7. Linear Second-Order Ordinary Differential Equations

In engineering, 2nd-order ODEs have been widely adopted to describe dynamic systems, such as an inverted pendulum and an object in simple harmonic motion. A 2nd-order ODE is generally expressed as

(7-1) $$F(x,y,y',y'')=0$$

For example, $y''+y^2y'-x^3=0$ and $xy''-ye^x=0$ are 2nd-order ODEs.

For the 2nd-order ODE (7-1) on an interval *I*, if there is a function $\varphi(x)$ that satisfies

(7-2) $$F(x,\varphi,\varphi',\varphi'')=0 \quad \text{for all } x \text{ in } I$$

then $y=\varphi(x)$ is a solution of (7-2). For example,

(7-3) $$\varphi(x)=x\cos 2x$$

is a solution of

(7-4) $$x^2y''-2xy'+2(2x^2+1)y=0$$

for *x*>0. This can be verified by substituting $y=\varphi(x)$ into (7-4).

Since it is more complicated to solve a 2nd-order ODE than 1st-order, this topic will only focus on the simpler case, the linear 2nd-order ODEs.

In general, a linear 2nd-order ODE in an interval *I* is often represented by the following form

(7-5) $$W(x)y''+P(x)y'+Q(x)y=R(x)$$

where $W(x)\neq 0$ and *W*(*x*), *P*(*x*), *Q*(*x*) and *R*(*x*) are continuous in the interval *I*. Taking monic process on (7-5) results in

(7-6) $$y''+p(x)y'+q(x)y=r(x)$$

and in what follows, (7-6) will be adopted for discussion.

Similar to 1st-order ODE, it is required to ensure the existence and uniqueness of the solution of (7-6). Let's consider the following example

(7-7) $$y''=6x$$

whose solution is

(7-8) $$y(x)=x^3+Cx+K$$

where *C* and *K* are two arbitrary constants. Clearly, this is different from the

1st-order case whose solution only has one arbitrary constant. If we are given one initial condition, such as $y(0)=3$, then $K=3$, i.e.,

(7-9) $$y(x)=x^3+Cx+3$$

To determine C, obviously, we require an extra condition, such as $y'(0)=-1$, which results in $C=-1$. That means under the conditions $y(0)=3$ and $y'(0)=-1$, the solution (7-8) can be uniquely determined as

(7-10) $$y(x)=x^3-x+3$$

Since both $y(0)=3$ and $y'(0)=-1$ are given at the initial point $x=0$, the above problem is known as an IVP. On the other hand, if we are given $y(0)=3$ and $y(1)=6$, then $K=3$ and $C=2$; clearly, the solution is still unique and obtained as

(7-11) $$y(x)=x^3+2x+3$$

Since the condition $y(1)=6$ is not given at the initial point $x=0$, it is not an IVP; instead, we call it the boundary value problem or BVP for short.

Next, let's discuss the existence and uniqueness of the IVP of a linear 2nd-order ODE, described as below:

(7-12) $$y''+p(x)y'+q(x)y=r(x), \quad y(x_0)=y_0 \text{ and } y'(x_0)=y'_0$$

Consider the simpler case that $r(x)=0$ for all x and the initial conditions $y(x_0)=y_0$ and $y'(x_0)=y'_0$ are neglected. Then, we have

(7-13) $$y''+p(x)y'+q(x)y=0$$

which is known as the homogeneous equation of (7-12). Assume $y_1(x)$ and $y_2(x)$ are two nonzero solutions of (7-13) and they are linearly independent, i.e., $y_1(x)\neq ky_2(x)$ for $k\neq 0$. Hence, we have

(7-14) $$y_i''+p(x)y_i'+q(x)y_i=0, \quad i=1,2$$

Let $\varphi(x)=c_1y_1(x)+c_2y_2(x)$, which is a linear combinations of $y_1(x)$ and $y_2(x)$ with constant coefficients c_1 and c_2. It is easy to check that

(7-15) $$\varphi''+p(x)\varphi'+q(x)\varphi=\sum_{i=1}^{2}c_i\left(y_i''+p(x)y_i'+q(x)y_i\right)=0$$

Therefore, $\varphi(x)=c_1y_1(x)+c_2y_2(x)$ is the homogeneous solution of (7-13).

