8. Power Series Method

Power Series

In mathematics, usually a function can be expressed by a power series of the form

(8-1) $$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

where a_n represents the coefficient of the n^{th} term and x_0 is the center of the series. For example, the Taylor series of an infinitely differentiable $f(x)$ is

(8-2) $$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

where $a_n=\dfrac{f^{(n)}(x_0)}{n!}$. Let $x_0=0$, then

(8-3) $$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

which is called the Maclaurin series.

The power series certainly converges at x=x_0 and may converge in three possible cases:

C1. The series may converge only at x=x_0.

C2. The series may converge for all real number $-\infty<x<\infty$.

C3. The series may converge for $|x-x_0|<r$ and diverge for $|x-x_0|>r$, where r is called the radius of convergence.
At $x=x_0+r$ and $x=x_0-r$, the series may or may not converge.

In conclusion, a power series may converge if $|x-x_0|<r$ and diverges if $|x-x_0|>r$ with the radius of convergence $0\le r<\infty$. In addition, the series may or may not converge at $x=x_0+r$ and $x=x_0-r$. Sometimes, we can find the radius of convergence r by the ratio test:

(8-4) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L$$

where the series converges absolutely if L<1 and diverges if L>1. However, when L=1, the test reaches no conclusion. For example, consider the function

$f(x)=\sum_{n=0}^{\infty} n!\, x^n$. From the ratio test, we have

(8-5) $$\lim_{n\to\infty}\left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \begin{cases} \infty & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Clearly, the function $f(x)=\sum_{n=0}^{\infty} n!\, x^n$ only converges at x=0.

Next, let's consider the function $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, then from the ratio test we have

(8-6) $$\lim_{n\to\infty}\left|\frac{n!\, x^{n+1}}{x^n (n+1)!}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 \qquad \text{for all } |x| < \infty$$

which means the power series converges for all x.

As for the function $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)9^n}(x-2)^{2n}$, using the ratio test obtains

(8-7) $$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+2)9^{n+1}}(x-2)^{2n+2}}{\frac{(-1)^n}{(n+1)9^n}(x-2)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{-(n+1)}{(n+2)9}(x-2)^2\right| = \frac{1}{9}(x-2)^2$$

It is clear that, the series converges if $\frac{1}{9}(x-2)^2 < 1$, which is $|x-2| < 3$ or $-1 < x < 5$. The radius of convergence is r=3 and the center point is x_0=2.

Suppose that within an interval of convergence, $f(x)$ and $g(x)$ can be expressed in power series as below:

(8-8) $$f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

(8-9) $$g(x)=\sum_{n=0}^{\infty} b_n (x-x_0)^n$$

Their algebraic operations are

(8-10) $$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

(8-11) $$f(x)-g(x)=\sum_{n=0}^{\infty}(a_n-b_n)(x-x_0)^n$$

(8-12) $$kf(x)=\sum_{n=0}^{\infty}ka_n(x-x_0)^n$$

(8-13) $$f(x)g(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$$

where k is a constant and $c_n=\sum_{j=0}^{n}a_j b_{n-j}$. Besides, the derivatives of $f(t)$ are listed as below:

(8-14) $$f'(x)=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}=\sum_{n=1}^{\infty}\frac{n!}{(n-1)!}a_n(x-x_0)^{n-1}$$

(8-15) $$f''(x)=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}=\sum_{n=2}^{\infty}\frac{n!}{(n-2)!}a_n(x-x_0)^{n-2}$$

and so on. In general, the k^{th} derivative is represented as

(8-16) $$f^{(k)}(x)=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n(x-x_0)^{n-k}\ ,\ k=1,2,3,\ldots$$

and its radius of convergence is the same as that of $f(t)$. As for the integral of $f(t)$, it is given as

(8-17) $$\int f(x)dx=\sum_{n=0}^{\infty}a_n\int(x-x_0)^n dx=\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1}+c$$

with c constant.

It should be noticed that the indices of the k^{th} derivative in (8-16) do not start with zero. By shifting indices as $m=n-k$, the k^{th} derivative can be modified as $f^{(k)}(x)=\sum_{m=0}^{\infty}\frac{(m+k)!}{m!}a_{m+k}(x-x_0)^m$, or

(8-18) $$f^{(k)}(x)=\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}a_{n+k}(x-x_0)^n$$

The shifting of indices is one of the common operations in power series.

Power Series Method of Linear Differential Equations

In this section, we will introduce the power series method to deal with the IVP of linear ODEs. First, consider the following example

(8-19) $$y'+p(x)y=q(x),\qquad y(x_0)=y_0$$

where $p(x)$ and $q(x)$ are analytic at x_0. The condition of analyticity requires that $p(x)$ and $q(x)$ are infinitely differentiable at x_0. This condition also implies that $p(x)$ and $q(x)$ can be represented by power series in some open interval about x_0. Most importantly, when $p(x)$ and $q(x)$ are analytic, the solution is analytic as well, and usually expressed as

(8-20) $$y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n=\sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$$

where $c_n=\dfrac{y^{(n)}(x_0)}{n!}$. For an IVP of a linear higher order ODE, it is also true that the solution is analytic when its coefficients are all analytic.

For example, let's consider an IVP of a linear 1st-order ODE, which is expressed as

(8-21) $$y'+2e^{-x}y=x,\qquad y(0)=1$$

To determine $y(x)$, we can directly apply $y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$. First, derive the higher derivatives below:

(8-22) $$\begin{aligned}&y''+2e^{-x}y'-2e^{-x}y=1\\&y'''+2e^{-x}y''-4e^{-x}y'+2e^{-x}y=0\\&y^{(4)}+2e^{-x}y'''-6e^{-x}y''+6e^{-x}y'-2e^{-x}y=0\end{aligned}$$

and take x=0 for them, i.e.,

(8-23) $$\begin{aligned}&y'(0)+2y(0)=0\\&y''(0)+2y'(0)-2y(0)=1\\&y'''(0)+2y''(0)-4y'(0)+2y(0)=0\\&y^{(4)}(0)+2y'''(0)-6y''(0)+6y'(0)-2y(0)=0\end{aligned}$$

Then, we have

(8-24) $$\begin{aligned}&y'(0)=-2y(0)=-2\\&y''(0)=-2y'(0)+2y(0)+1=4+2+1=7\\&y'''(0)=-2y''(0)+4y'(0)-2y(0)=-14-8-2=-24\\&y^{(4)}(0)=-2y'''(0)+6y''(0)-6y'(0)+2y(0)=104\end{aligned}$$

Substitute them into the Taylor series as below:

(8-25) $$y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n=1-2x+\frac{7}{2}x^2-4x^3+\frac{13}{3}x^4+\cdots$$

which is the solution of the IVP of (8-21).

Similarly, the same process can be applied to an IVP of a 2nd-order linear ODE, which is expressed as

(8-26) $$y'' + 2e^x y' + xy = x^2, \quad y(0)=1, \quad y'(0) = 0$$

The higher derivatives are

(8-27) $$\begin{aligned} &y''' + 2e^x y'' + 2e^x y' + xy' + y = 2x \\ &y^{(4)} + 2e^x y''' + (4e^x + x)y'' + (2e^x + 2)y' = 2 \end{aligned}$$

By setting x=0, the above equations become

(8-28) $$\begin{aligned} &y''(0) + 2y'(0) = 0 \\ &y'''(0) + 2y''(0) + 2y'(0) + y(0) = 0 \\ &y^{(4)}(0) + 2y'''(0) + 4y''(0) + 4y'(0) = 2 \end{aligned}$$

and the results are

(8-29) $$\begin{aligned} &y''(0) = -2y'(0) = 0 \\ &y'''(0) = -2y''(0) - 2y'(0) - y(0) = -1 \\ &y^{(4)}(0) = -2y'''(0) - 4y''(0) - 4y'(0) + 2 = 2 + 2 = 4 \end{aligned}$$

Hence, the Taylor series is

(8-30) $$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = 1 - \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots$$

which is the solution of (8-26).

