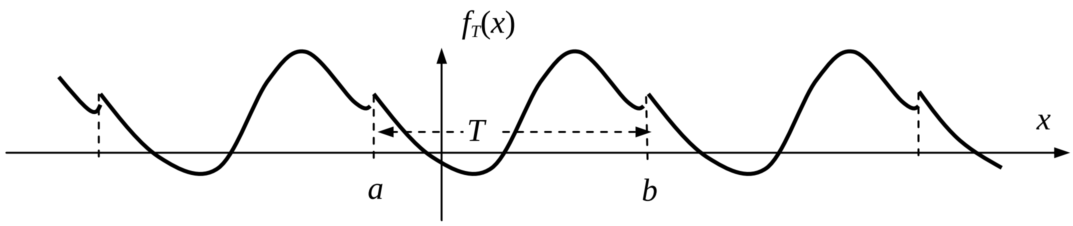
9. Special Topics

A. Fourier Series

The Fourier series was proposed by French mathematician Jean Baptiste Joseph Fourier (1768-1830) and mainly applied to periodic functions and extended to finite duration function.



The figure shows is a periodic function $f_T(x)$, $-\infty<x<\infty$, with period $T=b-a$, which can be expressed as

(1) $$f_T(x)=A_0+\sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi x}{T}+B_k sin\frac{2k\pi x}{T}\right)$$

or

(2) $$f_T(x)=A_0+\sum_{k=1}^{\infty}\left(A_k cosk\omega_0 x+B_k sink\omega_0 x\right)$$

where $x\in(-\infty,\ \infty)$ and $\omega_0=\dfrac{2\pi}{T}$ is the fundamental frequency. For the coefficients, they can be expressed as

(3) $$A_0=\frac{1}{T}\int_a^b f_T(x)d\,x=\frac{1}{T}\int_T f_T(x)d\,x$$

(4) $$A_k=\frac{2}{T}\int_T f_T(x)cos\frac{2k\pi x}{T}d\,x=\frac{2}{T}\int_T f_T(x)cos\,k\omega_0 x\,d\,x$$

(5) $$B_k=\frac{2}{T}\int_T f_T(x)sin\frac{2k\pi x}{T}d\,x=\frac{2}{T}\int_T f_T(x)sin\,k\omega_0 x\,d\,x$$

It has been proved that Fourier series could represent a periodic function if during one period T, the periodic function $f_T(x)$ satisfies the Dirichlet conditions:

(C1) The number of discontinuous points is finite.
(C2) The number of maximum and minimum points is finite.
(C3) The function is absolutely integrable.

One interesting property should be highlighted here. Although each of the infinite number of sinusoidal functions described in (1) is continuous, their sum may be discontinuous just like the function $f_T(x)$ depicted in the figure.

Let's derive the coefficients A_0, A_k and B_k, shown in (2), (3) and (4). For A_0, taking the integral of (1) in one period yields

(6) $$\int_a^b f_P(x)dx = \int_a^b A_0 dt + \sum_{k=1}^{\infty} A_k \int_a^b \cos k\omega_0 x\,dx + \sum_{k=1}^{\infty} B_k \int_a^b \sin k\omega_0 x\,dx$$

Since $\int_a^b \cos k\omega_0 x\,dx = 0$ and $\int_a^b \sin k\omega_0 x\,dx = 0$, we have

(7) $$\int_a^b f_P(x)dx = \int_a^b A_0 dx = A_0(b-a) = A_0 T$$

which results in

(8) $$A_0 = \frac{1}{T}\int_a^b f_T(x)dx = \frac{1}{T}\int_T f_T(x)dx$$

Here, we have denoted $\int_a^b$ as $\int_T$ to emphasize that the integration can be calculated in any duration of one period T, not just in $[a,b]$. Clearly, A_0 is the mean value of $f_T(x)$ for one period T. For A_k, $k>0$, we take the following integration in one period, which is expressed as

(9) $$\begin{aligned}\int_T f_T(x)\cos m\omega_0 x\,dx &= \int_T A_0 \cos m\omega_0 x\,dx \\ &\quad + \sum_{k=1}^{\infty} A_k \int_T \cos k\omega_0 x \cdot \cos m\omega_0 x\,dx \\ &\quad + \sum_{k=1}^{\infty} B_k \int_T \sin k\omega_0 x \cdot \cos m\omega_0 x\,dx\end{aligned}$$

where $m \in N$. It is known that

(10) $$\int_T A_0 \cos m\omega_0 x\,dx = 0$$

(11) $$\begin{aligned}&\int_T \cos k\omega_0 x \cdot \cos m\omega_0 x\,dx \\ &= \frac{1}{2}\int_T \left(\cos(k+m)\omega_0 x + \cos(k-m)\omega_0 x\right)dx \\ &= \frac{1}{2}\int_T \cos(k-m)\omega_0 x\,dx = \begin{cases} T/2, & k=m \\ 0, & k \neq m \end{cases}\end{aligned}$$

(12) $$\begin{aligned}&\int_T \sin k\omega_0 x \cdot \cos m\omega_0 x\,dx \\ &= \frac{1}{2}\int_T \left(\sin(k+m)\omega_0 x + \sin(k-m)\omega_0 x\right)dx = 0\end{aligned}$$

Hence, (9) becomes

(13) $$\int_T f_T(x)\cos m\omega_0 x\,dx = \sum_{k=1}^{\infty} A_k \int_T \cos k\omega_0 x \cdot \cos m\omega_0 x\,dx = \frac{T}{2}A_m$$

where the integration does not vanish only for $k=m$. As a result, we have $A_m = \frac{2}{T}\int_T f_T(x)\cos m\omega_0 x\,dx$, for m=1,2,3,…, same as the expression in (4).

