1) = K^*_{\boldsymbol{X},\boldsymbol{Y}}(t_1, t_2) &\implies K_{\boldsymbol{X},\boldsymbol{Y}}(\tau) = K^*_{\boldsymbol{Y},\boldsymbol{X}}(-\tau) \end{aligned}$$

Gaussian random process

Definition

A random process $\{\boldsymbol{X}(t), t \in \mathcal{T}\}$ is said to be Gaussian if for any integer $k > 0$ and for any $t_1, \cdots, t_k \in \mathcal{T}$, the *finite-dimensional* *joint cdf*

$$F_{\boldsymbol{X}}(t_1, \cdots, t_k; x_1, \cdots, x_k) = \Pr\left[\boldsymbol{X}(t_1) \le x_1, \cdots, \boldsymbol{X}(t_k) \le x_k\right]$$

is Gaussian.

Remark

The joint cdf of a Gaussian process is fully determined by its mean function and its autocovariance function.

Gaussian random process

Definition

Two real random processes $\{X(t), t \in \mathcal{T}_X\}$ *and* $\{Y(t), t \in \mathcal{T}_Y\}$ *are said to be jointly Gaussian if for any integers* $j, k > 0$ *and for any* $s_1, \cdots, s_j \in \mathcal{T}_{\mathbf{X}}$ *and* $t_1, \cdots, t_k \in \mathcal{T}_{\mathbf{Y}}$*, the finite-dimensional joint cdf*

$$\Pr\left[\boldsymbol{X}(s_1) \le x_1, \cdots, \boldsymbol{X}(s_j) \le x_j, \boldsymbol{Y}(t_1) \le y_1, \cdots, \boldsymbol{Y}(t_k) \le y_k\right]$$

is Gaussian.

Definition

A complex process is Gaussian if the real and imaginary processes are jointly Gaussian.

Gaussian random process

Remark

For joint (in general complex) Gaussian processes, "uncorrelatedness", defined as

$$R_{\boldsymbol{X},\boldsymbol{Y}}(t_1,t_2) = \boxed{\mathbb{E}[\boldsymbol{X}(t_1)\boldsymbol{Y}^*(t_2)]} = \boxed{\mathbb{E}[\boldsymbol{X}(t_1)]\mathbb{E}[\boldsymbol{Y}^*(t_2)]} = m_{\boldsymbol{X}}(t_1)m_{\boldsymbol{Y}}^*(t_2),$$

implies "independence", i.e.,

$$\begin{aligned}&\Pr\left[\boldsymbol{X}(s_1)\le x_1,\cdots,\boldsymbol{X}(s_j)\le x_j,\boldsymbol{Y}(t_1)\le y_1,\cdots,\boldsymbol{Y}(t_k)\le y_k\right]\\ =&\Pr\left[\boldsymbol{X}(s_1)\le x_1,\cdots,\boldsymbol{X}(s_k)\le x_k\right]\cdot\Pr\left[\boldsymbol{Y}(t_1)\le y_1,\cdots,\boldsymbol{Y}(t_k)\le y_k\right]\end{aligned}$$

Theorem

If a Gaussian random process $\boldsymbol{X}(t)$ is WSS, then it is SSS.

Idea behind the Proof:

For any $k > 0$, consider the sampled random vector

$$\vec{\boldsymbol{X}}_k = \begin{bmatrix} \boldsymbol{X}(t_1) \\ \boldsymbol{X}(t_2) \\ \vdots \\ \boldsymbol{X}(t_k) \end{bmatrix}.$$

The mean vector and covariance matrix of $\vec{\boldsymbol{X}}_k$ are respectively

$$m_{\vec{\boldsymbol{X}}_k} = \mathbb{E}[\vec{\boldsymbol{X}}_k] = \begin{bmatrix} \mathbb{E}[\boldsymbol{X}(t_1)] \\ \mathbb{E}[\boldsymbol{X}(t_2)] \\ \vdots \\ \mathbb{E}[\boldsymbol{X}(t_k)] \end{bmatrix} = m_{\boldsymbol{X}}(0) \cdot \vec{\mathbf{1}}$$

and

$$R_{\vec{\boldsymbol{X}}} = \mathbb{E}[\vec{\boldsymbol{X}}_k \vec{\boldsymbol{X}}_k^H] = \begin{bmatrix} K_{\boldsymbol{X}}(0) & K_{\boldsymbol{X}}(t_1 - t_2) & \cdots \\ K_{\boldsymbol{X}}(t_2 - t_1) & K_{\boldsymbol{X}}(0) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

It can be shown that for a new sampled random vector

$$\begin{bmatrix} \boldsymbol{X}(t_1 + \tau) \\ \boldsymbol{X}(t_2 + \tau) \\ \vdots \\ \boldsymbol{X}(t_k + \tau) \end{bmatrix}$$

the mean vector and covariance matrix remain the same.
Hence, $\boldsymbol{X}(t)$ is SSS. □

Power spectral density

Definition

Let $R_{\boldsymbol{X}}(\tau)$ be the correlation function of a WSS random process $\boldsymbol{X}(t)$. The power spectral density (PSD) or power spectrum of $\boldsymbol{X}(t)$ is defined as

$$S_{\boldsymbol{X}}(f) = \int_{-\infty}^{\infty} R_{\boldsymbol{X}}(\tau) e^{-\imath 2\pi f \tau}\, d\tau.$$

Let $R_{\boldsymbol{X},\boldsymbol{Y}}(\tau)$ be the cross-correlation function of two jointly WSS random process $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$; then the cross spectral density (CSD) is

$$S_{\boldsymbol{X},\boldsymbol{Y}}(f) = \int_{-\infty}^{\infty} R_{\boldsymbol{X},\boldsymbol{Y}}(\tau) e^{-\imath 2\pi f \tau}\, d\tau.$$

Properties of PSD

- PSD (in units of watts per Hz) describes the density of power as a function of frequency.
  - Analogously, probability density function (pdf) describes the density of probability as a function of outcome.
  - The integration of PSD gives power of the random process over the considered range of frequency. Analogously, the integration of pdf gives probability over the considered range of outcome.

Theorem

$S_{\mathbf{X}}(f)$ is non-negative and real (which matches that the power of a signal cannot be negative or complex-valued).

Proof: $S_{\mathbf{X}}(f)$ is real because

$$
\begin{aligned}
S_{\mathbf{X}}(f) &= \int_{-\infty}^{\infty} R_{\mathbf{X}}(\tau) e^{-\imath 2\pi f \tau}\, d\tau \\
&= \int_{-\infty}^{\infty} R_{\mathbf{X}}(-s) e^{\imath 2\pi f s}\, ds \quad (s = -\tau) \\
&= \int_{-\infty}^{\infty} R^{*}_{\mathbf{X}}(s) e^{\imath 2\pi f s}\, ds \\
&= \left(\int_{-\infty}^{\infty} R_{\mathbf{X}}(s) e^{-\imath 2\pi f s}\, ds \right)^{*} \\
&= S^{*}_{\mathbf{X}}(f)
\end{aligned}
$$

$S_{\boldsymbol{X}}(f)$ is non-negative because of the following (we only prove this based on that $\mathcal{T} \subset \mathbb{R}$ and $\boldsymbol{X}(t) = 0$ outside $[-T, T]$).

$$\begin{aligned} S_{\boldsymbol{X}}(f) &= \int_{-\infty}^{\infty} \mathbb{E}[\boldsymbol{X}(t+\tau)\boldsymbol{X}^*(t)]e^{-\imath 2\pi f \tau}\, d\tau \\ &= \mathbb{E}\left[\boldsymbol{X}^*(t)\int_{-\infty}^{\infty} \boldsymbol{X}(t+\tau)e^{-\imath 2\pi f \tau}\, d\tau\right] \quad (s = t+\tau) \\ &= \mathbb{E}\left[\boldsymbol{X}^*(t)\int_{-\infty}^{\infty} \boldsymbol{X}(s)e^{-\imath 2\pi f (s-t)}\, ds\right] \\ &= \mathbb{E}\left[\boldsymbol{X}^*(t)\tilde{\boldsymbol{X}}(f)e^{\imath 2\pi f t}\right] \end{aligned}$$

In notation, $\tilde{\boldsymbol{X}}(f) = \mathcal{F}\{\boldsymbol{X}(t)\}$.