Next, let's apply the form of $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ to solve an IVP of a 2nd-order linear ODE, which is given as

(8-31) $$y'' + xy' + x^2 y = 0, \quad y(0) = y_0, \quad y'(0) = y_1$$

Let the solution be

(8-32) $$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

then $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting them into (8-31) leads to $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n = 0$ or

(8-33) $$\begin{aligned} &\sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2}x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=2}^{\infty} c_{n-2}x^n \\ &= 2c_2 + (6c_3 + c_1)x + \sum_{n=2}^{\infty} [(n+1)(n+2)c_{n+2} + nc_n + c_{n-2}]x^n = 0 \end{aligned}$$

Hence,

(8-34) $$c_2 = 0,\ c_3 = -\frac{c_1}{6} \text{ and } c_{n+2} = -\frac{nc_n + c_{n-2}}{(n+1)(n+2)} \text{ for } n = 2,3,\cdots$$

Since $y(0) = y_0$ and $y'(0) = y_1$, we have $c_0 = y_0$ and $c_1 = y_1$. Now, the coefficients can be separated into two groups, c_{2k} and c_{2k+1}, for $k = 0,1,2,\cdots$. For the coefficients c_{2k}, we have

(8-35) $$c_0 = y_0,\ c_2 = 0,\ c_4 = -\frac{y_0}{12},\ c_6 = \frac{4y_0}{360}, \ldots..$$

and for the coefficients c_{2k+1}, we have

(8-36) $$c_1 = y_1,\ c_3 = -\frac{y_1}{6},\ c_5 = -\frac{y_1}{40},\ c_7 = \frac{7y_1}{1008}, \ldots..$$

Hence,

(8-37) $$y(x) = \sum_{n=0}^{\infty} c_n x^n = y_0\left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 + \cdots\right) + y_1\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{7}{1008}x^7 + \cdots\right)$$

which is the solution of (8-31).

Singular Points and the Method of Frobenius

Here, we will introduce the method of Frobenius to deal with a linear ODE which possesses singular points. Consider the ODE

(8-38) $$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

If $P(x) \neq 0$, then it can be written as

(8-39) $$y'' + q(x)y' + r(x)y = f(x)$$

where $q(x) = \frac{Q(x)}{P(x)}$, $r(x) = \frac{R(x)}{P(x)}$ and $f(x) = \frac{F(x)}{P(x)}$. If $q(x)$, $r(x)$ and $f(x)$ are analytic in some open interval about x_0, then we can determine a power series solution of (8-39) by methods introduced before. Here, x_0 is called an ordinary point.

If $P(x_0) = 0$ or any one of $q(x)$, $r(x)$ and $f(x)$ is not analytic at x_0, then x_0 is a singular point. To solve an ODE with singular points, it is required to try some other methods, such as the method of Frobenius.

First, let's take some examples for the ordinary points and singular points.

Consider the ODE

(8-40) $$x^3(x-2)^2 y'' + 5(x+2)(x-2)y' + 3x^2 y = 0$$

where $P(x) = x^3(x-2)^2$. Since $P(0) = P(2) = 0$, there are two singular points at 0 and 2. All the other real numbers are ordinary points.

In an interval about a singular point, the solutions may be quite different from what we have seen about an ordinary point. In order to understand the behavior of solutions near a singular point, we will concentrate on the homogeneous equation

(8-41) $$P(x)y'' + Q(x)y' + R(x)y = 0$$

After the homogeneous equation is solved, it is not difficult for us to solve the nonhomogeneous equation (8-38).

There are two kinds of singular points, regular singular point (RSP) and irregular singular points (IRSP). If x_0 is a singular point and both $(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at x_0, then x_0 is an RSP, otherwise it is an IRSP. For example, consider the ODE

(8-42) $$x^3(x-2)^2 y'' + 5(x+2)(x-2)y' + 3x^2 y = 0$$

which has singular points at 0 and 2. For x_0=0, we have

(8-43) $$(x-x_0)\frac{Q(x)}{P(x)} = x\frac{5(x+2)(x-2)}{x^3(x-2)^2} = \frac{5(x+2)}{x^2(x-2)}$$

which is not analytic at x_0=0, we say that x_0=0 is an IRSP. For x_0=2, we have

(8-44) $$(x-x_0)\frac{Q(x)}{P(x)} = (x-2)\frac{5(x+2)(x-2)}{x^3(x-2)^2} = \frac{5(x+2)}{x^3}$$

and

(8-45) $$(x-x_0)^2\frac{R(x)}{P(x)} = (x-2)^2\frac{3x^2}{x^3(x-2)^2} = \frac{3}{x}$$

Because both of them are analytic at x_0=2, we say that x_0=2 is an RSP.

If (8-41) has an RSP at x_0, we can solve it by choosing a possible solution expressed as

(8-46)
$$y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha}$$

which is called a Frobenius series. Note that a Frobenius series is not necessary to be a power series since α may be negative or no integer. The method using (8-46) to solve (8-41) is called the method of Frobenius.

Method of Frobenius

Suppose x_0 is an RSP of the 2nd-order homogeneous equation, which is expressed as

(8-47)
$$P(x)y''+Q(x)y'+R(x)y=0$$

where $q_p(x)=(x-x_0)\dfrac{Q(x)}{P(x)}$ and $r_p(x)=(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic and converge in an open interval (x_0–h, x_0+h). Let the solution be a Frobenius series given as below:

(8-48)
$$y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha}$$

with α a real number. Its first and second derivatives are

(8-49)
$$\begin{cases} y'(x)=\displaystyle\sum_{n=0}^{\infty}c_n(n+\alpha)(x-x_0)^{n+\alpha-1} \\ y''(x)=\displaystyle\sum_{n=0}^{\infty}c_n(n+\alpha)(n+\alpha-1)(x-x_0)^{n+\alpha-2} \end{cases}$$

Substituting them into (8-47) yields

(8-50)
$$\begin{aligned}&P(x)\sum_{n=0}^{\infty}c_n(n+\alpha)(n+\alpha-1)(x-x_0)^{n+\alpha-2}\\&+Q(x)\sum_{n=0}^{\infty}c_n(n+\alpha)(x-x_0)^{n+\alpha-1}+R(x)\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha}=0\end{aligned}$$

Further multiply $P^{-1}(x)(x-x_0)^2$, and obtain

(8-51)
$$\begin{aligned}&\sum_{n=0}^{\infty}c_n(n+\alpha)(n+\alpha-1)(x-x_0)^{n+\alpha}\\&+q_p(x)\sum_{n=0}^{\infty}c_n(n+\alpha)(x-x_0)^{n+\alpha}+r_p(x)\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha}=0\end{aligned}$$

where $q_p(x)=(x-x_0)\dfrac{Q(x)}{P(x)}$ and $r_p(x)=(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at x=x_0.

Hence,

(8-52) $$\sum_{n=0}^{\infty} c_n\left[(n+\alpha)(n+\alpha-1)+q_p(x)(n+\alpha)+r_p(x)\right](x-x_0)^{n+\alpha}=0$$

where $q_p(x)$ and $r_p(x)$ can be expanded by the Taylor series as

(8-53) $$q_p(x)=\sum_{m=0}^{\infty}\frac{q_p^{(m)}(x_0)}{m!}(x-x_0)^m$$

(8-54) $$r_p(x)=\sum_{m=0}^{\infty}\frac{r_p^{(m)}(x_0)}{m!}(x-x_0)^m$$

Apply (8-53) and (8-54) to (8-52), and we will obtain

(8-55) $$\sum_{i=0}^{\infty}\phi_i\cdot(x-x_0)^{\alpha+i}=0$$

where the coefficient ϕ_i is determined by α, n and c_n for $i,n=0,1,2,\cdots$. Since (8-55) is true in some interval of $(x_0-h,\ x_0+h)$, we can conclude that $\phi_i=0$ for $i=0,1,2,\cdots$. Next, let's discuss how to solve c_n from the coefficients $\phi_i=0$ for $i=0,1,2,\cdots$.