Similarly, B_k can be determined from the following integration

(14) $$\begin{aligned}\int_T f_T(x)\sin m\omega_0 x\,dx &= \int_T A_0 \sin m\omega_0 x\,dx \\ &+ \sum_{k=1}^{\infty} A_k \int_T \cos k\omega_0 x \cdot \sin m\omega_0 x\,dx \\ &+ \sum_{k=1}^{\infty} B_k \int_T \sin k\omega_0 x \cdot \sin m\omega_0 x\,dx\end{aligned}$$

where $m \in N$. It is known that

(15) $$\int_T A_0 \sin m\omega_0 x\,dx = 0$$

(16) $$\begin{aligned}&\int_T \cos k\omega_0 x \cdot \sin m\omega_0 x\,dx \\ &= \frac{1}{2}\int_T \left(\sin(k+m)\omega_0 x - \sin(k-m)\omega_0 x\right)dx = 0\end{aligned}$$

(17) $$\begin{aligned}&\int_T \sin k\omega_0 x \cdot \sin m\omega_0 x\,dx \\ &= \frac{1}{2}\int_T \left(\cos(k-m)\omega_0 x - \cos(k+m)\omega_0 x\right)dx \\ &= \frac{1}{2}\int_T \cos(k-m)\omega_0 x\,dx = \begin{cases} T/2, & k=m \\ 0, & k \neq m\end{cases}\end{aligned}$$

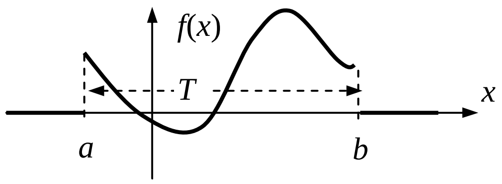
Hence, (14) becomes

(18) $$\int_T f_T(x)\sin m\omega_0 x\,dx = \sum_{k=1}^{\infty} B_k \int_T \sin k\omega_0 x \cdot \sin m\omega_0 x\,dx = \frac{T}{2}B_m$$

where the integration does not vanish only for $k=m$. As a result, we have $B_m = \frac{2}{T}\int_T f_T(x)\sin m\omega_0 x\,dx$, for m=1,2,3,…, same as the expression in (5).

In conclusion, any periodic function $f_T(x)$ satisfying the Dirichlet conditions (C1) to (C3) can be expressed as the Fourier series (1) or (2). On the other hand, the coefficents A_0, A_k and B_k in the Fourier series can be also used to represent $f_T(x)$, which will be further extended to the frequency spectrum of $f_T(x)$.

Actually, Fourier series (1) can be also used to represent a function with finite time duration, such as the function $f(x)$ shown in the figure to the right. Since $f(x)$ is equal to $f_T(x)$ for $x\in(a,b)$, it can be expressed as



(19) $$f(x) = A_0 + \sum_{k=1}^{\infty}(A_k cosk\omega_0 x + B_k sink\omega_0 x), \quad \text{for } x\in(a,\, b)$$

which is the same as (1) except that x is limited in the duration (a, b).

If $f(x)$ is an eve function, then its Fourier series only possesses the terms of cosine function, called the Fourier cosine series and shown as

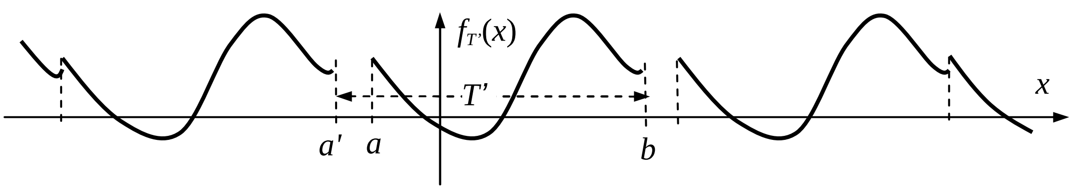
(20) $$f(x) = A_0 + \sum_{k=1}^{\infty} A_k cosk\omega_0 x, \quad \text{for } x\in(a,\, b)$$

On the other hand, if $f(x)$ is an odd function, it can be represented by the Fourier sine series as below:

(21) $$f(x) = \sum_{k=1}^{\infty} B_k sink\omega_0 x, \quad \text{for } x\in(a,\, b)$$

Later, they will be applied to the boundary value problem (BVP) of partial differential equations.

However, (19) is not a unique expression for the finite duration function $f(x)$. This can be seen from the figure below, in which $f(x)$ is also a part of the periodic function $f_{T'}(x)$, different to $f_T(x)$.



From the figure, since the period of $f_{T'}(x)$ is $T'=b-a'$, not $T=b-a$, we can write $f_{T'}(x)$ as

(22) $$f_{T'}(t) = A'_0 + \sum_{k=1}^{\infty}\left(A'_k cos\frac{2k\pi x}{T'} + B'_k sin\frac{2k\pi x}{T'}\right) \quad \text{for } x\in(-\infty,\, \infty)$$

and thus,

(23) $$f(x)=A'_0+\sum_{k=1}^{\infty}\left(A'_k cos\frac{2k\pi x}{T'}+B'_k sin\frac{2k\pi x}{T'}\right) \text{ for } x\in(a, b)$$

From (19) and (23), we can conclude that the Fourier series of a finite duration function is not unique.