Since the above is a constant with respect to t (by WSS),

$$\begin{aligned} S_{\boldsymbol{X}}(f) &= \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left[\boldsymbol{X}^*(t)\tilde{\boldsymbol{X}}(f)e^{\imath 2\pi f t}\right] dt \\ &= \frac{1}{2T}\mathbb{E}\left[\tilde{\boldsymbol{X}}(f)\int_{-T}^{T} \boldsymbol{X}^*(t)e^{\imath 2\pi f t} dt\right] \\ &= \frac{1}{2T}\mathbb{E}\left[\tilde{\boldsymbol{X}}(f)\tilde{\boldsymbol{X}}^*(f)\right] = \frac{1}{2T}\mathbb{E}\left[|\tilde{\boldsymbol{X}}(f)|^2\right] \geq 0. \end{aligned}$$

□

Wiener-Khintchine theorem

Theorem (Wiener-Khintchine)

Let $\{\boldsymbol{X}(t), t \in \mathbb{R}\}$ be a WSS random process. Define

$$\boldsymbol{X}_T(t) = \begin{cases} \boldsymbol{X}(t) & \text{if } t \in [-T, T] \\ 0, & \text{otherwise.} \end{cases}$$

and set

$$\tilde{\boldsymbol{X}}_T(f) = \int_{-\infty}^{\infty} \boldsymbol{X}_T(t) e^{-\imath 2\pi f t}\, dt = \int_{-T}^{T} \boldsymbol{X}(t) e^{-\imath 2\pi f t}\, dt.$$

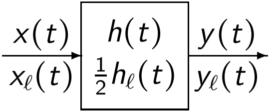
If $S_{\boldsymbol{X}}(f)$ exists (i.e., $R_{\boldsymbol{X}}(\tau)$ has a Fourier transform), then

$$S_{\boldsymbol{X}}(f) = \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left\{ \left| \tilde{\boldsymbol{X}}_T(f) \right|^2 \right\}$$

Variations of PSD definitions

- Power density spectrum : Alternative definition
  - Fourier transform of auto-covariance function (e.g., Robert M. Gray and Lee D. Davisson, Random Processes: A Mathematical Approach for Engineers, p. 193)
- I remark that from the viewpoint of digital communications, the text's definition is more appropriate since
  - the auto-covariance function is independent of a mean-shift; however, random signals with different "means" consume different "powers."

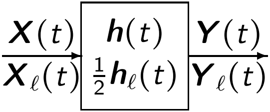
- What can we say about, e.g., the PSD of stochastic system input and output?



$$\begin{cases} \square(t) = \mathbf{Re}\{\square_\ell(t)e^{\imath 2\pi f_c t}\} \\ \square_\ell(t) = (\square(t) + \imath\,\hat{\square}(t))e^{-\imath 2\pi f_c t} \end{cases}$$

where "$\square$" can be x, y or h.

$$\Downarrow$$



$$\begin{cases} \square(t) = \mathbf{Re}\{\square_\ell(t)e^{\imath 2\pi f_c t}\} \\ \square_\ell(t) = (\square(t) + \imath\,\hat{\square}(t))e^{-\imath 2\pi f_c t} \end{cases}$$

where "$\square$" can be $\boldsymbol{X}$, $\boldsymbol{Y}$ or $\boldsymbol{h}$.

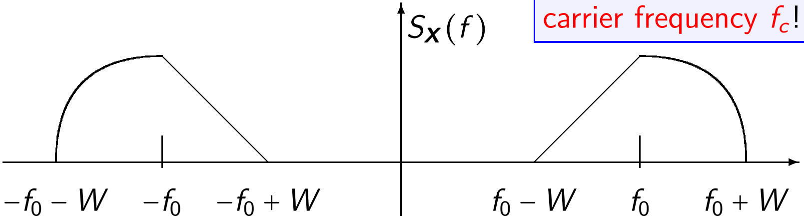
2.9 Bandpass and lowpass random processes

Definition (Bandpass random signal)

A bandpass (WSS) stochastic signal $\boldsymbol{X}(t)$ is a real random process whose PSD is located around central frequency f_0, i.e.

$$S_{\boldsymbol{X}}(f) = 0 \quad \text{for all } |f \pm f_0| > W$$

f_0 may not be the carrier frequency f_c!



- We know $\begin{cases} \boldsymbol{X}(t) = \textbf{Re}\left\{\boldsymbol{X}_\ell(t) e^{\imath 2\pi f_c t}\right\} \\ \boldsymbol{X}_\ell(t) = (\boldsymbol{X}(t) + \imath \hat{\boldsymbol{X}}(t)) e^{-\imath 2\pi f_c t} \end{cases}$

Assumption (Fundamental assumption)

The bandpass signal $\boldsymbol{X}(t)$ *is WSS.*
In addition, its complex lowpass equivalent process $\boldsymbol{X}_\ell(t)$ *is WSS. In other words,*

- $\boldsymbol{X}_i(t)$ *and* $\boldsymbol{X}_q(t)$ *are WSS.*
- $\boldsymbol{X}_i(t)$ *and* $\boldsymbol{X}_q(t)$ *are jointly WSS.*

Under this **fundamental assumption**, we obtain the following properties:

P1) If $\boldsymbol{X}(t)$ zero-mean, both $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ zero-mean because $m_{\boldsymbol{X}} = m_{\boldsymbol{X}_i}\cos(2\pi f_c t) - m_{\boldsymbol{X}_q}\sin(2\pi f_c t)$.

P2) $$\begin{cases} R_{\boldsymbol{X}_i}(\tau) = R_{\boldsymbol{X}_q}(\tau) \\ R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(\tau) = -R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau) \end{cases}$$

Proof of P2):

$$
\begin{aligned}
& R_{\boldsymbol{X}}(\tau) \\
&= \mathbb{E}\left[\boldsymbol{X}(t+\tau)\boldsymbol{X}(t)\right] \\
&= \mathbb{E}\left[\mathbf{Re}\left\{\boldsymbol{X}_\ell(t+\tau)e^{\imath 2\pi f_c(t+\tau)}\right\}\mathbf{Re}\left\{\boldsymbol{X}_\ell(t)e^{\imath 2\pi f_c t}\right\}\right] \\
&= \mathbb{E}\left[(\boldsymbol{X}_i(t+\tau)\cos(2\pi f_c(t+\tau)) - \boldsymbol{X}_q(t+\tau)\sin(2\pi f_c(t+\tau)))\right. \\
&\quad \left.(\boldsymbol{X}_i(t)\cos(2\pi f_c t) - \boldsymbol{X}_q(t)\sin(2\pi f_c t))\right] \\
&= \frac{R_{\boldsymbol{X}_i}(\tau) + R_{\boldsymbol{X}_q}(\tau)}{2}\cos(2\pi f_c \tau) \\
&\quad + \frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(\tau) - R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau)}{2}\sin(2\pi f_c \tau) \\
&\quad + \frac{R_{\boldsymbol{X}_i}(\tau) - R_{\boldsymbol{X}_q}(\tau)}{2}\cos(2\pi f_c(2t+\tau)) \quad (=0) \\
&\quad - \frac{R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(\tau) + R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau)}{2}\sin(2\pi f_c(2t+\tau)) \quad (=0)
\end{aligned}
$$

□

P3) $R_{\mathbf{X}}(\tau) = \mathbf{Re}\left\{\frac{1}{2}R_{\mathbf{X}_\ell}(\tau)e^{\imath 2\pi f_c\tau}\right\}$.