To use the linear combinations $\varphi(x)=c_1y_1(x)+c_2y_2(x)$ as the solution, it is required that $y_1(x)$ and $y_2(x)$ are independent. There is a test, called

Wronskian Test, to tell whether $y_1(x)$ and $y_2(x)$ are linear independent or not. Let's define the Wronskian of $y_1(x)$ and $y_2(x)$ as below:

(7-16) $$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x)-y_2(x)y_1'(x)$$

and its derivative is

(7-17) $$\begin{aligned} W'(x) &= y_1'y_2'+y_1y_2''-y_2'y_1'-y_2y_1''=y_1y_2''-y_2\,y_1'' \\ &= y_1(-p(x)y_2'-q(x)y_2)-y_2(-p(x)y_1'-q(x)y_1) \\ &= -p(x)(y_1y_2'-y_2y_1') = -p(x)W(x) \end{aligned}$$

Hence,

(7-18) $$W(x)=Ae^{-\int p(x)dx}$$

where *A* is a constant and $e^{-\int p(x)dx}\neq 0$ for all *x* in *I*. Clearly, if there exists $W(x_0)=0$ at a point $x=x_0$ in *I*, then we have *A*=0 which means $W(x)=0$ for all *x* in *I*. On the other hand, if there exists $W(x_0)\neq 0$ at a point $x=x_0$ in *I*, then we have *A*≠0 which means $W(x)\neq 0$ for all *x* in *I*. Hence, the above analysis comes to the conclusion: Either *W*(*x*)=0 or $W(x)\neq 0$ for all *x* in *I*.

Moreover, it is easy to check that if $y_1(x)$ and $y_2(x)$ are linearly dependent, i.e., $y_1(x)=ky_2(x)$, then $y_1'(x)=ky_2'(x)$ and

(7-19) $$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1y_2'-y_2y_1'=0$$

Hence, if $W(x)\neq 0$ then $y_1(x)\neq ky_2(x)$, or $y_1(x)$ and $y_2(x)$ are linearly independent. Conversely, if $W(x)=y_1y_2'-y_2y_1'=0$ then $y_1y_2'=y_2y_1'$ or $\dfrac{y_1'}{y_1}=\dfrac{y_2'}{y_2}$, which is equivalent to $ln\,y_1=ln\,y_2+c$ or $y_1=ky_2$ with $k=e^c$. That means if $y_1(x)\neq ky_2(x)$ then $W(x)\neq 0$.

To sum up, the linear independency of $y_1(x)$ and $y_2(x)$ can be checked by the Wronskian $W(x_0)$ at any specific point $x=x_0$. If $W(x_0)\neq 0$, then $y_1(x)$ and $y_2(x)$ are linearly independent, otherwise they are linearly dependent.

Consider $y'' - 4y = 0$, which has two solutions $y_1 = e^{2x}$ and $y_2 = e^{-2x}$. The Wronskian is

(7-20) $$W(x) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \neq 0$$

which means $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ are linearly independent. Thus, their linear combination $\varphi(x) = c_1 e^{2x} + c_2 e^{-2x}$ with constant coefficients c_1 and c_2 is also a solution of $y'' - 4y = 0$.

Further consider $y'' - xy = 0$. It seems simple, but actually it is not easy to get the solution. Under such situation, we often solve it by the power series method, which will be introduced later, and obtain two solutions

(7-21) $$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} x^{3k}$$

(7-22) $$y_2(x) = x + \sum_{k=1}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3k-1)}{(3k+1)!} x^{3k+1}$$

It is not so easy to evaluate he Wronskian of $y_1(x)$ and $y_2(x)$ at any x in I. However, if we calculate the Wronskian at x=0, we will find that

(7-23) $$W(0) = y_1(0) y_2'(0) - y_2(0) y_1'(0) = 1 \times 1 - 0 \times 0 = 1 \neq 0$$

Since $W(0) \neq 0$ implies $W(x) \neq 0$ for all x, we know that $y_1(x)$ and $y_2(x)$ are linearly independent.

Next, consider the case of $r(x) \neq 0$ in (7-12) and neglect the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Then, we have

(7-24) $$y'' + p(x) y' + q(x) y = r(x)$$

which is called the nonhomogeneous equation. Similar to the 1st-order linear ODE, first we determine $y_1(x)$ and $y_2(x)$ for $r(x) = 0$ and obtain the homogeneous solution $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$, which satisfies

(7-25) $$y_h'' + p(x) y_h' + q(x) y_h = 0$$

Then, find a particular solution $y_p(x)$, which satisfies

(7-26) $$y_p'' + p(x) y_p' + q(x) y_p = r(x)$$

From (7-25) and (7-26), it can be obtained that

(7-27) $$\varphi'' + p(x) \varphi' + q(x) \varphi = r(x)$$

where

(7-28) $$\varphi(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

Clearly, $\varphi(x)$ is the solution of the nonhomogeneous equation (7-24).

Order Reduction Method for Homogeneous Equations

For a 2nd-order linear homogeneous ODE, there are two independent solutions. Most importantly, if a solution is obtained first, then the second solution can be determined by the order reduction method, which will be introduced below.