The coefficient ϕ_0 related to the term x^{α} can be obtained from (8-52), (8-53) and (8-54) by setting $n=m=0$, i.e.,

(8-56) $$\phi_0=c_0\left[\alpha(\alpha-1)+\alpha q_p(x_0)+r_p(x_0)\right]=0$$

Assume $c_0\neq 0$, then $\alpha(\alpha-1)+\alpha q_p(x_0)+r_p(x_0)=0$, or expressed as the so-called indicial equation

(8-57) $$I(\alpha)=\alpha^2+\alpha\left(q_p(x_0)-1\right)+r_p(x_0)=0$$

where $I(\alpha)$ is named as the indicial function. Since the indicial equation $I(\alpha)=0$ is used to determine the possible real number α, the following condition must be satisfied:

(8-58) $$\left(q_p(x_0)-1\right)^2-4r_p(x_0)\geq 0$$

Otherwise, there is no Frobenius solution.

Under the assumption that $c_0\neq 0$ and (8-58) is satisfied, there at least exists a solution $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha}$ with α a root of $I(\alpha)=0$, which is called the Frobenius solution expressed by a Frobenius series.

Due to the fact that the indicial equation has two roots, we may have two Frobenius solutions. Let the roots be α_1 and α_2, where $\alpha_1 \geq \alpha_2$. We assign the first Frobenius solution for the larger root α_1, i.e.,

(8-59) $$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+\alpha_1} = c_0 (x-x_0)^{\alpha_1} + \sum_{n=1}^{\infty} c_n (x-x_0)^{n+\alpha_1}$$

with $c_0 \neq 0$. To determine $y_1(x)$, it is required to find c_n for n=1,2,3,⋯. From (8-52) to (8-55), we have

(8-60) $$\begin{aligned} & c_0 \left[\alpha_1(\alpha_1 - 1) + \alpha_1 q_p(x_0) + r_p(x_0)\right](x-x_0)^{\alpha_1} \\ & + c_0 \sum_{m=1}^{\infty} \left(\alpha_1 \frac{q_p^{(m)}(x_0)}{m!} + \frac{r_p^{(m)}(x_0)}{m!} \right)(x-x_0)^{m+\alpha_1} \\ & + \sum_{n=1}^{\infty} c_n (n+\alpha_1)(n+\alpha_1-1)(x-x_0)^{n+\alpha_1} \\ & + \sum_{k=1}^{\infty} \sum_{n=1}^{k} c_n \left((n+\alpha_1) \frac{q_p^{(k-n)}(x_0)}{(k-n)!} + \frac{r_p^{(k-n)}(x_0)}{(k-n)!} \right)(x-x_0)^{k+\alpha_1} = 0 \end{aligned}$$

i.e.,

(8-61) $$\begin{aligned} & c_0 \left[\alpha_1(\alpha_1 - 1) + \alpha_1 q_p(x_0) + r_p(x_0)\right](x-x_0)^{\alpha_1} \\ & + \sum_{k=1}^{\infty} c_k (k+\alpha_1)(k+\alpha_1-1)(x-x_0)^{k+\alpha_1} \\ & + \sum_{k=1}^{\infty} \sum_{n=0}^{k} c_n \left((n+\alpha_1) \frac{q_p^{(k-n)}(x_0)}{(k-n)!} + \frac{r_p^{(k-n)}(x_0)}{(k-n)!} \right)(x-x_0)^{k+\alpha_1} = 0 \end{aligned}$$

It can be further written as

(8-62) $$\sum_{k=0}^{\infty} \phi_k (x-x_0)^{k+\alpha_1} = 0$$

where

(8-63) $$\phi_0 = c_0 \left[\alpha_1(\alpha_1 - 1) + \alpha_1 q_p(x_0) + r_p(x_0)\right] = 0$$

and for k=1,2,3,…

(8-64) $$\phi_k = c_k (k+\alpha_1)(k+\alpha_1-1) + \sum_{n=0}^{k} c_n \left((n+\alpha_1) \frac{q_p^{(k-n)}(x_0)}{(k-n)!} + \frac{r_p^{(k-n)}(x_0)}{(k-n)!} \right)$$

Since $\phi_k = 0$ for k=1,2,3,⋯, we have

(8-65) $$c_k (k+\alpha_1)(k+\alpha_1-1) + \sum_{n=0}^{k} c_n \left((n+\alpha_1) \frac{q_p^{(k-n)}(x_0)}{(k-n)!} + \frac{r_p^{(k-n)}(x_0)}{(k-n)!} \right) = 0$$

From the recursive relation in (8-65), c_n is solved and proportional to c_0.

Then, the first Frobenius solution $y_1(x)$ in (8-59) is obtained.

For the solution of the second root α_2, which is not greater than α_1, there are three possible cases: (C1) $\alpha_1-\alpha_2$ is not an integer, (C2) $\alpha_1-\alpha_2=0$ and (C3) $\alpha_1-\alpha_2=N$ is a positive integer.

(C1) $\alpha_1-\alpha_2$ is not an integer

In this case, the second Frobenius solution related to α_2 is similarly given as

(8-66) $$y_2(x)=\sum_{n=0}^{\infty}c_n^*(x-x_0)^{n+\alpha_2}=c_0^*(x-x_0)^{\alpha_2}+\sum_{n=1}^{\infty}c_n^*(x-x_0)^{n+\alpha_2}$$

with $c_0^*\neq 0$. Since $\alpha_1-\alpha_2$ is not an integer, we know that $(x-x_0)^{n+\alpha_1}$ and $(x-x_0)^{n+\alpha_2}$ are independent and then $y_2(x)$ is independent to $y_1(x)$. Therefore, the total solution of (8-47) with $c_0\neq 0$ and $c_0^*\neq 0$ is

(8-67) $$\begin{aligned}y(x)&=A_1y_1(x)+A_2y_2(x)\\&=A_1\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha_1}+A_2\sum_{n=0}^{\infty}c_n^*(x-x_0)^{n+\alpha_2}\end{aligned}$$

where A_1 and A_2 are arbitrary constants. For simplicity, we often choose $c_0=c_0^*=1$.

For example, let's consider the following homogeneous equation, expressed as

(8-68) $$x^2y''+\frac{x}{2}y'+\frac{x^2}{2}y=0$$

which has a singular point at x_0=0. It is easy to check that

(8-69) $$q_p(x)=(x-x_0)\frac{Q(x)}{P(x)}=x\frac{x/2}{x^2}=\frac{1}{2}$$

(8-70) $$r_p(x)=(x-x_0)^2\frac{R(x)}{P(x)}=x^2\frac{x^2/2}{x^2}=\frac{x^2}{2}$$

Both are analytic at x_0=0, and thus, x_0=0 is an RSP. Let the solution be a Frobenius series shown as below:

(8-71) $$y(x)=\sum_{n=0}^{\infty}c_nx^{n+\alpha}$$

where α is a real number. Then, the derivatives are

(8-72)
$$\begin{cases} y'(x)=\sum_{n=0}^{\infty} c_n(n+\alpha)x^{n+\alpha-1} \\ y''(x)=\sum_{n=0}^{\infty} c_n(n+\alpha)(n+\alpha-1)x^{n+\alpha-2} \end{cases}$$

Substituting them into (8-68) results in

(8-73)
$$\sum_{n=0}^{\infty} c_n(n+\alpha)(n+\alpha-1)x^{n+\alpha}+\frac{1}{2}\sum_{n=0}^{\infty} c_n(n+\alpha)x^{n+\alpha}+\frac{x^2}{2}\sum_{n=0}^{\infty} c_n x^{n+\alpha}=0$$

i.e.,

(8-74)
$$\sum_{k=0}^{\infty} c_k(k+\alpha)(k+\alpha-1)x^{k+\alpha}+\sum_{k=0}^{\infty}\frac{c_k}{2}(k+\alpha)x^{k+\alpha}+\sum_{k=2}^{\infty}\frac{c_{k-2}}{2}x^{k+\alpha}=0$$