Proof: Observe from *P2)*,

$$
\begin{aligned}
R_{\mathbf{X}_\ell}(\tau) &= \mathbb{E}[\mathbf{X}_\ell(t+\tau)\mathbf{X}_\ell^*(t)] \\
&= \mathbb{E}[(\mathbf{X}_i(t+\tau)+\imath\mathbf{X}_q(t+\tau))(\mathbf{X}_i(t)-\imath\mathbf{X}_q(t))] \\
&= R_{\mathbf{X}_i}(\tau)+R_{\mathbf{X}_q}(\tau)-\imath R_{\mathbf{X}_i,\mathbf{X}_q}(\tau)+\imath R_{\mathbf{X}_q,\mathbf{X}_i}(\tau) \\
&= 2R_{\mathbf{X}_i}(\tau)+\imath 2R_{\mathbf{X}_q,\mathbf{X}_i}(\tau).
\end{aligned}
$$

Hence, also from *P2)*,

$$
\begin{aligned}
R_{\mathbf{X}}(\tau) &= R_{\mathbf{X}_i}(\tau)\cos(2\pi f_c\tau)-R_{\mathbf{X}_q,\mathbf{X}_i}(\tau)\sin(2\pi f_c\tau) \\
&= \mathbf{Re}\left\{\frac{1}{2}R_{\mathbf{X}_\ell}(\tau)e^{\imath 2\pi f_c\tau}\right\}
\end{aligned}
$$

P4) $S_{\mathbf{X}}(f) = \frac{1}{4}\left[S_{\mathbf{X}_\ell}(f - f_c) + S^*_{\mathbf{X}_\ell}(-f - f_c)\right].$

Proof: A direct consequence of *P3)*. □

Note:

- Amplitude $\tilde{\boldsymbol{X}}(f) = \frac{1}{2}\left[\tilde{\boldsymbol{X}}_\ell(f - f_c) + \tilde{\boldsymbol{X}}^*_\ell(-f - f_c)\right]$
- Amplitude square

$$\begin{aligned} \left|\tilde{\boldsymbol{X}}(f)\right|^2 &= \frac{1}{4}\left|\tilde{\boldsymbol{X}}_\ell(f - f_c) + \tilde{\boldsymbol{X}}^*_\ell(-f - f_c)\right|^2 \\ &= \frac{1}{4}\left(\left|\tilde{\boldsymbol{X}}_\ell(f - f_c)\right|^2 + \left|\tilde{\boldsymbol{X}}^*_\ell(-f - f_c)\right|^2\right) \end{aligned}$$

- Wiener-Khintchine: $S_{\mathbf{X}}(f) \equiv \left|\tilde{\boldsymbol{X}}(f)\right|^2.$

P5) $\boldsymbol{X}_i(t)$ and $\boldsymbol{X}_q(t)$ uncorrelated if one of them has zero-mean.

Proof: From *P2)*,

$$R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(\tau) = -R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau) = -R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(-\tau).$$

Hence, $R_{\boldsymbol{X}_i,\boldsymbol{X}_q}(0) = 0$ (i.e.,

$$\mathbb{E}[\boldsymbol{X}_i(t)\boldsymbol{X}_q(t)] = 0 = \mathbb{E}[\boldsymbol{X}_i(t)]\mathbb{E}[\boldsymbol{X}_q(t)]).$$

□

P6) If $S_{\boldsymbol{X}_\ell}(-f) = S^*_{\boldsymbol{X}_\ell}(f) (= S_{\boldsymbol{X}_\ell}(f))$ symmetric, then $\boldsymbol{X}_i(t+\tau)$ and $\boldsymbol{X}_q(t)$ uncorrelated for any τ, provided one of them has zero-mean.

Proof: From *P3)*,

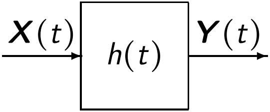
$$R_{\boldsymbol{X}_\ell}(\tau) = 2R_{\boldsymbol{X}_i}(\tau) + \imath 2R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau).$$

$S_{\boldsymbol{X}_\ell}(-f) = S^*_{\boldsymbol{X}_\ell}(f)$ implies $R_{\boldsymbol{X}_\ell}(\tau)$ is real;

hence, $R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau) = 0$ for any τ. □

Note that $S_{\boldsymbol{X}_\ell}(-f) = S^*_{\boldsymbol{X}_\ell}(f)$ iff $R_{\boldsymbol{X}_\ell}(\tau)$ real iff $R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau) = 0$ for any τ.

We next discuss the PSD of a system.



$$Y(t) = \int_{-\infty}^{\infty} h(\tau)\boldsymbol{X}(t-\tau)\,d\tau$$

$$m_{\boldsymbol{Y}} = m_{\boldsymbol{X}} \int_{-\infty}^{\infty} h(\tau)\,d\tau$$

$$\begin{aligned}
R_{\boldsymbol{X},\boldsymbol{Y}}(\tau) &= \mathbb{E}\left[\boldsymbol{X}(t+\tau)\left(\int_{-\infty}^{\infty} h(u)\boldsymbol{X}(t-u)\,du\right)^*\right] \\
&= \int_{-\infty}^{\infty} h^*(u)R_{\boldsymbol{X}}(\tau+u)\,du = \int_{-\infty}^{\infty} h^*(-v)R_{\boldsymbol{X}}(\tau-v)\,dv \\
&= R_{\boldsymbol{X}}(\tau) \star h^*(-\tau)
\end{aligned}$$

$$\begin{aligned}
R_{\boldsymbol{Y}}(\tau) &= \mathbb{E}\left[\left(\int_{-\infty}^{\infty} h(u)\boldsymbol{X}(t+\tau-u)\,du\right)\left(\int_{-\infty}^{\infty} h(v)\boldsymbol{X}(t-v)\,dv\right)^*\right] \\
&= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} h^*(v)R_{\boldsymbol{X}}((\tau-u)+v)\,dv\right)du \\
&= \int_{-\infty}^{\infty} h(u)R_{\boldsymbol{X},\boldsymbol{Y}}(\tau-u)\,du \\
&= R_{\boldsymbol{X},\boldsymbol{Y}}(\tau) \star h(\tau) = R_{\boldsymbol{X}}(\tau) \star h^*(-\tau) \star h(\tau).
\end{aligned}$$

Thus,

$$S_{\mathbf{X},\mathbf{Y}}(f) = S_{\mathbf{X}}(f)H^*(f) \text{ since } \int_{-\infty}^{\infty} h^*(-\tau)e^{-\imath 2\pi f\tau}d\tau = H^*(f)$$

and

$$S_{\mathbf{Y}}(f) = S_{\mathbf{X},\mathbf{Y}}(f)H(f) = S_{\mathbf{X}}(f)|H(f)|^2.$$

White process

Definition (White process)

A (WSS) process $\boldsymbol{W}(t)$ is called a white process if its PSD is constant for all frequencies:

$$S_{\boldsymbol{W}}(f) = \frac{N_0}{2}$$

- This constant is usually denoted by $\frac{N_0}{2}$ because the PSD is two-sided ($-\infty \leftarrow 0$ and $0 \rightarrow \infty$). So, the power spectral density is actually N_0 per Hz ($N_0/2$ at $f = -f_0$ and $N_0/2$ at $f = f_0$).
- The autocorrelation function $R_{\boldsymbol{W}}(\tau) = \frac{N_0}{2}\delta(\cdot)$, where $\delta(\cdot)$ is the Dirac delta function.

Why negative frequency?

- It is an imaginarily convenient way created by Human to correspond to the "imaginary" domain of a complex signal (that is why we call it "imaginary part").
- By giving respectively the spectrum for f_0 and $-f_0$ (which may not be symmetric), we can specify the amount of real part and imaginary part in time domain corresponding to this frequency.
- For example, if the spectrum is conjugate symmetric, we know imaginary part (in time domain) = 0.
- Notably, in communications, imaginary part is the part that will be modulated by (or transmitted with carrier) $\sin(2\pi f_c t)$; on the contrary, real part is the part that will be modulated by (or transmitted with carrier) $\cos(2\pi f_c t)$.