For convenience, the Fourier series (1) in trigonometric form is often changed into the complex form based on

(24) $$cos\, k\omega_0 x=\frac{1}{2}\left(e^{jk\omega_0 x}+e^{-jk\omega_0 x}\right)$$

(25) $$sin\, k\omega_0 x=\frac{1}{2j}\left(e^{jk\omega_0 x}-e^{-jk\omega_0 x}\right)$$

Substituting them into (1) yields

(26) $$f_T(x)=A_0+\sum_{k=1}^{\infty}\left(\frac{A_k}{2}\left(e^{jk\omega_0 x}+e^{-jk\omega_0 x}\right)+\frac{B_k}{2j}\left(e^{jk\omega_0 x}-e^{-jk\omega_0 x}\right)\right)$$

or

(27) $$f_T(x)=c_0+\sum_{k=1}^{\infty}\left(c_k e^{jk\omega_0 x}+c_{-k}e^{-jk\omega_0 x}\right)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 x}$$

where

(28) $$c_k=\frac{1}{2}(A_k-jB_k)=|c_k|\,e^{j\phi_k}$$

where $|c_k|=|c_{-k}|=\frac{1}{2}\sqrt{A_k^2+B_k^2}$, $\phi_k=-\phi_{-k}=-tan^{-1}\left(\frac{B_k}{A_k}\right)$ and $c_{-k}=c_k^*$.

Note that the amplitude $|c_k|$ is an even function and the phase ϕ_k is an odd function.

The amplitude $|c_k|$ and the phase ϕ_k are respectively related to the frequency $\omega=k\omega_0$. If we draw the function $|c_k|$ with respect to $\omega=k\omega_0$ then we have the amplitude spectrum of $f_T(x)$. Similarly, if we draw the function ϕ_k with respect to $\omega=k\omega_0$ then we have the phase spectrum of $f_T(x)$. Both spectra are unique and can be used to represent the function $f_T(x)$. Since $|c_k|$ and ϕ_k only exist at integer k, their spectra are called discrete frequency spectrum.

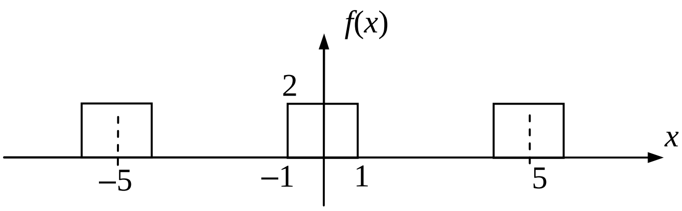
In addition to (28), the coefficient c_k can be also obtained by the following process:

(29) $$\int_T f_T(x)e^{-jm\omega_0 x}dx = \sum_{k=-\infty}^{\infty} c_k \int_T e^{j(k-m)\omega_0 x}dx = Tc_m$$

where $\int_P e^{j(k-m)\omega_0 x}dx = 0$ for $k \neq m$. Hence,

(30) $$c_k = \frac{1}{T}\int_T f_T(x)e^{-jk\omega_0 x}dx$$

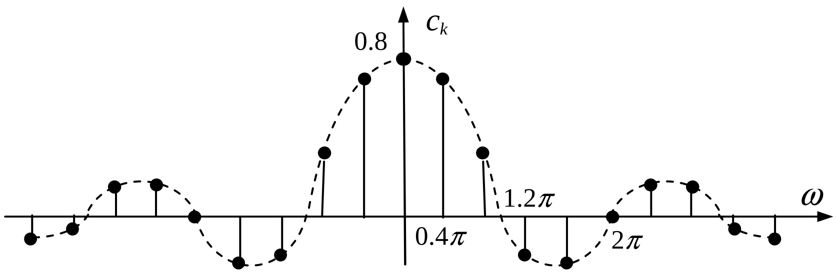
which is often used to calculate the coefficients of Fourier series in complex form.

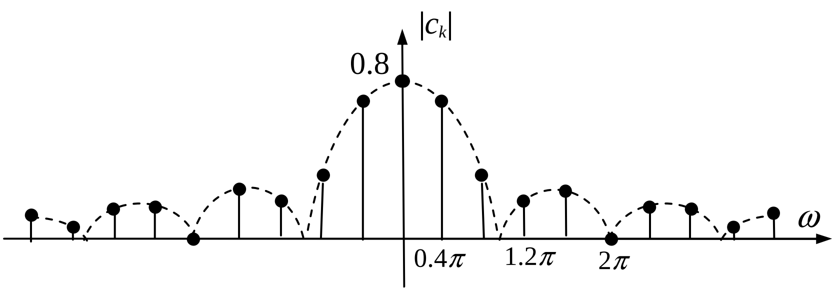


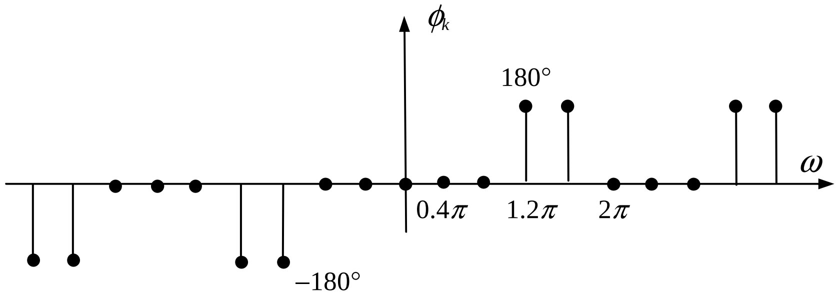
Above shows a periodic function. Since the period T=5, we have $\omega_0=2\pi/T=0.4\pi$. From (30), it can be attained that