The coefficients of $x^{k+\alpha}$ are

(8-75)
$$\phi_0=c_0\left(\alpha(\alpha-1)+\frac{1}{2}\alpha\right)=c_0\alpha\left(\alpha-\frac{1}{2}\right)$$

(8-76)
$$\phi_1=c_1\left(\alpha(\alpha+1)+\frac{1}{2}(\alpha+1)\right)=c_1(\alpha+1)\left(\alpha+\frac{1}{2}\right)$$

(8-77)
$$\phi_k=c_k(k+\alpha)\left(k+\alpha-\frac{1}{2}\right)+\frac{c_{k-2}}{2}=0\,,\quad k=2,3,4,\ldots$$

Assume $c_0\neq 0$, from (8-57) we obtain the indicial equation as

(8-78)
$$\alpha^2+\alpha\left(q_p(x_0)-1\right)+r_p(x_0)=\alpha^2-\frac{1}{2}\alpha=0$$

The roots are $\alpha_1=1/2$ and $\alpha_2=0$. Clearly, this is the case of (C1). For $\alpha_1=1/2$, the first Frobenius solution is

(8-79)
$$y_1(x)=\sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}}$$

From (8-76) and $\phi_1=0$, we choose $c_1=0$. From (8-77) and $\phi_k=0$, we have

(8-80)
$$c_k=-\frac{c_{k-2}}{2k\left(k+\frac{1}{2}\right)}=-\frac{1}{k(2k+1)}c_{k-2}\,,\quad \text{for}\quad k=2,3,4\ldots.$$

Hence,

(8-81)
$$c_1=0,\ c_3=0,\ \dots,\ c_{2k-1}=0,\ \dots\dots$$
$$c_2=-\frac{1}{2\cdot 5}c_0,\ c_4=-\frac{1}{4\cdot 9}c_2=\frac{1}{(2\cdot 4)\cdot(5\cdot 9)}c_0,\ \dots\dots$$
$$c_{2k}=(-1)^k\frac{1}{(2\cdot 4\cdot\dots\cdot 2k)\cdot(5\cdot 9\cdot\dots\cdot(4k+1))}c_0$$

If we choose $c_0=1$, then

(8-82)
$$y_1(x)=\sum_{n=0}^{\infty}c_nx^{n+\frac{1}{2}}$$
$$=x^{\frac{1}{2}}\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2\cdot 4\cdot\dots\cdot 2n)\cdot(5\cdot 9\cdot\dots\cdot(4n+1))}x^{2n}\right)$$

For $\alpha_2=0$, the second Frobenius solution is

(8-83)
$$y_2(x)=\sum_{n=0}^{\infty}c_n^*x^n$$

From (8-76) and $\phi_1=0$, we must choose $c_1^*=0$. From (8-77) and $\phi_k=0$, we have

(8-84)
$$c_k^*=-\frac{c_{k-2}^*}{2k\left(k-\frac{1}{2}\right)}=-\frac{1}{k(2k-1)}c_{k-2}^*,\quad \text{for}\quad k=2,3,4\dots$$

Hence,

(8-85)
$$c_1^*=0,\ c_3^*=0,\ \dots,\ c_{2k-1}^*=0,\ \dots\dots$$
$$c_2^*=-\frac{1}{2\cdot 3}c_0^*,\ c_4^*=-\frac{1}{4\cdot 7}c_2^*=\frac{1}{(2\cdot 4)\cdot(3\cdot 7)}c_0^*,\ \dots\dots$$
$$c_{2k}^*=(-1)^k\frac{1}{(2\cdot 4\cdot\dots\cdot 2k)\cdot(3\cdot 7\cdot\dots\cdot(4k-1))}c_0^*$$

If we choose $c_0^*=1$, then

(8-86)
$$y_2(x)=\sum_{n=0}^{\infty}c_n^*x^n$$
$$=1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2\cdot 4\cdot\dots\cdot 2n)\cdot(3\cdot 7\cdot\dots\cdot(4n-1))}x^{2n}$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, the total solution is $y(x)=A_1y_1(x)+A_2y_2(x)$, where A_1 and A_2 are arbitrary constants.

(C2) $\alpha_1-\alpha_2=0$

In this case, from the indicial equation (8-57), we know that the repeated roots are

(8-87) $$\alpha_1 = \alpha_2 = \frac{1 - q_p(x_0)}{2}$$

With the method of Frobenius, the first Frobenius solution is

(8-88) $$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\alpha_1}$$

and the second Frobenius solution is proposed by the following form:

(8-89) $$y_2(x) = y_1(x) ln|x - x_0| + \sum_{n=1}^{\infty} c_n^* (x - x_0)^{n+\alpha_1} \quad \text{for} \quad x \neq x_0$$

whose derivatives are

(8-90) $$\begin{aligned} y_2'(x) = & \, y_1'(x) ln|x - x_0| + y_1(x)(x - x_0)^{-1} \\ & + \sum_{n=1}^{\infty} c_n^* (n + \alpha_1)(x - x_0)^{n+\alpha_1 - 1} \end{aligned}$$

(8-91) $$\begin{aligned} y_2''(x) = & \, y_1''(x) ln|x - x_0| + 2y_1'(x)(x - x_0)^{-1} - y_1(x)(x - x_0)^{-2} \\ & + \sum_{n=1}^{\infty} c_n^* (n + \alpha_1)(n + \alpha_1 - 1)(x - x_0)^{n+\alpha_1 - 2} \end{aligned}$$

Substituting them into (8-47) yields

(8-92) $$\begin{aligned} & [P(x)y_1''(x) + Q(x)y_1'(x) + R(x)y_1(x)] ln|x - x_0| \\ & + P(x)\left[2y_1'(x)(x - x_0)^{-1} - y_1(x)(x - x_0)^{-2}\right] + Q(x)y_1(x)(x - x_0)^{-1} \\ & + P(x)\sum_{n=1}^{\infty} c_n^* (n + \alpha_1)(n + \alpha_1 - 1)(x - x_0)^{n+\alpha_1 - 2} \\ & + Q(x)\sum_{n=1}^{\infty} c_n^* (n + \alpha_1)(x - x_0)^{n+\alpha_1 - 1} + R(x)\sum_{n=1}^{\infty} c_n^* (x - x_0)^{n+\alpha_1} = 0 \end{aligned}$$

Since $P(x)y_1''(x) + Q(x)y_1'(x) + R(x)y_1(x) = 0$, we have

(8-93) $$\begin{aligned} & + P(x)\left[2y_1'(x)(x - x_0)^{-1} - y_1(x)(x - x_0)^{-2}\right] + Q(x)y_1(x)(x - x_0)^{-1} \\ & + P(x)\sum_{n=1}^{\infty} c_n^* (n + \alpha_1)(n + \alpha_1 - 1)(x - x_0)^{n+\alpha_1 - 2} \\ & + Q(x)\sum_{n=1}^{\infty} c_n^* (n + \alpha_1)(x - x_0)^{n+\alpha_1 - 1} + R(x)\sum_{n=1}^{\infty} c_n^* (x - x_0)^{n+\alpha_1} = 0 \end{aligned}$$

Further premultiplying $P^{-1}(x)(x - x_0)^2$ yields

(8-94)
$$\left[2y_1'(x)(x-x_0)-y_1(x)\right]+q_p(x)y_1(x) + \sum_{n=1}^{\infty} c_n^*(n+\alpha_1)(n+\alpha_1-1)(x-x_0)^{n+\alpha_1} + q_p(x)\sum_{n=1}^{\infty} c_n^*(n+\alpha_1)(x-x_0)^{n+\alpha_1} + r_p(x)\sum_{n=1}^{\infty} c_n^*(x-x_0)^{n+\alpha_1} = 0$$