Why $\delta(\cdot)$ function?

Definition (Dirac delta function)

Define the Dirac delta function $\delta(t)$ as

$$\delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & t \neq 0 \end{cases},$$

which satisfies the replication property, i.e., for every **continuous** *point of $g(t)$,*

$$g(t) = \int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau.$$

Hence, by replication property,

$$\int_{-\infty}^{\infty} \delta(u)du = \int_{-\infty}^{\infty} \delta(t-\tau)d\tau = \int_{-\infty}^{\infty} 1 \cdot \delta(t-\tau)d\tau = 1.$$

- Note that it seems $\delta(t) = 2\delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & t \neq 0 \end{cases}$; but with $g_1(t) = 1$ and $g_2(t) = 2$ continuous at all points,

$$1 = \int_{-\infty}^{\infty} g_1(\tau)\delta(t-\tau)d\tau \neq \int_{-\infty}^{\infty} g_2(\tau)\delta(t-\tau)d\tau = 2.$$

- So, it is not "well-defined" and contradicts the below intuition: With $f(t) = \delta(t)$ and $g(t) = 2\delta(t)$,

$$f(t) = g(t) \text{ for } t \in \mathbb{R} \text{ except for countably many points}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{\infty} g(t)dt \quad \left(\text{if } \int_{-\infty}^{\infty} f(t)dt \text{ is finite}\right).$$

- Hence, $\delta(t)$ and $2\delta(t)$ are two "different" Dirac delta functions by definition. (Their multiplicative constant cannot be omitted!)
- What is the problem saying $f(t) = g(t)$ for $t \in \mathbb{R}$?

- **Comment:** $\boxed{x + a = y + a \Rightarrow x = y}$ is incorrect if $a = \infty$. As a result, saying $\boxed{\infty = \infty}$ (or $\boxed{\delta(t) = 2\delta(t)}$) is not a "rigorously defined" statement.

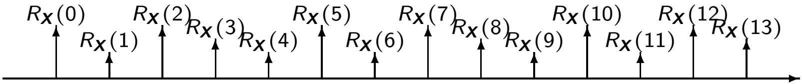
- **Summary:** The Dirac delta function, like "∞", is simply a concept *defined* only through its *replication property*.

- Hence, a white process $\boldsymbol{W}(t)$ that has autocorrelation function $R_{\boldsymbol{W}}(\tau) = \frac{N_0}{2}\delta(\tau)$ is just a convenient and simplified notion for theoretical research about real world phenomenon. Usually, $N_0 = KT$, where T is the ambient temperature in kelvins and k is Boltzman's constant.

Discrete-time random processes

- The property of a time-discrete process $\{\boldsymbol{X}[n], n \in \mathbb{Z}^+\}$ can be "obtained" using sampling notion via the Dirac delta function.
- $\boldsymbol{X}[n] = \boldsymbol{X}(nT)$, a sample at $t = nT$ from a time-continuous process $\boldsymbol{X}(t)$, where we assume $T = 1$ for convenience.
- The autocorrelation function of a time-discrete process is given by:

$$\begin{aligned} R_{\boldsymbol{X}}[m] &= \mathbb{E}\{\boldsymbol{X}[n+m]\boldsymbol{X}^*[n]\} \\ &= \mathbb{E}\{\boldsymbol{X}(n+m)\boldsymbol{X}^*(n)\} \\ &= R_{\boldsymbol{X}}(m), \text{ a sample from } R_{\boldsymbol{X}}(t). \end{aligned}$$



$$
\begin{aligned}
S_{\mathbf{X}}[f] &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} R_{\mathbf{X}}(t)\delta(t-n) \right) e^{-\imath 2\pi f t} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\mathbf{X}}(t) e^{-\imath 2\pi f t} \delta(t-n) dt \\
&= \sum_{n=-\infty}^{\infty} R_{\mathbf{X}}(n) e^{-\imath 2\pi f n} \text{ (Replication Property)} \\
&= \sum_{n=-\infty}^{\infty} R_{\mathbf{X}}[n] e^{-\imath 2\pi f n} \text{ (Fourier Series)}
\end{aligned}
$$

Hence, by Fourier sesies,

$$
R_{\mathbf{X}}[n] = \int_{-1/2}^{1/2} S_{\mathbf{X}}[f] e^{\imath 2\pi f m} df \left(= R_{\mathbf{X}}(n) = \int_{-\infty}^{\infty} S_{\mathbf{X}}(f) e^{\imath 2\pi f m} df \right).
$$

2.8 Series expansion of random processes

2.8-1 Sampling band-limited random process

Deterministic case

- A deterministic signal $x(t)$ is called band-limited if $X(f) = 0$ for all $|f| > W$.

- Shannon-Nyquist theorem: If the sampling rate $f_s \geq 2W$, then $x(t)$ can be perfectly reconstructed from samples.

  An example of such reconstruction is

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{f_s}\right) \text{sinc}\left[f_s\left(t - \frac{n}{f_s}\right)\right].$$

- Note that the above is only sufficient, not necessary.

Stochastic case

- A WSS stochastic process $\boldsymbol{X}(t)$ is said to be band-limited if its PSD $S_{\boldsymbol{X}}(f) = 0$ for all $|f| > W$.

- It follows that

$$R_{\boldsymbol{X}}(\tau) = \sum_{n=-\infty}^{\infty} R_{\boldsymbol{X}}\left(\frac{n}{2W}\right) \operatorname{sinc}\left[2W\left(\tau - \frac{n}{2W}\right)\right].$$

- In fact, this random process $\boldsymbol{X}(t)$ can be reconstructed by its random samples in the sense of mean square.

Theorem

$$\mathbb{E}\left|\boldsymbol{X}(t) - \sum_{n=-\infty}^{\infty} \boldsymbol{X}\left(\frac{n}{2W}\right) sinc\left[2W\left(t - \frac{n}{2W}\right)\right]\right|^2 = 0$$

The random samples

- Problems of using these random samples
  - These random samples $\left\{\boldsymbol{X}\left(\frac{n}{2W}\right)\right\}_{n=-\infty}^{\infty}$ are in general correlated unless $\boldsymbol{X}(t)$ is zero-mean white.

$$\begin{aligned}\mathbb{E}\left\{\boldsymbol{X}\left(\frac{n}{2W}\right)\boldsymbol{X}^*\left(\frac{m}{2W}\right)\right\} &= R_{\boldsymbol{X}}\left(\frac{n-m}{2W}\right)\\ &\neq \mathbb{E}\left\{\boldsymbol{X}\left(\frac{n}{2W}\right)\right\}\mathbb{E}\left\{\boldsymbol{X}^*\left(\frac{m}{2W}\right)\right\} = |m_{\boldsymbol{X}}|^2\end{aligned}$$

  - If $\boldsymbol{X}(t)$ is zero-mean white,

$$\begin{aligned}\mathbb{E}\left\{\boldsymbol{X}\left(\frac{n}{2W}\right)\boldsymbol{X}^*\left(\frac{m}{2W}\right)\right\} &= R_{\boldsymbol{X}}\left(\frac{n-m}{2W}\right) = \frac{N_0}{2}\delta\left(\frac{n-m}{2W}\right)\\ &= \mathbb{E}\left\{\boldsymbol{X}\left(\frac{n}{2W}\right)\right\}\mathbb{E}\left\{\boldsymbol{X}^*\left(\frac{m}{2W}\right)\right\} = |m_{\boldsymbol{X}}|^2 = 0 \text{ except } n = m.\end{aligned}$$

  - Thus, we will introduce the uncorrelated KL expansions in Slide 2-87.