(31) $$\begin{aligned} c_k &= \frac{1}{T}\int_T f(x)e^{-jk\omega_0 x}\,d\,x = \frac{1}{5}\int_{-1}^{1}\left(2e^{-j0.4k\pi x}\right)d\,x \\ &= \frac{2\,sin(0.4k\pi)}{k\pi} = 0.8\,sin\,c(0.4k\pi) \end{aligned}$$

where c_k is a real number. The coefficients c_k, the amplitude $|c_k|$ and the phase ϕ_k are depicted in the following figures.







The power of a periodic function $f_T(x)$ is defined as $f_T^2(x)$. Thus, the average power of $f_T(x)$ is often described by the mean-square value, given as

(32) $$P=\frac{1}{T}\int_T f_T^2(x)dx$$

Based on the Fourier series, we have

(33) $$\begin{aligned} P&=\frac{1}{T}\int_T\left(\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 x}\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_0 x}\right)dx \\ &=\frac{1}{T}\int_T\left(\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 x}\sum_{m=-\infty}^{\infty}c_{-m} e^{-jm\omega_0 x}\right)dx \end{aligned}$$

It can be rearranged as

(34) $$\begin{aligned} P&=\frac{1}{T}\int_T\left(\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}c_k c_{-m} e^{j(k-m)\omega_0 x}\right)dx \\ &=\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{T}\int_T c_k c_{-m} e^{j(k-m)\omega_0 x}dx \end{aligned}$$

It is known that $\int_T c_k c_{-m} e^{j(k-m)\omega_0 x}dx = 0$ for $k\neq m$. Hence, (34) can be rewritten as

(35) $$P=\sum_{k=-\infty}^{\infty}\frac{1}{T}\int_T c_k c_{-k}dx=\sum_{k=-\infty}^{\infty}c_k c_{-k}=\sum_{k=-\infty}^{\infty}\left|c_k\right|^2$$

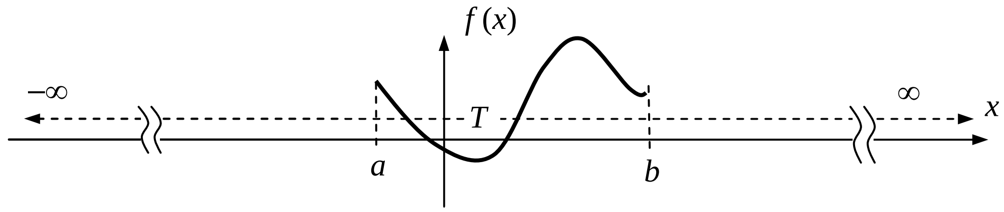
Finally, we obtain the Parseval's theorem as

(36) $$P=\frac{1}{T}\int_T f_T^2(x)dx=\sum_{k=-\infty}^{\infty}\left|c_k\right|^2$$

which is the average power of a periodic function.

B. Fourier Transform

Previously, we have shown that there is an infinite number of Fourier series to express a finite duration function. Now, one question is raised: Can we find a unique expression for any function which satisfies Dirichlet conditions? The answer is "Yes".



Any finite duration function $f(x)$ can be treated as a periodic function with period T=∞. Then, according to Fourier series we have

(1) $$f(x)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 x}$$

where

(2) $$c_k=\frac{1}{T}\int_T f(x)e^{-jk\omega_0 x}\,dx$$

with $T=\dfrac{2\pi}{\omega_0}\to\infty$. Hence, we can rewrite (1) as

(3) $$f(x)=\sum_{\substack{k=-\infty\\T\to\infty}}^{\infty}\left(\frac{\omega_0}{2\pi}\int_{-T/2}^{T/2} f(\tau)e^{-jk\omega_0\tau}\,d\tau\right)e^{jk\omega_0 x}$$

Under the assumption $T\to\infty$, let $\omega_0=\Delta\omega\to 0$ and $k\omega_0=k\Delta\omega\to\omega$ where ω is a continuous variable, then (3) can be changed into

(4) $$f(x)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\,d\tau\right)e^{j\omega x}\Delta\omega$$

Further taking $\Delta\omega$ as $d\omega$, (4) can be written as the following integral form

(5) $$f(x)=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\,d\tau\right)e^{j\omega x}d\omega$$

If we define

(6) $$F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx$$

then

(7) $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}\, d\,\omega$$

Usually, $F(\omega)$ in (6) is called the Fourier transform of $f(x)$, denoted as

(8) $$\Im\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\, d\,x$$

and $f(x)$ is the inverse Fourier transform of $F(\omega)$, denoted as

(9) $$\Im^{-1}\{F(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}\, d\,\omega$$

Moreover, from (7), we know that $f(x)$ is composed of an infinite number of terms $\frac{1}{2\pi}F(\omega)$, each corresponding to a frequency ω.