Hence, we have

(8-95)
$$\sum_{n=0}^{\infty} c_n\left(2(n+\alpha_1)-1\right)(x-x_0)^{n+\alpha_1} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n \frac{q_p^{(m)}(x_0)}{m!}(x-x_0)^{n+m+\alpha_1} + \sum_{n=1}^{\infty} c_n^*(n+\alpha_1)(n+\alpha_1-1)(x-x_0)^{n+\alpha_1} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_n^* \frac{q_p^{(m)}(x_0)}{m!}(n+\alpha_1)(x-x_0)^{n+m+\alpha_1} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_n^* \frac{r_p^{(m)}(x_0)}{m!}(x-x_0)^{n+m+\alpha_1} = 0$$

which can be rewritten as

(8-96)
$$2c_0\left(\alpha_1 - \frac{1-q_p(x_0)}{2}\right)(x-x_0)^{\alpha_1} + \sum_{m=1}^{\infty} c_0 \frac{q_p^{(m)}(x_0)}{m!}(x-x_0)^{m+\alpha_1} + \sum_{n=1}^{\infty} c_n\left(2(n+\alpha_1)-1\right)(x-x_0)^{n+\alpha_1} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_n \frac{q_p^{(m)}(x_0)}{m!}(x-x_0)^{n+m+\alpha_1} + \sum_{n=1}^{\infty} c_n^*(n+\alpha_1)(n+\alpha_1-1)(x-x_0)^{n+\alpha_1} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_n^* \frac{q_p^{(m)}(x_0)}{m!}(n+\alpha_1)(x-x_0)^{n+m+\alpha_1} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_n^* \frac{r_p^{(m)}(x_0)}{m!}(x-x_0)^{n+m+\alpha_1} = 0$$

The first term can be deleted since $\alpha_1 - \frac{1-q_p(x_0)}{2}$, and then

(8-97)
$$\sum_{k=1}^{\infty} c_0 q_p^{(k)}(x_0)(x-x_0)^{k+\alpha_1} + \sum_{k=1}^{\infty} c_k\left(2(k+\alpha_1)-1\right)(x-x_0)^{k+\alpha_1} + \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} c_{k-m} \frac{q_p^{(m)}(x_0)}{m!}(x-x_0)^{k+\alpha_1} + \sum_{k=1}^{\infty} c_k^*(k+\alpha_1)(k+\alpha_1-1)(x-x_0)^{k+\alpha_1} + \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} c_{k-m}^* \frac{q_p^{(m)}(x_0)}{m!}(k-m+\alpha_1)(x-x_0)^{k+\alpha_1} + \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} c_{k-m}^* \frac{r_p^{(m)}(x_0)}{m!}(x-x_0)^{k+\alpha_1} = 0$$

or

(8-98) $$\sum_{k=1}^{\infty}\phi_k\cdot(x-x_0)^{k+\alpha_1}=0$$

where the coefficients ϕ_k, k=1,2,⋯, are given as

(8-99) $$\begin{aligned}\phi_k &= c_0\frac{q_p^{(k)}(x_0)}{k!}+c_k\left(2(k+\alpha_1)-1\right)+\sum_{m=0}^{k-1}c_{k-m}\frac{q_p^{(m)}(x_0)}{m!}\\ &+c_k^*(k+\alpha_1)(k+\alpha_1-1)+\sum_{m=0}^{k-1}c_{k-m}^*\frac{q_p^{(m)}(x_0)}{m!}(k-m+\alpha_1)+\sum_{m=0}^{k-1}c_{k-m}^*\frac{r_p^{(m)}(x_0)}{m!}\end{aligned}$$

From (8-98), we know that $\phi_k=0$, i.e.,

(8-100) $$\begin{aligned}&c_0q_p^{(k)}(x_0)+c_k\left(2(k+\alpha_1)-1\right)+\sum_{m=0}^{k-1}c_{k-m}\frac{q_p^{(m)}(x_0)}{m!}+c_k^*(k+\alpha_1)(k+\alpha_1-1)\\ &+\sum_{m=0}^{k-1}c_{k-m}^*\frac{q_p^{(m)}(x_0)}{m!}(k-m+\alpha_1)+\sum_{m=0}^{k-1}c_{k-m}^*\frac{r_p^{(m)}(x_0)}{m!}=0\end{aligned}$$

for k=1,2,3,⋯⋯. We can solve c_n^* and then obtain the second Frobenius solution $y_2(x)$ in (8-89) for x in some interval of (x_0–h, x_0+h) and $x\neq x_0$. Therefore, the total solution of the case (C2) $\alpha_1-\alpha_2=0$ with $c_0\neq 0$ is

(8-101) $$\begin{aligned}y(x)&=A_1y_1(x)+A_2y_2(x)\\ &=\left(A_1+A_2\,ln|x-x_0|\right)\sum_{n=0}^{\infty}c_n(x-x_0)^{n+\alpha_1}+A_2\sum_{n=1}^{\infty}c_n^*(x-x_0)^{n+\alpha_1}\end{aligned}$$

where A_1 and A_2 are arbitrary constants. For simplicity, we often choose $c_0=1$.

Let's consider the following homogeneous equation as an example, which is expressed as

(8-102) $$x^2y''+xy'+xy=0$$

which has a singular point at x_0=0. It is easy to check that

(8-103) $$q_p(x)=(x-x_0)\frac{Q(x)}{P(x)}=x\frac{x}{x^2}=1$$

(8-104) $$r_p(x)=(x-x_0)^2\frac{R(x)}{P(x)}=x^2\frac{x}{x^2}=x$$

Both are analytic at x_0=0, and thus, x_0=0 is an RSP. Let the solution be a Frobenius series shown as below:

(8-105) $$y(x)=\sum_{n=0}^{\infty}c_nx^{n+\alpha}$$

where α is a real number. Then, the derivatives are

(8-106)
$$\begin{cases} y'(x) = \sum_{n=0}^{\infty} c_n (n+\alpha) x^{n+\alpha-1} \\ y''(x) = \sum_{n=0}^{\infty} c_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2} \end{cases}$$

Substituting them into (8-102) results in

(8-107)
$$\sum_{n=0}^{\infty} c_n (n+\alpha)(n+\alpha-1) x^{n+\alpha} + \sum_{n=0}^{\infty} c_n (n+\alpha) x^{n+\alpha} + x\sum_{n=0}^{\infty} c_n x^{n+\alpha} = 0$$

i.e.,

(8-108)
$$\sum_{k=0}^{\infty} c_k (k+\alpha)(k+\alpha-1) x^{k+\alpha} + \sum_{k=0}^{\infty} c_k (k+\alpha) x^{k+\alpha} + \sum_{k=1}^{\infty} c_{k-1} x^{k+\alpha} = 0$$

Hence, we have

(8-109)
$$\sum_{k=0}^{\infty} \phi_k x^{k+\alpha} = 0$$

where

(8-110)
$$\phi_0 = c_0(\alpha(\alpha-1)+\alpha) = c_0\alpha^2 = 0$$

(8-111)
$$\phi_k = c_k (k+\alpha)^2 + c_{k-1} = 0, \quad k=1,2,3,\ldots$$

Assume $c_0 \neq 0$, from (8-110) we know the indicial equation is $\alpha^2 = 0$, which has repeated roots $\alpha_1 = \alpha_2 = 0$. Clearly, this is the case of (C2) and let the first Frobenius solution be

(8-112)
$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+\alpha_1} = c_0 + \sum_{n=1}^{\infty} c_n x^n$$

Then, from (8-111) we have $\phi_k = c_k(k+\alpha_1)^2 + c_{k-1} = c_k k^2 + c_{k-1} = 0$, i.e.,

(8-113)
$$c_k = -\frac{c_{k-1}}{k^2}, \quad \text{for } k=1,2,3\ldots.$$

Hence, $c_1 = -\frac{1}{1^2}c_0$, $c_2 = -\frac{1}{2^2}c_1 = \frac{1}{(1\cdot 2)^2}c_0, \ldots, c_k = (-1)^k \frac{1}{(k!)^2}c_0$, and so forth.