2.9 Bandpass and lowpass random processes (revisited)

Definition (Filtered white noise)

A process $\boldsymbol{N}(t)$ is called a filtered white noise if its PSD equals

$$S_{\boldsymbol{N}}(f) = \begin{cases} \frac{N_0}{2}, & |f \pm f_c| < W \\ 0, & \text{otherwise} \end{cases}$$

- Applying *P4)* $S_{\boldsymbol{X}}(f) = \frac{1}{4}\left[S_{\boldsymbol{X}_\ell}(f - f_c) + S^*_{\boldsymbol{X}_\ell}(-f - f_c)\right]$, we learn the PSD of the lowpass equivalent process $\boldsymbol{N}_\ell(t)$ of $\boldsymbol{N}(t)$ is

$$S_{\boldsymbol{N}_\ell}(f) = \begin{cases} 2N_0, & |f| < W \\ 0, & \text{otherwise} \end{cases}$$

- From *P6)*, $S_{\boldsymbol{N}_\ell}(-f) = S^*_{\boldsymbol{N}_\ell}(f)$ implies $\boldsymbol{N}_i(t+\tau)$ and $\boldsymbol{N}_q(t)$ are uncorrelated for any τ if one of them has zero mean.

Now we explore more properties for PSD of bandlimited $\boldsymbol{X}(t)$ and complex $\boldsymbol{X}_\ell(t)$.

P0-1) By **fundamental assumption** on Slide 2-52, we obtain that $\boldsymbol{X}(t)$ and $\hat{\boldsymbol{X}}(t)$ are jointly WSS.

> $R_{\boldsymbol{X},\hat{\boldsymbol{X}}}(\tau)$ and $R_{\hat{\boldsymbol{X}}}(\tau)$ are only functions of τ because $\hat{\boldsymbol{X}}(t)$ is the Hilbert transform of $\boldsymbol{X}(t)$, i.e., $R_{\boldsymbol{X},\hat{\boldsymbol{X}}}(\tau) = R_{\boldsymbol{X}}(\tau) \star h^*(-\tau) = -R_{\boldsymbol{X}}(\tau) \star h(\tau)$ (since $h^*(-\tau) = -h(\tau)$) and $R_{\hat{\boldsymbol{X}}}(\tau) = R_{\boldsymbol{X},\hat{\boldsymbol{X}}}(\tau) \star h(\tau)$.

P0-2) $\boldsymbol{X}_i(t) = \mathbf{Re}\left\{(\boldsymbol{X}(t) + \imath\hat{\boldsymbol{X}}(t))e^{-\imath 2\pi f_c t}\right\}$ is WSS by **fundamental assumption**.

P2′) $\begin{cases} R_{\boldsymbol{X}}(\tau) = R_{\hat{\boldsymbol{X}}}(\tau) \\ R_{\boldsymbol{X},\hat{\boldsymbol{X}}}(\tau) = -R_{\hat{\boldsymbol{X}},\boldsymbol{X}}(\tau) \end{cases}$

($\boldsymbol{X}(t) + \imath\hat{\boldsymbol{X}}(t)$ is the "lowpass equivalent" signal of $\boldsymbol{X}_i(t)$!)

($\boldsymbol{X}_i(t) + \imath\boldsymbol{X}_q(t)$ is the lowpass equivalent signal of $\boldsymbol{X}(t)$!)

> Thus, $R_{\hat{\boldsymbol{X}},\boldsymbol{X}}(\tau) = -R_{\boldsymbol{X},\hat{\boldsymbol{X}}}(\tau) = R_{\boldsymbol{X}}(\tau) \star h(\tau) = \hat{R}_{\boldsymbol{X}}(\tau)$ is the Hilbert transform output due to input $R_{\boldsymbol{X}}(\tau)$.

$P3'$) $R_{\mathbf{X}_i}(\tau) = \mathbf{Re}\left\{\frac{1}{2}R_{(\mathbf{X}+\imath\hat{\mathbf{X}})}(\tau)e^{-\imath 2\pi f_c \tau}\right\}$

$$\begin{aligned} R_{\mathbf{X}_i}(\tau) &= \mathbf{Re}\left\{\frac{1}{2}R_{(\mathbf{X}+\imath\hat{\mathbf{X}})}(\tau)e^{-\imath 2\pi f_c\tau}\right\} \\ &= \mathbf{Re}\left\{(R_{\mathbf{X}}(\tau) + \imath R_{\hat{\mathbf{X}},\mathbf{X}}(\tau))e^{-\imath 2\pi f_c\tau}\right\} \\ &= R_{\mathbf{X}}(\tau)\cos(2\pi f_c\tau) + \hat{R}_{\mathbf{X}}(\tau)\sin(2\pi f_c\tau) \end{aligned}$$

Note that $\hat{S}_{\mathbf{X}}(f) = S_{\mathbf{X}}(f)H_{\text{Hilbert}}(f) = S_{\mathbf{X}}(f)(-\imath\,\text{sgn}(f))$.

$P4'$) $S_{\mathbf{X}_i}(f)\ \left(= S_{\mathbf{X}_q}(f)\right) = S_{\mathbf{X}}(f - f_c) + S_{\mathbf{X}}(f + f_c)$ for $|f| < f_c$

$$\begin{aligned} \boxed{S_{\mathbf{X}_i}(f)} &= \frac{1}{2}\left(S_{\mathbf{X}}(f-f_c) + S_{\mathbf{X}}(f+f_c)\right) \\ &+ \frac{1}{2\imath}\left(-\imath\,\text{sgn}(f-f_c)S_{\mathbf{X}}(f-f_c) + \imath\,\text{sgn}(f+f_c)S_{\mathbf{X}}(f+f_c)\right) \\ &= \boxed{S_{\mathbf{X}}(f-f_c) + S_{\mathbf{X}}(f+f_c)} \quad \text{for } |f| < f_c \end{aligned}$$

P4″) $S_{\boldsymbol{X}_q,\boldsymbol{X}_i}(f) = \imath\left[S_{\boldsymbol{X}}(f-f_c) - S_{\boldsymbol{X}}(f+f_c)\right]$ for $|f| < f_c$

Terminologies & relations

- $$\begin{cases} R_{\boldsymbol{X}}(\tau) = \mathbf{Re}\left\{ \frac{1}{2}R_{\boldsymbol{X}_\ell}(\tau)\, e^{\imath 2\pi f_c \tau}\right\} & (P3) \\ \underbrace{R_{\hat{\boldsymbol{X}},\boldsymbol{X}}(\tau) = R_{\boldsymbol{X}}(\tau) \star h_{\text{Hilbert}}(\tau)}_{\text{P0-1}} = \mathbf{Im}\left\{ \frac{1}{2}R_{\boldsymbol{X}_\ell}(\tau)\, e^{\imath 2\pi f_c \tau}\right\} \end{cases}$$

- Then: $\underbrace{\frac{1}{2}R_{\boldsymbol{X}_\ell}(\tau) = R_{\boldsymbol{X}_i}(\tau) + \imath R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau)}_{\text{Proof of } P3} = (R_{\boldsymbol{X}}(\tau) + \imath R_{\hat{\boldsymbol{X}},\boldsymbol{X}}(\tau))e^{-\imath 2\pi f_c \tau}$

- $$\begin{cases} R_{\boldsymbol{X}_i}(\tau) = \mathbf{Re}\left\{ (R_{\boldsymbol{X}}(\tau) + \imath R_{\hat{\boldsymbol{X}},\boldsymbol{X}}(\tau))\, e^{-\imath 2\pi f_c \tau}\right\} & (P3') \\ R_{\boldsymbol{X}_q,\boldsymbol{X}_i}(\tau) = \mathbf{Im}\left\{ (R_{\boldsymbol{X}}(\tau) + \imath R_{\hat{\boldsymbol{X}},\boldsymbol{X}}(\tau))\, e^{-\imath 2\pi f_c \tau}\right\} = R_{\boldsymbol{X}_i}(\tau) \star h_{\text{Hilbert}}(\tau) \end{cases}$$

Proof (of P4″): Hence,

$$\begin{aligned} R_{\mathbf{X}_q,\mathbf{X}_i}(\tau) &= \mathbf{Im}\left\{\left(R_{\mathbf{X}}(\tau) + \imath R_{\hat{\mathbf{X}},\mathbf{X}}(\tau)\right)e^{-\imath 2\pi f_c \tau}\right\} \\ &= -R_{\mathbf{X}}(\tau)\sin(2\pi f_c\tau) + R_{\hat{\mathbf{X}},\mathbf{X}}(\tau)\cos(2\pi f_c\tau) \\ &= -R_{\mathbf{X}}(\tau)\sin(2\pi f_c\tau) + \hat{R}_{\mathbf{X}}(\tau)\cos(2\pi f_c\tau). \end{aligned}$$

Then we can prove *P4″* by following similar procedure to the proof of *P4′*.