Suppose that we have found a homogeneous solution $y_1(x) \neq 0$ for the 2nd-order ODE

(7-29) $$y'' + p(x)y' + q(x)y = 0$$

Then choose the second solution as

(7-30) $$y_2(x) = \mu(x) y_1(x)$$

whose first and second derivatives are

(7-31) $$y_2' = \mu' y_1 + \mu y_1'$$

(7-32) $$y_2'' = \mu'' y_1 + 2\mu' y_1' + \mu y_1''$$

Since $y_2'' + p(x)y_2' + q(x)y_2 = 0$, we have

(7-33) $$(\mu'' y_1 + 2\mu' y_1' + \mu y_1'') + p(x)(\mu' y_1 + \mu y_1') + q(x)\mu y_1 = 0$$

i.e.,

(7-34) $$\mu'' y_1 + \mu'(2y_1' + p(x)y_1) + \mu(y_1'' + p(x)y_1' + q(x)y_1) = 0$$

Due to the fact that $y_1'' + p(x)y_1' + q(x)y_1 = 0$, (7-34) can be written as

(7-35) $$\mu'' y_1 + \mu'(2y_1' + p(x)y_1) = 0$$

Since $y_1 \neq 0$, it can be changed into

(7-36) $$\mu'' + \beta(x)\mu' = 0$$

where $\beta(x) = p(x) + 2\dfrac{y_1'}{y_1}$. Let $z(x) = \mu'(x)$, then

(7-37) $$z' + \beta(x) z = 0$$

Clearly, the 2nd-order ODE (7-29) is reduced to the 1st-order ODE (7-37). That is why we call the above process as the order-reduction method. The solution of (7-37) can be solved as

(7-38) $$z(x) = Ce^{-\int \beta(x)dx}$$

with C constant. Since $z(x) = \mu'(x)$ and $y_2(x) = \mu(x)y_1(x)$, we have

(7-39) $$\mu = \int z(x)dx = C\int e^{-\int \beta(x)dx}dx$$

(7-40) $$y_2(x) = \mu(x)y_1(x) = Cy_1(x)\int e^{-\int \beta(x)dx}dx$$

Here, $y_1(x)$ and $y_2(x)$ are linearly independent since the Wronskian is

(7-41) $$W(x) = y_1y_2' - y_1'y_2 = y_1(\mu'y_1 + \mu y_1') - y_1'(\mu y_1) = \mu' y_1^2 = zy_1^2$$

Clearly, $W(x) \neq 0$ because $z(x) = Ce^{-\int \beta(x)dx} \neq 0$ and $y_1 \neq 0$. Thus, $y_1(x)$ and $y_2(x)$ are linearly independent.

Consider the 2nd-order CODE $y'' + 2\omega y' + \omega^2 y = 0$ which has a repeated eigenvalue ω and two homogeneous solutions $y_1(x) = e^{-\omega x}$ and $y_2(x) = xe^{-\omega x}$. Here, we assume $y_1(x) = e^{-\omega x}$ is given and then use the order-reduction method to determine the second solution $y_2(x) = xe^{-\omega x}$.

According to the order-reduction method, the second solution is defined as $y_2(x) = \mu(x)y_1(x) = \mu(x)e^{-\omega x}$. Then, its first and second derivatives are obtained as $y_2' = \mu'y_1 + \mu y_1'$ and $y_2'' = \mu''y_1 + 2\mu'y_1' + \mu y_1''$. Substitute them into the CODE $y'' + 2\omega y' + \omega^2 y = 0$ and we have

(7-42) $$\mu''y_1 + 2\mu'y_1' + 2\omega\mu'y_1 = \mu''y_1 - 2\omega\mu'y_1 + 2\omega\mu'y_1 = 0$$

i.e., $\mu''y_1 = 0$. Since $y_1 \neq 0$, we have $\mu'' = 0$. Then, $\mu(x) = cx + d$ and the second solution is

(7-43) $$y_2(x) = \mu(x)y_1 = \mu(x)e^{-\omega x} = cxe^{-\omega x} + de^{-\omega x}$$

Since only the term not in the form of $y_1(x) = e^{-\omega x}$ is needed, we choose c=1 and d=0, i.e., the second solution is shown as below:

(7-44) $$y_2(x) = xe^{-\omega x}$$

The Wronskian is

(7-45) $$W(x) = y_1y_2' - y_1'y_2 = e^{-\omega x}\left(e^{-\omega x} - \omega xe^{-\omega x}\right) + \omega xe^{-2\omega x} = e^{-2\omega x} \neq 0$$

for all x. Hence, $y_1(x) = e^{-\omega x}$ and $y_2(x) = xe^{-\omega x}$ are linearly independent, and form a fundamental set of solutions for all x. Finally, the solution of the homogeneous equation is

(7-46) $$y(x) = c_1 e^{-\omega x} + c_2 x e^{-\omega x}$$

with arbitrary constants c_1 and c_2.