If we choose $c_0 = 1$, then $c_k = (-1)^k \frac{1}{(k!)^2}$ for k=0,1,2,3,… and

(8-114)
$$y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^n = 1 - x + \frac{1}{(2!)^2}x^2 - \frac{1}{(3!)^2}x^3 + \ldots\ldots$$

For the second solution $y_2(x)$, from (8-89) we have

(8-115)
$$y_2(x) = y_1(x) ln|x| + \sum_{n=1}^{\infty} c_n^* x^n$$

and the derivatives are

(8-116) $$y_2'(x) = y_1'(x) ln|x| + \frac{y_1(x)}{x} + \sum_{n=1}^{\infty} nc_n^* x^{n-1}$$

(8-117) $$y_2''(x) = y_1''(x) ln|x| + 2\frac{y_1'(x)}{x} - \frac{y_1(x)}{x^2} + \sum_{n=1}^{\infty} n(n-1)c_n^* x^{n-2}$$

Substituting them into (8-102) yields

(8-118) $$\begin{aligned}&\left(x^2 y_1''(x) + xy_1'(x) + xy_1(x)\right) ln|x| + 2xy_1'(x) \\ &+ \sum_{n=1}^{\infty} n(n-1)c_n^* x^n + \sum_{n=1}^{\infty} nc_n^* x^n + \sum_{n=1}^{\infty} c_n^* x^{n+1} = 0\end{aligned}$$

i.e.,

(8-119) $$2xy_1'(x) + \sum_{n=1}^{\infty} n(n-1)c_n^* x^n + \sum_{n=1}^{\infty} nc_n^* x^n + \sum_{n=2}^{\infty} c_{n-1}^* x^n = 0$$

From (8-114), we have $y_1'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!\,(n-1)!} x^{n-1}$ and then (8-119) is written as

(8-120) $$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n!\,(n-1)!} x^n + \sum_{n=1}^{\infty} n(n-1)c_n^* x^n + \sum_{n=1}^{\infty} nc_n^* x^n + \sum_{n=2}^{\infty} c_{n-1}^* x^n = 0$$

i.e.,

(8-121) $$\left(c_1^* - 2\right)x + \sum_{n=2}^{\infty} \left((-1)^n \frac{2}{n!\,(n-1)!} + n^2 c_n^* + c_{n-1}^* \right) x^n = 0$$

Hence, $c_1^* = 2$ and $(-1)^n \frac{2}{n!\,(n-1)!} + n^2 c_n^* + c_{n-1}^* = 0$, for n=1,2,3,…, i.e.,

(8-122) $$c_n^* = -\frac{1}{n^2} c_{n-1}^* - (-1)^n \frac{2}{n(n!)^2}, \quad \text{for } n=1,2,3,\ldots$$

then

(8-123) $$\begin{aligned}y_2(x) = &\left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^n \right) ln|x| \\ &+ 2x - \frac{3}{4}x^2 + \frac{11}{108}x^3 - \frac{25}{3456}x^4 + \frac{137}{432000}x^5 + \cdots\end{aligned}$$

The solution is $y(x) = A_1 y_1(x) + A_2 y_2(x)$ with A_1 and A_2 arbitrary constants.

(C3) $\alpha_1 - \alpha_2 = N$ is a positive integer

In this case, if we still assume the second Frobenius solution is similar to the first Frobenius solution, i.e., (8-124) $$y_2(x)=\sum_{n=0}^{\infty}c_n^*(x-x_0)^{n+\alpha_2}=c_0^*(x-x_0)^{\alpha_2}+\sum_{n=1}^{\infty}c_n^*(x-x_0)^{n+\alpha_2}$$

with $c_0^*\neq 0$, then from (8-65) we have

(8-125) $$c_k^*(k+\alpha_2)(k+\alpha_2-1)+\sum_{n=0}^{k}c_n^*\left[(n+\alpha_2)\frac{q_p^{(k-n)}(x_0)}{(k-n)!}+\frac{r_p^{(k-n)}(x_0)}{(k-n)!}\right]=0$$

for k=1,2,3,$\cdots$, which can be further rewritten as

(8-126) $$\begin{aligned}&c_k^*(k+\alpha_2)(k+\alpha_2-1)+\sum_{n=1}^{k}c_n^*\left[(n+\alpha_2)\frac{q_p^{(k-n)}(x_0)}{(k-n)!}+\frac{r_p^{(k-n)}(x_0)}{(k-n)!}\right]\\&=-c_0^*\left[\alpha_2\frac{q_p^{(k)}(x_0)}{k!}+\frac{r_p^{(k)}(x_0)}{k!}\right]\end{aligned}$$

Since α_1 and α_2 are the roots of $\alpha^2+\alpha(q_p(x_0)-1)+r_p(x_0)=0$, we have

(8-127) $$\alpha_2^2+\alpha_2(q_p(x_0)-1)+r_p(x_0)=0$$

(8-128) $$\alpha_1+\alpha_2=1-q_p(x_0)$$

Hence, the coefficient of c_k^* in (8-126) is

(8-129) $$\begin{aligned}&(k+\alpha_2)(k+\alpha_2-1)+q_p(x_0)(k+\alpha_2)+r_p(x_0)\\&=k(k+2\alpha_2-1+q_p(x_0))+\alpha_2^2+\alpha_2(q_p(x_0)-1)+r_p(x_0)\\&=k(k+2\alpha_2-(\alpha_1+\alpha_2))=k(k-(\alpha_1-\alpha_2))=k(k-N)\end{aligned}$$

Hence, (8-126) becomes

(8-130) $$\begin{aligned}&c_k^*k(k-N)+\sum_{n=1}^{k-1}c_n^*\left[(n+\alpha_2)\frac{q_p^{(k-n)}(x_0)}{(k-n)!}+\frac{r_p^{(k-n)}(x_0)}{(k-n)!}\right]\\&=-c_0^*\left[\alpha_2\frac{q_p^{(k)}(x_0)}{k!}+\frac{r_p^{(k)}(x_0)}{k!}\right]\end{aligned}$$

For k=1,2,…,N–1, we have

(8-131) $$c_1^*=-\left[\frac{\alpha_2q_p^{(1)}(x_0)+r_p^{(1)}(x_0)}{1-N}\right]c_0^*=\frac{\sigma_{10}}{1-N}c_0^*=\beta_1c_0^*$$

(8-132) $$\begin{aligned}c_2^*&=-\left[\frac{(1+\alpha_2)q_p^{(1)}(x_0)+r_p^{(1)}(x_0)}{2(2-N)}\right]c_1^*-\left[\frac{\alpha_2\,q_p^{(2)}(x_0)/2+r_p^{(2)}(x_0)/2}{2(2-N)}\right]c_0^*\\&=\frac{\sigma_{21}}{2(2-N)}c_1^*+\frac{\sigma_{20}}{2(2-N)}c_0^*=\frac{\sigma_{21}}{2(2-N)}\beta_1c_0^*+\frac{\sigma_{20}}{2(2-N)}c_0^*\\&=\left(\frac{\sigma_{21}}{2(2-N)}\beta_1+\frac{\sigma_{20}}{2(2-N)}\right)c_0^*=\beta_2c_0^*\end{aligned}$$

and so on. This implies that

(8-133) $$c_k^* = \beta_k c_0^* \quad \text{for } k=1,2,\ldots,N-1$$

If $k=N$, we can conclude from that (8-130) that

(8-134) $$c_N^* N(N-N) + \left(\sum_{i=1}^{N-1} \sigma_{Ni}\beta_i + \sigma_{N0}\right) c_0^* = c_N^* N(N-N) + \rho_N c_0^* = 0$$

where $\rho_N = \sum_{i=1}^{N-1} \sigma_{Ni}\beta_i + \sigma_{N0}$. If $\rho_N \neq 0$, c_0^* must be zero, which violates the assumption $c_0^* \neq 0$. If $\rho_N = 0$, we still cannot determine c_N^*. Hence, the second solution cannot be expressed as a Frobenius series.