2.2 Signal space representation

Key idea & motivation

- The low-pass equivalent representation removes the dependence of system performance analysis on carrier frequency.
- Equivalent vectorization of the (discrete or continuous) signals further removes the "waveform" redundancy in the analysis of system performance.

Vector space concepts

- Inner product: $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \sum_{i=1}^{n} v_{1,i} v_{2,i}^* = \boldsymbol{v}_2^H \boldsymbol{v}_1$
  ("H" denotes Hermitian transpose)
- Orthogonal if $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$
- Norm: $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$
- Orthonormal: $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$ and $\|\boldsymbol{v}_1\| = \|\boldsymbol{v}_2\| = 1$
- Linearly independent:

$$\sum_{i=1}^{k} a_i \boldsymbol{v}_i = \boldsymbol{0} \text{ iff } a_i = 0 \text{ for all } i$$

Vector space concepts

- Triangle inequality

$$\left\| \boldsymbol{v}_1 + \boldsymbol{v}_2 \right\| \leq \left\| \boldsymbol{v}_1 \right\| + \left\| \boldsymbol{v}_2 \right\|$$

- Cauchy-Schwartz inequality

$$\left| \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle \right| \leq \left\| \boldsymbol{v}_1 \right\| \left\| \boldsymbol{v}_2 \right\| .$$

  Equality holds iff $\boldsymbol{v}_1 = a\boldsymbol{v}_2$ for some a.

- Norm square of sum:

$$\left\| \boldsymbol{v}_1 + \boldsymbol{v}_2 \right\|^2 = \left\| \boldsymbol{v}_1 \right\|^2 + \left\| \boldsymbol{v}_2 \right\|^2 + \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle + \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle$$

- Pythagorean: if $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$, then

$$\left\| \boldsymbol{v}_1 + \boldsymbol{v}_2 \right\|^2 = \left\| \boldsymbol{v}_1 \right\|^2 + \left\| \boldsymbol{v}_2 \right\|^2$$

Eigen-decomposition

1. Matrix transformation w.r.t. matrix A

$$\hat{\boldsymbol{v}} = A\boldsymbol{v}$$

2. Eigenvalues of square matrix A are solutions $\{\lambda\}$ of characteristic polynomial

$$\det(A - \lambda I) = 0$$

3. Eigenvectors for eigenvalue λ is solution $\boldsymbol{v}$ of

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

Signal space concept

How to extend the signal space concept to a (complex) function/signal $z(t)$ defined over $[0, T)$?

Answer: We can start by defining the inner product for complex functions.

- Inner product: $\langle z_1(t), z_2(t)\rangle = \int_0^T z_1(t) z_2^*(t)\, dt$
- Orthogonal if $\langle z_1(t), z_2(t)\rangle = 0$.
- Norm: $\|z(t)\| = \sqrt{\langle z(t), z(t)\rangle}$
- Orthonormal: $\langle z_1(t), z_2(t)\rangle = 0$ and $\|z_1(t)\| = \|z_2(t)\| = 1$.
- Linearly independent: $\sum_{i=1}^k a_i z_i(t) = 0$ iff $a_i = 0$ for all $a_i \in \mathbb{C}$

- Triangle Inequality

$$\|z_1(t) + z_2(t)\| \le \|z_1(t)\| + \|z_2(t)\|$$

- Cauchy Schwartz inequality

$$|\langle z_1(t), z_2(t)\rangle| \le \|z_1(t)\| \cdot \|z_2(t)\|$$

  Equality holds iff $z_1(t) = a \cdot z_2(t)$ for some $a \in \mathbb{C}$.

- Norm square of sum:

$$\|z_1(t) + z_2(t)\|^2 = \|z_1(t)\|^2 + \|z_2(t)\|^2 + \langle z_1(t), z_2(t)\rangle + \langle z_2(t), z_1(t)\rangle$$

- Pythagorean property: if $\langle z_1(t), z_2(t)\rangle = 0$,

$$\|z_1(t) + z_2(t)\|^2 = \|z_1(t)\|^2 + \|z_2(t)\|^2$$

- Transformation w.r.t. a function $C(t,s)$

$$\hat{z}(t) = \int_0^T C(t,s)z(s)\,ds$$

This is in parallel to

$$\hat{\boldsymbol{v}} \quad \left(\hat{v}_t = \sum_{s=1}^{n} A_{t,s} v_s\right) \quad = A\boldsymbol{v}.$$

Eigenvalues and eigenfunctions

Given a complex continuous function $C(t,s)$ over $[0,T)^2$, the eigenvalues and eigenfunctions are $\{\lambda_k\}$ and $\{\varphi_k(t)\}$ such that

$$\int_0^T C(t,s)\varphi_k(s)\,ds = \lambda_k\varphi_k(t) \quad \text{(In parallel to } A\boldsymbol{v} = \lambda\boldsymbol{v}\text{)}$$

Mercer's theorem

Theorem (Mercer's theorem)

Give a complex continuous function $C(t,s)$ over $[0,T]^2$ that is Hermitian symmetric (i.e., $C(t,s) = C^*(s,t)$) and nonnegative definite (i.e., $\sum_i \sum_j a_i C(t_i,t_j) a_j^* \geq 0$ for any $\{a_i\}$ and $\{t_i\}$). Then the eigenvalues $\{\lambda_k\}$ are reals, and $C(t,s)$ has the following eigen-decomposition

$$C(t,s) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(t) \varphi_k^*(s).$$

Karhunen-Loève theorem

Theorem (Karhunen-Loève theorem)

Let $\{Z(t), t \in [0, T)\}$ *be a zero-mean random process with a continuous autocorrelation function* $R_Z(t,s) = \mathbb{E}[Z(t)Z^*(s)]$. *Then* $Z(t)$ *can be written as*

$$Z(t) \overset{\mathcal{M}_2}{=} \sum_{k=1}^{\infty} Z_k \cdot \varphi_k(t) \quad 0 \le t < T$$

where "=" is in the sense of mean-square,

$$Z_k = \langle Z(t), \varphi_k(t) \rangle = \int_0^T Z(t)\varphi_k^*(t)\,dt$$

and $\{\varphi_k(t)\}$ *are orthonormal eigenfunctions of* $R_Z(t,s)$.

- Merit of KL expansion: $\{Z_k\}$ are uncorrelated. (But samples $\{Z(\,k/(2W)\,)\}$ are not uncorrelated even if $Z(t)$ is bandlimited!)

Proof.

$$
\begin{aligned}
\mathbb{E}[Z_i Z_j^*] &= \mathbb{E}\left[\left(\int_0^T Z(t)\varphi_i^*(t)dt\right)\left(\int_0^T Z(s)\varphi_j^*(s)ds\right)^*\right] \\
&= \int_0^T \left(\int_0^T R_Z(t,s)\varphi_j(s)ds\right)\varphi_i^*(t)dt \\
&= \int_0^T \lambda_j \varphi_j(t)\varphi_i^*(t)dt \\
&= \begin{cases} \lambda_j & \text{if } i = j \\ 0 \quad \left(= \mathbb{E}[Z_i]E[Z_j^*]\right) & \text{if } i \neq j \end{cases}
\end{aligned}
$$

□

Lemma

For a given orthonormal set $\{\phi_k(t)\}$, how to minimize the energy of error signal $e(t) = s(t) - \hat{s}(t)$ for $\hat{s}(t)$ spanned by (i.e., expressed as a linear combination of) $\{\phi_k(t)\}$?