The Fourier transform $F(\omega)$ in (6) is a complex number and can be expressed as

(10) $$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\, d\,x = |F(\omega)|\, e^{j\phi(\omega)}$$

It is obvious that

(11) $$F(-\omega) = \int_{-\infty}^{\infty} f(x)e^{j\omega x}\, d\,x = F^*(\omega)$$

which leads to

(12) $$|F(-\omega)|\, e^{j\phi(-\omega)} = |F(\omega)|\, e^{-j\phi(\omega)}$$

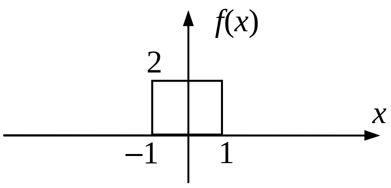
It is clear that

(13) $$|F(-\omega)| = |F(\omega)|$$

(14) $$\phi(-\omega) = -\phi(\omega)$$

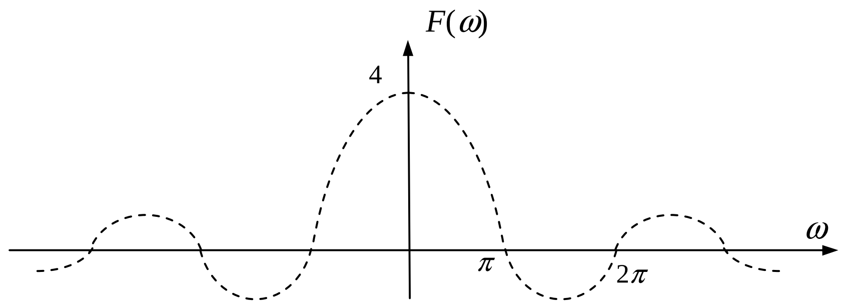
which means the amplitude $|F(\omega)|$ is an even function and the phase $\phi(\omega)$ is an odd function.

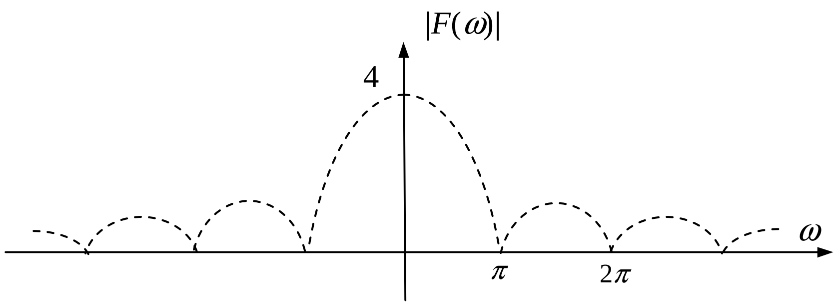
Consider the pulse function $f(x)$ shown in the figure to the right. Its Fourier transform is given as

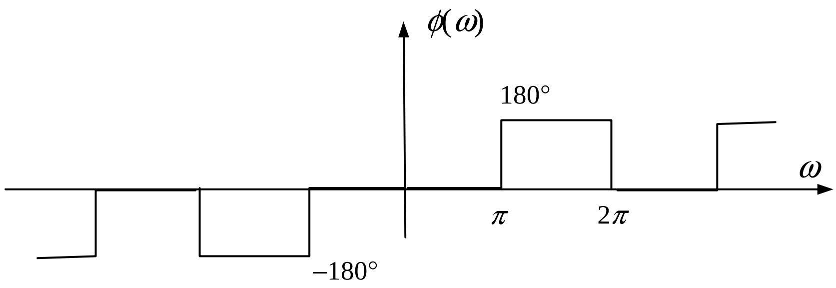


(15) $$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\, d\,x \\ &= \int_{-1}^{1} 2e^{-j\omega x}\, d\,x = \frac{4\sin\omega}{\omega} = 4\,\mathrm{sinc}(\omega) \end{aligned}$$

which is a real number. Below show the figures of Fourier transform $F(\omega)$, its amplitude $|F(\omega)|$ and phase $\phi(\omega)$.







It has been shown that a periodic function can be represented by the Fourier series as

(16) $$f(x)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 x}$$

Then, what is the Fourier transform of a periodic function? Based on the definition, we have

(17) $$\begin{aligned}\Im\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx = \int_{-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 x}\right)e^{-j\omega x}\,dx \\ &= \sum_{k=-\infty}^{\infty}c_k\left(\int_{-\infty}^{\infty}e^{-j(\omega-k\omega_0)x}dx\right)\end{aligned}$$

Hence, it is required to find the expression of $\int_{-\infty}^{\infty}e^{-j(\omega-k\omega_0)x}dx$. Let's check the following inverse Fourier transform

(18) $$\Im^{-1}\{\delta(\omega-k\omega_0)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-k\omega_0)e^{j\omega x}\,d\,\omega=\frac{1}{2\pi}e^{jk\omega_0 x}$$

which implies

(19) $$\begin{aligned}\Im\left\{e^{jk\omega_0 x}\right\}&=\int_{-\infty}^{\infty}e^{jk\omega_0 x}e^{-j\omega x}\,d\,\omega=\int_{-\infty}^{\infty}e^{-j(\omega-k\omega_0)x}\,d\,\omega\\&=2\pi\delta(\omega-k\omega_0)\end{aligned}$$

Therefore, (17) can be rewritten as

(20) $$\Im\left\{\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 x}\right\}=2\pi\sum_{k=-\infty}^{\infty}c_k\delta(\omega-k\omega_0)$$

which is the expression of a periodic function in frequency domain. It consists of a sequence of impulses with weight $2\pi c_k$ and presents a discrete frequency spectrum.