Based on the above analysis, a second solution different to the Frobenius series is required and it is proposed as below:

(8-135) $$y_2(x) = b y_1(x) ln|x - x_0| + \sum_{n=0}^{\infty} c_n^* (x - x_0)^{n+\alpha_2}$$

where b may be zero or nonzero. Since the discussion of this case is complicated, let's just stop here.

Let's consider the following 2nd-order ODE as an example, which is expressed as

(8-136) $$xy'' - y = 0$$

which has a RSP at x_0=0. Let $y(x) = \sum_{n=0}^{\infty} c_n x^{n+\alpha}$, then

(8-137)
$$\begin{aligned} &\sum_{n=0}^{\infty} c_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-1} - \sum_{n=0}^{\infty} c_n x^{n+\alpha} \\ &= \sum_{n=0}^{\infty} c_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-1} - \sum_{n=1}^{\infty} c_{n-1} x^{n+\alpha-1} \\ &= c_0 \alpha(\alpha-1) x^{\alpha-1} + \sum_{n=1}^{\infty} \left(c_n (n+\alpha)(n+\alpha-1) - c_{n-1}\right) x^{n+\alpha-1} = 0 \end{aligned}$$

Hence,

(8-138) $$c_0 \alpha(\alpha-1) = 0$$

(8-139) $$c_n = \frac{1}{(n+\alpha)(n+\alpha-1)} c_{n-1}, \quad \text{for } n=1,2,3,\ldots.$$

Choose $c_0 = 1$, then the indicial equation is

(8-140) $$\alpha(\alpha-1) = 0$$

with roots $\alpha_1 = 1$ and $\alpha_2 = 0$. Since $\alpha_1 - \alpha_2 = 1$ is an integer, it is the third case (8-C3) and the first solution is related to $\alpha_1 = 1$ and from (8-139) the coefficients, for n=1,2,3,…, are

(8-141)
$$\begin{aligned} c_n &= \frac{1}{(n+1)n} c_{n-1} = \frac{(n-1)!}{(n+1)!} c_{n-1} = \frac{(n-1)!}{(n+1)!} \frac{(n-2)!}{n!} c_{n-2} \\ &= \frac{(n-1)!}{(n+1)!} \frac{(n-2)!}{n!} \frac{(n-3)!}{(n-1)!} c_{n-3} = \cdots = \frac{(n-1)!}{(n+1)!} \frac{(n-2)!}{n!} \cdots \frac{0!}{1!} c_0 \\ &= \frac{1!\,0!}{(n+1)!\,n!} c_0 = \frac{1}{(n+1)!\,n!} \end{aligned}$$

This gives us a Frobenius solution

(8-142)
$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} c_n x^{n+1} = x + \frac{1}{2!1!} x^2 + \frac{1}{3!2!} x^3 + \frac{1}{4!3!} x^4 + \cdots \\ &= x + \frac{1}{2} x^2 + \frac{1}{12} x^3 + \frac{1}{144} x^4 + \cdots \end{aligned}$$

Next, for $\alpha_2 = 0$, if we still use (8-137) to find the second solution, then

(8-143) $$n(n-1)c_n = c_{n-1}, \text{ for } n=1,2,3,\ldots. \qquad \text{(8-25-97)}$$

It is easy to check that $n = 1$ leads to $c_0 = 0$, which is contrary to the assumption of $c_0 \neq 0$. Hence, to find the second solution, we have to adopt (8-135) as below:

(8-144) $$y_2(x) = b y_1(x) ln|x| + \sum_{n=0}^{\infty} c_n^* x^n$$

Substituting it into (8-136) gets

(8-145)
$$\begin{aligned} & b(x y_1''(x) - y_1(x)) ln|x| + 2b y_1'(x) - b y_1(x)\frac{1}{x} \\ & + \sum_{n=2}^{\infty} n(n-1) c_n^* x^{n-1} - \sum_{n=0}^{\infty} c_n^* x^n = 0 \end{aligned}$$

Since $x y_1''(x) - y_1(x) = 0$ and $y_1(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!\,n!} x^{n+1}$, we have

(8-146) $$\sum_{n=0}^{\infty} \left(\frac{2b}{(n!)^2} - \frac{b}{(n+1)!\,n!} \right) x^n + \sum_{n=2}^{\infty} n(n-1) c_n^* x^{n-1} - \sum_{n=0}^{\infty} c_n^* x^n = 0$$

i.e.,

(8-147) $$\sum_{n=0}^{\infty} \left(\frac{2b}{(n!)^2} - \frac{b}{(n+1)!\,n!} \right) x^n + \sum_{n=1}^{\infty} n(n+1) c_{n+1}^* x^n - \sum_{n=0}^{\infty} c_n^* x^n = 0$$

Then

(8-148) $$b-c_0^*+\sum_{n=1}^{\infty}\left(\frac{2b}{(n!)^2}-\frac{b}{(n+1)!\,n!}+n(n+1)c_{n+1}^*-c_n^*\right)x^n=0$$

which leads to

(8-149) $$c_0^*=b$$

(8-150) $$c_{n+1}^*=\frac{1}{n(n+1)}\left(c_n^*-\frac{2n+1}{n!\,(n+1)!}b\right),\quad \text{for } n=1,2,3,\ldots.$$

Here, we choose $c_0^*=1$ and then $b=1$. For (8-150), we choose $c_1^*=0$ then $c_2^*=-\frac{3}{4}$, $c_3^*=-\frac{7}{36}$, $c_4^*=-\frac{35}{1728}$, and so on. Hence,

(8-151) $$y_2(x)=y_1(x)ln|x|+1-\frac{3}{4}x^2-\frac{7}{36}x^3-\frac{35}{1728}x^4-\cdots$$

and the total solution is

(8-152) $$y(x)=A_1y_1(x)+A_2y_2(x)$$

where A_1 and A_2 are arbitrary constants.

Bessel's Equation

In applied mathematics, one of the most important ODEs is Bessel's equation, which is expressed as

(8-153) $$x^2y''+xy'+(x^2-\nu^2)y=0$$

where ν is assumed to be real and nonnegative, i.e., $\nu\ge 0$. It has been applied to a lot of fields, such as electric fields and heat conduction.

Bessel's equation can be solved by the Frobenius method and its solutions about the RSP x_0=0 are called Bessel functions. As mentioned, the solution is a Frobenius series represented by

(8-154) $$y(x)=\sum_{n=0}^{\infty}c_nx^{n+\alpha}$$

Substituting (8-154) and its derivatives into (8-153) gives

(8-155) $$\sum_{n=0}^{\infty}c_n(n+\alpha)(n+\alpha-1)x^{n+\alpha}+\sum_{n=0}^{\infty}c_n(n+\alpha)x^{n+\alpha}+\sum_{n=0}^{\infty}c_n(x^2-\nu^2)x^{n+\alpha}=0$$

which can be rearranged as

(8-156) $$\sum_{n=0}^{\infty}c_n\left((n+\alpha)^2-\nu^2\right)x^{n+\alpha}+\sum_{n=2}^{\infty}c_{n-2}x^{n+\alpha}=0$$

Hence,

(8-157) $$c_0\left(\alpha^2-\nu^2\right)=0$$

(8-158) $$c_1\left((\alpha+1)^2-\nu^2\right)=0$$

(8-159) $$c_n\left((n+\alpha)^2-\nu^2\right)+c_{n-2}=0\,,\quad \text{for } n=2,3,\ldots..$$

Assuming $c_0\neq 0$, (8-157) results in the indicial equation

(8-160) $$\alpha^2-\nu^2=0$$

with roots $\alpha=\pm\nu$. Let $\alpha=\nu>0$, then (8-158) reduces to $c_1(2\nu+1)=0$. Since $2\nu+1\neq 0$, we have $c_1=0$ and from (8-159) we obtain $c_{2n+1}=0$ for n=1,2,3,⋯. Therefore, only the coefficients c_{2n} exist and also from (8-159) we have

(8-161) $$c_{2n}=-\frac{1}{2^2n(n+\nu)}c_{2n-2}\,,\quad n=1,2,3,\cdots$$

which results in

(8-162) $$c_{2n}=(-1)^n\frac{1}{2^{2n}n!\,(1+\nu)(2+\nu)\cdots(n+\nu)}c_0\,,\quad \text{for } n=1,2,3,\cdots.$$

The Frobenius solution for $\nu\geq 0$ is then expressed as

(8-163) $$y(x)=c_0x^{\nu}+c_0\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^{2n}n!\,(1+\nu)(2+\nu)\cdots(n+\nu)}x^{2n+\nu}$$

with c_0 an arbitrary constant.