Assume $\hat{s}(t) = \sum_k a_k\phi_k(t)$; then

$$
\begin{aligned}
\|e(t)\|^2 &= \|s(t) - \hat{s}(t)\|^2 \\
&= \|s(t) - \textstyle\sum_k a_k\phi_k(t)\|^2 \\
&= \|s(t)\|^2 - \textstyle\sum_k\langle s(t), a_k\phi_k(t)\rangle - \sum_k\langle a_k\phi_k(t), s(t)\rangle + \sum_k|a_k|^2 \\
&= \|s(t)\|^2 - \textstyle\sum_k a_k^*\langle s(t), \phi_k(t)\rangle - \sum_k a_k(\langle s(t), \phi_k(t)\rangle)^* + \sum_k|a_k|^2 \\
&= \|s(t)\|^2 - \textstyle\sum_k|\langle s(t), \phi_k(t)\rangle|^2 + \sum_k\|a_k - \langle s(t), \phi_k(t)\rangle\|^2
\end{aligned}
$$

Thus, $a_k = \langle s(t), \phi_k(t)\rangle$ minimizes $\|e(t)\|^2$. □

Definition

If every finite energy signal $s(t)$ *(i.e.,* $\|s(t)\|^2 < \infty$*) satisfies*

$$\|e(t)\|^2 = \left\| s(t) - \sum_k \langle s(t), \phi_k(t) \rangle \phi_k(t) \right\|^2 = 0$$

equivalently,

$$s(t) \overset{\mathcal{L}_2}{=} \sum_k \langle s(t), \phi_k(t) \rangle \phi_k(t) = \sum_k a_k \cdot \phi_k(t)$$

(in the sense that the norm of the difference between left-hand-side and right-hand-side is zero), then the set of orthonormal functions $\{\phi_k(t)\}$ *is said to be complete.*

Example (Fourier series)

$$\left\{\sqrt{\frac{2}{T}}\cos\left(\frac{2\pi kt}{T}\right), \sqrt{\frac{2}{T}}\sin\left(\frac{2\pi kt}{T}\right) : 0 \le k \in \mathbb{Z}\right\}$$

is a complete orthonormal set for signals defined over $[0, T)$ *with finite number of discontinuities.* □

- For a complete orthonormal basis, the energy of $s(t)$ is equal to

$$\begin{aligned}
\|s(t)\|^2 &= \left\langle \sum_j a_j \phi_j(t), \sum_k a_k \phi_k(t) \right\rangle \\
&= \sum_j \sum_k a_j a_k^* \langle \phi_j(t), \phi_k(t) \rangle \\
&= \sum_j a_j a_j^* \\
&= \sum_j |a_j|^2
\end{aligned}$$

- Given a deterministic function $s(t)$, and a set of complete orthonormal basis $\{\phi_k(t)\}$ (possibly countably infinite), $s(t)$ can be written as

$$s(t) \stackrel{\mathcal{L}_2}{=} \sum_{k=0}^{\infty} a_k \phi_k(t), \quad 0 \le t \le T$$

where

$$a_k = \langle s(t), \phi_k(t) \rangle = \int_0^T s(t) \phi_k^*(t)\, dt.$$

In addition,

$$\|s(t)\|^2 = \sum_k |a_k|^2.$$

Remark

In terms of energy (and error rate):

- A bandpass signal $s(t)$ can be equivalently "analyzed" through lowpass equivalent signal $s_\ell(t)$ without the burden of carrier freq f_c;

- A lowpass equivalent signal $s_\ell(t)$ can be equivalently "analyzed" through (countably many) $\{a_k = \langle s_\ell(t), \phi_k(t)\rangle\}$ without the burden of continuous waveforms.

Gram-Schmidt procedure

Given a set of functions $v_1(t), v_2(t), \cdots, v_k(t)$

1. $\phi_1(t) = \frac{v_1(t)}{\|v_1(t)\|}$

2. Compute for $i = 2, 3, \cdots, k$ (or until $\|\phi_i(t)\| = 0$),

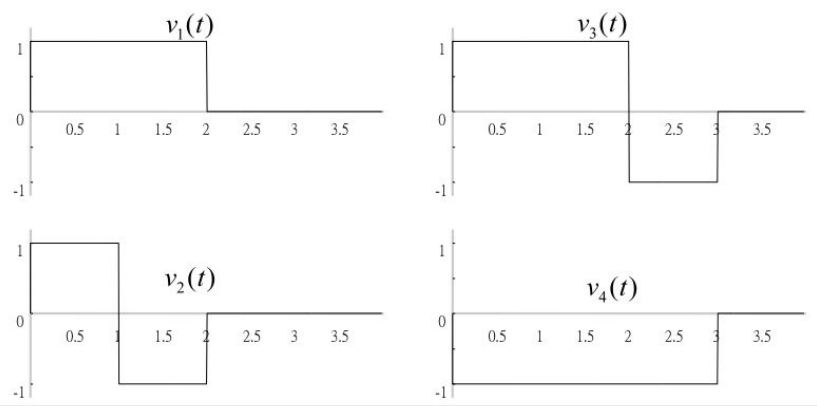
$$\gamma_i(t) = v_i(t) - \sum_{j=1}^{i-1} \langle v_i(t), \phi_j(t) \rangle \phi_j(t)$$

and set $\phi_i(t) = \frac{\gamma_i(t)}{\|\gamma_i(t)\|}$.

This gives an orthonormal basis $\phi_1(t), \phi_2(t), \cdots, \phi_{k'}(t)$, where $k' \leq k$.

Example

Find a Gram-Schmidt orthonormal basis of the following signals.



Sol.

- $\phi_1(t) = \frac{v_1(t)}{\|v_1(t)\|} = \frac{v_1(t)}{\sqrt{2}}$
- $$\begin{aligned}\gamma_2(t) &= v_2(t) - \langle v_2(t), \phi_1(t)\rangle \phi_1(t) \\ &= v_2(t) - \left(\int_0^3 v_2(t)\phi_1^*(t)\, dt\right)\phi_1(t) = v_2(t)\end{aligned}$$

  Hence $\phi_2(t) = \frac{\gamma_2(t)}{\|\gamma_2(t)\|} = \frac{v_2(t)}{\sqrt{2}}$.
- $$\begin{aligned}\gamma_3(t) &= v_3(t) - \langle v_3(t), \phi_1(t)\rangle \phi_1(t) - \langle v_3(t), \phi_2(t)\rangle \phi_2(t) \\ &= v_3(t) - \sqrt{2}\phi_1(t) - 0 \cdot \phi_2(t) = \begin{cases} -1, & 2 \le t < 3 \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

  Hence $\phi_3(t) = \frac{\gamma_3(t)}{\|\gamma_3(t)\|}$.

-

$$\begin{aligned}\gamma_4(t) &= v_4(t) - \langle v_4(t), \phi_1(t)\rangle \phi_1(t) - \langle v_4(t), \phi_2(t)\rangle \phi_2(t) \\ &\quad - \langle v_4(t), \phi_3(t)\rangle \phi_3(t) \\ &= v_4(t) - (-\sqrt{2})\phi_1(t) - (0)\phi_2(t) - \phi_3(t) = 0\end{aligned}$$

- Orthonormal basis$=\{\phi_1(t), \phi_2(t), \phi_3(t)\}$, where $3 \le 4$.

Example

Represent the signals in Slide 2-95 in terms of the orthonormal basis obtained in the same example.