Let $F(\omega)$ and $G(\omega)$ be the Fourier transform of $f(x)$ and $g(x)$, respectively. Some important properties often used in Fourier transform are listed below:

Linearity

(21) $$\Im\{a\cdot f(x)+b\cdot g(x)\}=a\cdot F(\omega)+b\cdot G(\omega)$$

Time-shifting

(22) $$\Im\{f(x-x_0)\}=e^{-j\omega x_0}\,F(\omega)$$

Frequency-shifting

(23) $$\Im\left\{e^{j\omega_0 x}f(x)\right\}=F(\omega-\omega_0)$$

Time compression and expansion

(24) $$\Im\{f(ax)\}=\frac{1}{a}F\left(\frac{\omega}{a}\right)$$

Convolution in time

(25) $$\Im\left\{\int_{-\infty}^{\infty}f(x-\tau)g(\tau)d\tau\right\}=F(\omega)G(\omega)$$

Multiplication in time

(26) $$\Im\{f(x)g(x)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega-\Omega)G(\Omega)d\Omega$$

Time duration and frequency bandwidth

Let TD be the time duration of function $f(x)$ and BW be the frequency bandwidth of $F(\omega)$, then TD is approximately proportional to $1/BW$, i.e.,

(27) $$TD \sim \frac{1}{BW}$$

Parseval theorem

The instantaneous power of a function $f(x)$ is often expressed as $f^2(x)$, or $f(x)f^*(x)$ when $f(x)$ is a complex number. Then, the total energy is given as

(28) $$E=\int_{-\infty}^{\infty} f(x)\, f^*(x)d\,x=\int_{-\infty}^{\infty}\left|f(x)\right|^2 d\,x$$

From (7), we have

(29) $$\begin{aligned}E&=\int_{-\infty}^{\infty} f(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t}\,d\,\omega\right)d\,x\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\left(\int_{-\infty}^{\infty} f(x)e^{-j\omega t}\,d\,x\right)d\,\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)d\,\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\,\omega\end{aligned}$$

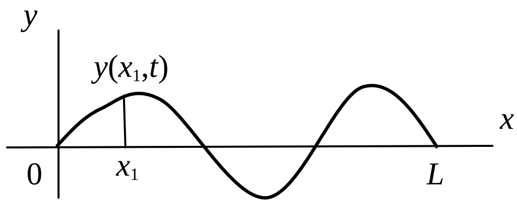
Therefore,

(30) $$\int_{-\infty}^{\infty}\left|f(x)\right|^2 d\,x=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\,\omega$$

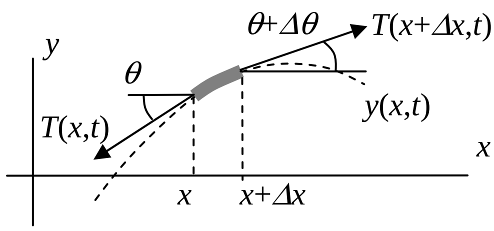
which is called the Parseval theorem.

C. Wave Equation

In general, vibration and oscillation induced in elastic material are governed by a PDE called the wave equation. For example, consider the motion of an elastic string on a guitar. If the string with a length L is placed along the x-axis and vibrates with small displacement in the x-y plane, then the shape of the string can be described by $y(x,t)$ at time t. The figure to the right shows the displacement $y(x_1,t)$ at $x=x_1$.



Assume the tension $T(x,$t) at time t acts tangentially to the string at x. Besides, the particle moves to $y(x,t)$ vertically and does not encounter any damping forces.



From the figure to the right, based on the Newton's second law of motion, the dynamic equation is given as

(1) $$T(x+\Delta x,t)sin(\theta+\Delta\theta)-T(x,t)sin\theta=\rho\Delta x\frac{\partial^2 y(\bar{x},t)}{\partial t^2}$$

where ρ is the length density of mass and $\bar{x}$ is the center of mass. Define the vertical component of the tension as $u(x,t)=T(x,t)sin\theta$, then (1) can be changed into

(2) $$\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}=\rho\frac{\partial^2 y(\bar{x},t)}{\partial t^2}$$

The limit of $\Delta x\to 0$ leads to $\bar{x}\to x$ and

(3) $$\frac{\partial u(x,t)}{\partial x}=\rho\frac{\partial^2 y(x,t)}{\partial t^2}$$

For simplicity, we often denote (3) as

(4) $$u_x(x,t)=\rho y_{tt}(x,t)$$

where $u_x\equiv\frac{\partial u}{\partial x}$ and $y_{tt}\equiv\frac{\partial^2 y}{\partial t^2}$. Let the horizontal component of the tension be defined as $h(x,t)=T(x,t)\cos\theta$, then

(5) $$u(x,t)=h(x,t)\tan\theta=h(x,t)\frac{\partial y(x,t)}{\partial x}=h(x,t)y_x(x,t)$$

Substituting it into (3) gets

(6) $$\frac{\partial}{\partial x}\left(h\frac{\partial y}{\partial x}\right)=\rho\frac{\partial^2 y}{\partial t^2}$$

Since the particle of the string only moves vertically, we have $\frac{\partial h}{\partial x}=0$, i.e., *h* is constant along the *x* axis and then (6) is rewritten as

(7) $$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$

or

(8) $$y_{tt}=c^2 y_{xx}$$

where $c^2=\frac{h}{\rho}$. This is known as the one-dimension wave equation.