Now, let's consider a specific case that $\nu=m$ is an integer and $c_0=\dfrac{1}{2^m m!}$.

Then, (8-163) becomes

(8-164) $$c_{2n}=(-1)^n\frac{1}{2^{2n+m}n!\,(n+m)!}\,,\quad n=0,1,2,3,\cdots$$

Note that this expression is also available for n=0. In general, the solution is purposely denoted as $y(x)=J_m(x)$ where

(8-165) $$J_m(x)=x^m\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+m}n!\,(n+m)!}x^{2n}$$

Interchanging m and n, we obtain

(8-166) $$J_n(x)=x^n\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m+n}m!\,(n+m)!}$$

which is called the Bessel function of the first kind of order n. Based on ratio test, this series converges very fast for all x because of the factorials in the denominator. For n=0, the Bessel function of order 0, $J_0(x)$, is given as

(8-167)
$$J_0(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m}(m!)^2}$$
$$=1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\cdots$$

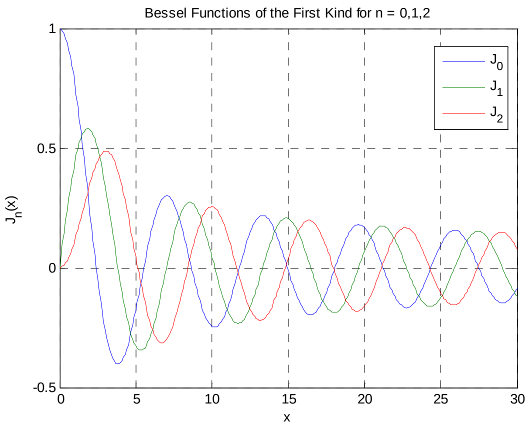
For n=1, the Bessel function of order 1, $J_1(x)$, is given as

(8-168)
$$J_1(x)=x\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m+1}m!(m+1)!}$$
$$=\frac{x}{2}-\frac{x^3}{2^3(1!)(2!)}+\frac{x^5}{2^5(2!)(3!)}-\frac{x^7}{2^7(3!)(4!)}+\cdots$$

The Bessel functions of J_0, J_1 and J_2 are shown below.

<u>Matlab commands "besseleq.m"</u>

```
x = 0:0.1:30;
J = zeros(8-5,301);
for i = 0:4
    J(8-i+1,:) = besselj(8-i,x);
plot(8-x,J)
axis(8-[0 30 -.5 1])
grid on
xlabel(8-'x')
ylabel(8-'J_n(8-x)')
legend(8-'J_0','J_1','J_2','Location','Best')
title(8-'Bessel Functions of the First Kind for n = 0,1,2')
```



The Frobenius solution for $\nu\geq 0$, not an integer, has been given in (8-163). Here, we will introduce the gamma function first, which is related to the Bessel function and defined as

(8-169)
$$\Gamma(\nu)=\int_0^{\infty}e^{-t}t^{\nu-1}dt$$

Then, we have

(8-170)
$$\Gamma(\nu+1)=\int_0^{\infty}e^{-t}t^{\nu}dt=-e^{-t}t^{\nu}\Big|_0^{\infty}+\nu\int_0^{\infty}e^{-t}t^{\nu-1}dt=\nu\Gamma(\nu)$$

If ν=1, from (8-169) we have

(8-171)
$$\Gamma(1)=\int_0^{\infty}e^{-t}dt=1$$

and from (8-170), we have $\Gamma(2)=1\cdot\Gamma(1)=1!$, $\Gamma(3)=2\cdot\Gamma(2)=2!$, and so on. Clearly,

(8-172) $$\Gamma(n+1)=n!\,,\qquad n=0,1,2,\cdots$$

This shows the gamma function generalizes the factorial function.

If $\nu=n$ is a nonnegative integer, the solution is a Bessel function given in (8-166) which can be represented as

(8-173) $$J_n(x)=x^n\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m+n}m!\,\Gamma(m+n+1)}$$

If ν is a not an integer, the general solution is given in (8-163). Let the coefficient $c_0=\dfrac{1}{2^\nu\Gamma(\nu+1)}$, then (8-163) becomes $y(x)=J_\nu(x)$, where

(8-174) $$\begin{aligned}J_\nu(x)&=\frac{1}{2^\nu\Gamma(\nu+1)}x^\nu\\&\quad+\frac{1}{2^\nu\Gamma(\nu+1)}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+\nu}}{2^{2n}n!(1+\nu)(2+\nu)\cdots(n+\nu)}\\&=\frac{1}{2^\nu 0!\,\Gamma(\nu+1)}x^\nu+\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+\nu}}{2^{2n+\nu}n!\,\Gamma(n+\nu+1)}\\&=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+\nu}}{2^{2n+\nu}n!\,\Gamma(n+\nu+1)}\end{aligned}$$

For the general solution of Bessel equation with ν not an integer, the first solution is $J_\nu(x)$ and the second solution is $J_{-\nu}(x)$, expressed as

(8-175) $$J_{-\nu}(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n-\nu}}{2^{2n-\nu}n!\,\Gamma(n-\nu+1)}$$

Since $J_\nu(x)$ and $J_{-\nu}(x)$ are linear independent, the total solution is

(8-176) $$y(x)=A_1J_\nu(x)+A_2J_{-\nu}(x)$$

where A_1 and A_2 are constant.

For the case that $\nu=n$ is a nonnegative integer, a question is raised: Can we just simply assign

(8-177) $$J_{-n}(x)=x^{-n}\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m-n}m!\,\Gamma(m-n+1)}$$

as the second solution? The answer is NO! Let's explain it here. First, from the truth $\Gamma(m-n+1)=\infty$ for $m<n$, we rewrite (8-177) as

(8-178) $$J_{-n}(x) = x^{-n}\sum_{m=n}^{\infty}(-1)^m \frac{x^{2m}}{2^{2m-n}\, m!\, \Gamma(m-n+1)}$$ (8-26-28)

Let *m*=*n*+*s*, then

(8-179)
$$\begin{aligned} J_{-n}(x) &= x^{-n}\sum_{s=0}^{\infty}(-1)^{n+s} \frac{x^{2n+2s}}{2^{2n+2s-n}(n+s)!\, \Gamma(s+1)} \\ &= (-1)^n x^n \sum_{s=0}^{\infty}(-1)^s \frac{x^{2s}}{2^{2s+n}(n+s)!\, s!} = (-1)^n J_n(x) \end{aligned}$$

That means $J_{-n}(x)$ is not independent to $J_n(x)$. Hence, (8-177) is not the second solution and we have to determine the second solution by the method of Frobenius of the case (C3). The resulted second solution is called the Bessel function of the second kind. Since the derivation of the second solution is too complicated, we will stop the discussion here.