Sol.

$$
\begin{aligned}
v_1(t) &= \sqrt{2}\phi_1(t) + 0\cdot\phi_2(t) + 0\cdot\phi_3(t) \Longrightarrow \left[\sqrt{2}, 0, 0\right] \\
v_2(t) &= 0\cdot\phi_1(t) + \sqrt{2}\cdot\phi_2(t) + 0\cdot\phi_3(t) \Longrightarrow \left[0, \sqrt{2}, 0\right] \\
v_3(t) &= \sqrt{2}\phi_1(t) + 0\cdot\phi_2(t) + \cdot\phi_3(t) \Longrightarrow \left[\sqrt{2}, 0, 1\right] \\
v_4(t) &= -\sqrt{2}\phi_1(t) + 0\cdot\phi_2(t) + 1\cdot\phi_3(t) \Longrightarrow \left[-\sqrt{2}, 0, 1\right]
\end{aligned}
$$

□

The vectors are named signal space representations or constellations of the signals.

Remark

The orthonormal basis is not unique.
For example, for $k = 1, 2, 3$, re-define

$$\phi_k(t) = \begin{cases} 1, & k-1 \le t < k \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} v_1(t) &\stackrel{\Phi}{\Longrightarrow} (+1, +1, 0) \\ v_2(t) &\stackrel{\Phi}{\Longrightarrow} (+1, -1, 0) \\ v_3(t) &\stackrel{\Phi}{\Longrightarrow} (+1, +1, -1) \\ v_4(t) &\stackrel{\Phi}{\Longrightarrow} (-1, -1, -1) \end{aligned}$$

Euclidean distance

$$\begin{aligned} s_1(t) &\Longrightarrow (a_1, a_2, \cdots, a_n) \text{ for some complete basis} \\ s_2(t) &\Longrightarrow (b_1, b_2, \cdots, b_n) \text{ for the same complete basis} \end{aligned}$$

$$\begin{aligned} d_{12} &= \text{Euclidean distance between } s_1(t) \text{ and } s_2(t) \\ &= \sqrt{\sum_{i=1}^{n} |a_i - b_i|^2} \\ &= \|s_1(t) - s_2(t)\| \quad \left(= \sqrt{\int_0^T |s_1(t) - s_2(t)|^2 dt} \right) \end{aligned}$$

Bandpass and lowpass orthonormal basis

- Now let's change our focus from $[0, T)$ to $(-\infty, \infty)$

- A time-limited signal cannot be bandlimited to W.
- A bandlimited signal cannot be time-limited to T.

Hence, in order to talk about the ideal bandlimited signal, we have to deal with signals with unlimited time span.

- Re-define the inner product as:

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)g^*(t)\, dt$$

- Let $s_{1,\ell}(t)$ and $s_{2,\ell}(t)$ be lowpass equivalent signals of the bandpass $s_1(t)$ and $s_2(t)$, satisfying

$$S_{1,\ell}(f) = S_{2,\ell}(f) = 0 \text{ for } |f| > f_B$$
$$s_i(t) = \mathbf{Re}\left\{s_{i,\ell}(t)e^{\imath 2\pi f_c t}\right\} \text{ where } f_c \gg f_B$$

Then, as we have proved in Slide 2-24,

$$\boxed{\langle s_1(t), s_2(t)\rangle} = \boxed{\frac{1}{2}\mathbf{Re}\left\{\langle s_{1,\ell}(t), s_{2,\ell}(t)\rangle\right\}}.$$

Proposition

If $\langle s_{1,\ell}(t), s_{2,\ell}(t)\rangle = 0$, then $\langle s_1(t), s_2(t)\rangle = 0$.

Proposition

If $\{\phi_{n,\ell}(t)\}$ is a complete basis for the set of lowpass signals, then

$$\begin{cases} \phi_n(t) = \mathbf{Re}\left\{\left(\sqrt{2}\phi_{n,\ell}(t)\right)e^{\imath 2\pi f_c t}\right\} \\ \tilde{\phi}_n(t) = -\mathbf{Im}\left\{\left(\sqrt{2}\phi_{n,\ell}(t)\right)e^{\imath 2\pi f_c t}\right\} \\ \qquad\quad = \mathbf{Re}\left\{\left(\imath\sqrt{2}\phi_{n,\ell}(t)\right)e^{\imath 2\pi f_c t}\right\} \end{cases}$$

is a complete orthonormal set for the set of bandpass signals.

Proof: First, orthonormality can be proved by

$$\langle \phi_n(t), \phi_m(t)\rangle = \frac{1}{2}\mathbf{Re}\left\{\left\langle \sqrt{2}\phi_{n,\ell}(t), \sqrt{2}\phi_{m,\ell}(t)\right\rangle\right\} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\langle \tilde{\phi}_n(t), \tilde{\phi}_m(t)\rangle = \frac{1}{2}\mathbf{Re}\left\{\left\langle \imath\sqrt{2}\phi_{n,\ell}(t), \imath\sqrt{2}\phi_{m,\ell}(t)\right\rangle\right\} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

and

$$\begin{aligned}\left\langle \tilde{\phi}_n(t), \phi_m(t) \right\rangle &= \frac{1}{2}\mathbf{Re}\left\{\left\langle \imath\sqrt{2}\phi_{n,\ell}(t), \sqrt{2}\phi_{m,\ell}(t)\right\rangle\right\} \\ &= \mathbf{Re}\left\{ \imath \left\langle \phi_{n,\ell}(t), \phi_{m,\ell}(t)\right\rangle\right\} \\ &= \begin{cases} \mathbf{Re}\left\{ \imath \right\} = 0 & n = m \\ 0 & n \neq m \end{cases}\end{aligned}$$

Now, with $\begin{cases} s(t) = \mathbf{Re}\left\{ s_\ell(t) e^{\imath 2\pi f_c t}\right\} \\ \hat{s}(t) = \mathbf{Re}\left\{ \hat{s}_\ell(t) e^{\imath 2\pi f_c t}\right\} \\ \hat{s}_\ell(t) \stackrel{\mathcal{L}_2}{=} \sum_n a_{n,\ell}\phi_{n,\ell}(t) \text{ with } a_{n,\ell} = \langle s_\ell(t), \phi_{n,\ell}(t)\rangle \\ \left\| s_\ell(t) - \hat{s}_\ell(t) \right\|^2 = 0 \end{cases}$

we have

$$\left\| s(t) - \hat{s}(t) \right\|^2 = \frac{1}{2}\left\| s_\ell(t) - \hat{s}_\ell(t) \right\|^2 = 0$$

and

$$
\begin{aligned}
\hat{s}(t) &= \mathbf{Re}\left\{\hat{s}_\ell(t)e^{\imath 2\pi f_c t}\right\} \\
&= \mathbf{Re}\left\{\sum_n a_{n,\ell}\phi_{n,\ell}(t)e^{\imath 2\pi f_c t}\right\} \\
&= \sum_n \left(\mathbf{Re}\left\{\frac{a_{n,\ell}}{\sqrt{2}}\right\}\mathbf{Re}\left\{\sqrt{2}\phi_{n,\ell}(t)e^{\imath 2\pi f_c t}\right\}\right. \\
&\quad \left. +\mathbf{Im}\left\{\frac{a_{n,\ell}}{\sqrt{2}}\right\}\mathbf{Im}\left\{\left(-\sqrt{2}\phi_{n,\ell}(t)\right)e^{\imath 2\pi f_c t}\right\}\right) \\
&= \sum_n \left(\mathbf{Re}\left\{\frac{a_{n,\ell}}{\sqrt{2}}\right\}\phi_n(t) + \mathbf{Im}\left\{\frac{a_{n,\ell}}{\sqrt{2}}\right\}\tilde{\phi}_n(t)\right)
\end{aligned}
$$

□

What you learn from Chapter 2

- Random process
  - WSS
  - autocorrelation and crosscorrelation functions
  - PSD and CSD
  - White and filtered white
- Relation between (bandlimited) bandpass and lowpass equivalent deterministic signals
- Relation between (bandlimited) bandpass and lowpass equivalent random signals
  - Properties of autocorrelation and power spectrum density
- Role of Hilbert transform
- Signal space concept