Let's consider the following BVP related to the wave equation, which is described by (7) or (8), and the vertical displacement is subject to the boundary conditions for $0<x<L$ and $t>0$ as below:

(9) $$y(x,0)=f(x)$$

(10) $$y_t(x,0)=g(x)$$

(11) $$y(0,t)=0$$

(12) $$y(L,t)=0$$

where $f(x)$ is the shape of the string at $t=0$ and $g(x)$ is the vertical velocity of the particle at $t=0$. To solve the problem, usually we adopt the concept of separable variables and assume

(13) $$y(x,t)=X(x)T(t)$$

Substituting it into (7) gets

(14) $$X(x)T''(t)=c^2T(t)X''(x)$$

Define

(15) $$\frac{X''(x)}{X(x)}=\frac{T''(t)}{c^2T(t)}=-\lambda^2$$

then

(16) $$X''(x)+\lambda^2 X(x)=0$$

(17) $$T''(t)+\lambda^2c^2T(t)=0$$

Hence,

(18) $$X(x)=a_1\cos\lambda x+a_2\sin\lambda x$$

(19) $$T(t)=b_1\cos c\lambda t+b_2\sin c\lambda t$$

From (11), we obtain

(20) $$y(0,t)=X(0)T(t)=a_1T(t)=0$$

i.e., $a_1=0$ and

(21) $$X(x)=a_2\sin\lambda x$$

From (12), we have

(22) $$y(L,t)=X(L)T(t)=a_2\sin\lambda L\cdot T(t)=0$$

i.e., $\sin\lambda L=0$ or $\lambda=\frac{n\pi}{L}$, n=0,1,2,$\cdots$. That implies

(23) $$y(x,t)=X(x)T(t)=\sin\frac{n\pi}{L}x\left(A_n\cos\frac{n\pi c}{L}t+B_n\sin\frac{n\pi c}{L}t\right)$$

for n=1,2,3,…, where A_n and B_n are arbitrary constants. As a result, we can assume that the solution is a combination of (29-23) and expressed as

(24) $$y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left(A_n\cos\frac{n\pi c}{L}t+B_n\sin\frac{n\pi c}{L}t\right)$$

Further, if t=0, (9) results in

(25) $$y(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}x=f(x)$$

which implies that $f(x)$ is a Fourier sine series and in the form of

(26) $$f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{2n\pi}{P}x \quad \text{for } x\in(0,L)$$

with period P=2L. That means $f(x)$ is odd and only spans in half period from x=0 to x=L. Therefore, we still calculate A_n for one period as below:

(27) $$A_n = \frac{2}{P}\int_{-L}^{L} f(x) sin \frac{2n\pi}{P} x\, dx = \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi}{L} x\, dx = \frac{2}{L}\int_{0}^{L} f(x) sin \frac{n\pi}{L} x\, dx$$

To include (10), if t=0, it can be attained that

(28) $$y_t(x,0) = \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = \sum_{n=0}^{\infty} B_n \frac{n\pi c}{L} sin \frac{n\pi}{L} x = g(x)$$

which also implies that $g(x)$ is odd and spans in half period L. Similar to (27) we have

(29) $$\frac{n\pi c}{L} B_n = \frac{2}{L}\int_{0}^{L} g(x) sin \frac{n\pi}{L} x\, dx$$

i.e.,

(30) $$B_n = \frac{2}{n\pi c}\int_{0}^{L} g(x) sin \frac{n\pi}{L} x\, dx$$

This solves the boundary problem of the wave equation (7) and the solution is

(31) $$y(x,t) = \sum_{n=1}^{\infty} sin \frac{n\pi}{L} x \left(A_n cos \frac{n\pi c}{L} t + B_n sin \frac{n\pi c}{L} t \right)$$

or

(32) $$y(x,t) = \sum_{n=1}^{\infty} C_n sin\left(\frac{n\pi c}{L} t + \phi_n \right) sin \frac{n\pi}{L} x$$

where $C_n = \sqrt{A_n^2 + B_n^2}$ and $\phi_n = tan^{-1}(A_n / B_n)$. Clearly, the displacement is the linear combination of a set of standing waves. The nth mode of the displacement is

(33) $$y_n(x,t) = C_n sin\left(\frac{n\pi c}{L} t + \phi_n \right) sin \frac{n\pi}{L} x$$

with frequency $f_n = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{h}{\rho}}$. For n=1, we obtain the fundamental frequency $f_1 = \frac{1}{2L}\sqrt{\frac{h}{\rho}}$ when playing the guitar.

D. Laplace's Equation

Laplace's equation is a 2nd-order PDE named after Pierre-Simon Laplace and is often expressed as

(1) $$\nabla^2 u = 0$$

where ∇^2 is the Laplace operator and u is a scalar function. Here, we will discuss the Laplace's equation in two dimensions (x,y) which is defined for $u \equiv u(x,y)$ and written as

(2) $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

This is the simplest case of linear PDEs in two dimensions shown as

(3) $$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

with coefficients from A to G being functions of x and y. If $B^2 - \mathrm{AC} < 0$, (3) is called an elliptic PDE and the simplest one is

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g$$

which is the well-known Poisson's equation. For the homogeneous case g=0, we call it the Laplace's equation.

The general theory related to Laplace's equation is known as the potential theory and the solutions are harmonic functions, which have been widely applied to a diversity of fields such as electromagnetism, astronomy, and fluid dynamics.

Consider the Laplace's equation in (2) for 0<x<A and 0<y<B, which is subject to the boundary conditions

(5) $$u_x(0,y) = 0$$

(6) $$u_x(A,y) = 0$$

(7) $$u(x,0) = 0$$

(8) $$u(x,B) = f(x)$$

with $f(x)$ set for the boundary y=B. Assume

(9) $$u(x,y)=X(x)Y(y)$$

then from (2) we obtain

(10) $$X''(x)Y(y)=X(x)Y''(y)$$

which yields

(11) $$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda^2$$

or

(12) $$X''(x)+\lambda^2X(x)=0$$

(13) $$Y''(y)-\lambda^2Y(y)=0$$

Then, for λ=0

(14) $$X(x)=a_1+a_2x$$

(15) $$Y(y)=b_1+b_2y$$

and for $\lambda \neq 0$

(16) $$X(x)=a_1\cos\lambda x+a_2\sin\lambda x$$

(17) $$Y(y)=b_1\cosh\lambda y+b_2\sinh\lambda y$$

From (5) and (7), we obtain

(18) $$u_x(0,y)=X'(0)Y(y)=a_2Y(y)=0$$

(19) $$u(x,0)=X(x)b_1=0$$

i.e., $a_2=0$ and $b_1=0$. From (6), we have

(20) $$u_x(A,y)=X'(A)Y(y)=-a_1\sin\lambda A\cdot Y(y)=0$$

i.e., $\sin\lambda A=0$ or $\lambda=\frac{n\pi}{A}$, n=0,1,2,…. Hence,

(21) $$X(x)=a_1\cos\frac{n\pi}{A}x$$

(22) $$Y(y)=b_2\sinh\frac{n\pi}{A}y$$

That implies

(23) $$u(x,y)=X(x)Y(y)=c_n\cos\frac{n\pi}{A}x\cdot\sinh\frac{n\pi}{A}y$$

for n=1,2,3,⋯. From (8), it can be attained that

(24) $$u(x,B)=\sum_{n=0}^{\infty} c_n \cdot sinh\frac{n\pi}{A}B \cdot cos\frac{n\pi}{A}x = f(x)$$

which implies that $f(x)$ is a Fourier cosine series and in the form of

(25) $$f(x)=\sum_{n=1}^{\infty} c_n\, sinh\frac{n\pi}{A}B \cdot cos\frac{2n\pi}{P}x, \quad \text{for } x\in(0,A)$$

with period P=2A. That means $f(x)$ is even and only spans in half period. Therefore, we obtain

(26) $$c_n = \frac{2}{A}\left(sinh\frac{n\pi}{A}B\right)^{-1}\int_0^A f(x)cos\frac{n\pi x}{A}d\,x$$

such that the solution can be expressed as

(27) $$u(x,y)=\sum_{n=0}^{\infty} c_n \cdot cos\frac{n\pi}{A}x \cdot sinh\frac{n\pi}{A}y$$

for $0<x<A$ and $0<y<B$.

Next, let's determine the potential function $\phi(x,y)$ of an electrostatic field bounded in the range $0<x<A$ and $0<y<\infty$ with boundary conditions

(28) $$\phi(0,y)=0$$

(29) $$\phi(A,y)=0$$

(30) $$\phi(x,0)=V$$

Note that there is no boundary for $y>0$, i.e., the potential $\phi(x,y)$ is distributed from $y=0$ to $y=\infty$. According to the electrostatic theory, the potential is governed by the Laplace's equation, i.e.,

(31) $$\nabla^2\phi(x,y)=\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial y^2}=\phi_{xx}+\phi_{yy}=0$$

Assume

(32) $$\phi(x,y)=X(x)Y(y)$$

then from (31) we obtain

(33) $$X''(x)Y(y)=X(x)Y''(y)$$

Let

(34) $$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2$$

then

(35) $$X''(x) + \lambda^2 X(x) = 0$$

(36) $$Y''(y) - \lambda^2 Y(y) = 0$$

which leads to

(37) $$X(x) = a_1 \cos \lambda x + a_2 \sin \lambda x$$

(38) $$Y(y) = be^{-\lambda y}$$

for $\lambda>0$. From (28), we obtain

(39) $$\phi(0,y) = X(0)Y(y) = a_1 Y(y) = 0$$

i.e., $a_1 = 0$ and

(40) $$X(x) = a_2 \sin \lambda x$$

Further from (29), we have

(41) $$\phi(A,y) = a_2 b \sin \lambda A \cdot e^{-\lambda y} = 0$$

i.e., $\sin \lambda A = 0$ or $\lambda = \frac{n\pi}{A}$, n=0,1,2,$\cdots$. Hence,

(42) $$\phi(x,y) = c_n \sin \frac{n\pi}{A} x \cdot e^{-\frac{n\pi}{A} y}$$

for n=1,2,3,…. From (30), it can be attained that

(43) $$\phi(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{A} x = V, \qquad \text{for } x \in (0,A)$$

Referring to Fourier sine series, we rewrite (43) as

(44) $$V = \sum_{n=1}^{\infty} c_n \sin \frac{2n\pi}{P} x$$

That means the period is P=2A and we obtain

(45) $$c_n = \frac{2}{A}\int_0^A V \sin \frac{n\pi x}{A} d\,x = \frac{2V}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{4V}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Therefore, the potential is

(46) $$\phi(x,y) = \sum_{n=0}^{\infty} \frac{4V}{(2n+1)\pi} \sin \frac{(2n+1)\pi}{A} x \cdot e^{-\frac{(2n+1)\pi}{A} y}$$

for $0<x<A$ and $0<y<\infty$.