Let's consider the other example $y'' - \frac{1}{x+1} y' + \frac{1}{x(x+1)} y = 0$ for $x>0$, which has a solution $y_1(x) = x$. The other solution is defined as $y_2(x) = \mu(x)x$. Then, $y_2' = \mu' x + \mu$ and $y_2'' = \mu'' x + 2\mu'$. Substituting them into the ODE yields $(\mu'' x + 2\mu') - \frac{1}{x+1}(\mu' x + \mu) + \frac{1}{x(x+1)} \mu x = 0$. Let $z = \mu'$, then we have

(7-47) $$z' + \left(\frac{2}{x} - \frac{1}{x+1} \right) z = 0$$

which results in $z = x^{-2}(x+1)^{-1} = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$ and

(7-48) $$\mu = \int z dx = -\frac{1}{x} + ln\left(1 + \frac{1}{x}\right)$$

Therefore, the second solution is

(7-49) $$y_2(x) = \mu(x)x = -1 + x\, ln\left(1 + \frac{1}{x}\right)$$

Since $y_1(x) = x$ and $y_2(x) = -1 + x\, ln\left(1 + \frac{1}{x}\right)$ are linearly independent. The homogeneous solution is

(7-50) $$y(x) = c_1 x + c_2 \left(-1 + x\, ln \frac{x+1}{x} \right)$$

with arbitrary constants c_1 and c_2.

Cauchy-Euler Differential Equations

A Cauchy-Euler equation, or simply called Euler equation, is generally expressed as

(7-51) $$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$

which is defined on the half line $x>0$ or $x<0$. In this section, we will focus on the 2^{nd}-order case, given as

(7-52) $$x^2 y'' + a_1 x y' + a_0 y = 0$$

and defined on the right half line $x>0$.

The most common way is to transform (7-52) into a CODE by setting $x = e^t$ and $y(x) = y(e^t) = z(t)$. Then, the first derivative of $y(x)$ is

(7-53) $$y'(x) = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dz}{dt}\bigg/\frac{dx}{dt} = z'(t)/e^t = z'(t)e^{-t}$$

i.e.,

(7-54) $$xy'(x) = e^t y'(x) = z'(t)$$

The second derivative is

(7-55) $$y''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(z'(t)e^{-t}\right) = \frac{d\left(z'(t)e^{-t}\right)}{dt}\frac{dt}{dx} = \frac{d\left(z'(t)e^{-t}\right)}{dt}e^{-t}$$
$$= \left(z''(t)e^{-t} - z'(t)e^{-t}\right)e^{-t} = z''(t)e^{-2t} - z'(t)e^{-2t}$$

i.e.,

(7-56) $$x^2 y''(x) = e^{2t} y''(x) = z''(t) - z'(t)$$

Hence, (7-52) can be changed into

(7-57) $$z''(t) + (a_1 - 1)z'(t) + a_0 z(t) = 0$$

which is a 2nd-order CODE and $z(t)$ can be solved by the methods introduced before. Since $x = e^t$, i.e., $t = ln\,x$, the solution is then obtained as $y(x) = z(t)\big|_{t = ln\,x} = z(ln\,x)$, for $x>0$. Next, let's use some examples for demonstration.

Consider the 2nd-order ODE $x^2 y'' + 4xy' + 2y = 0$ for $x>0$. By setting $x = e^t$ and $y(x) = y(e^t) = z(t)$, the differential equation is transformed to

(7-58) $$z''(t) + 3z'(t) + 2z(t) = 0$$

whose characteristic equation is $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ with roots $\lambda = -1$ and $\lambda = -2$. Hence,

(7-59) $$z(t) = c_1 e^{-t} + c_2 e^{-2t}$$

From $x = e^t$, we have $t = ln\,x$ and then

(7-60) $$y(x) = z(ln\,x) = c_1 e^{-ln\,x} + c_2 e^{-2\,ln\,x} = c_1 x^{-1} + c_2 x^{-2}$$

which is the solution of $x^2 y'' + 4xy' + 2y = 0$ for $x>0$.

Consider the 2nd-order ODE $x^2 y'' - 3xy' + 4y = 0$ for $x>0$. Let $x = e^t$ and $y(x) = y(e^t) = z(t)$, the differential equation is transformed to

(7-61) $$z''(t) - 4z'(t) + 4z(t) = 0$$

whose characteristic equation is $\lambda^2-4\lambda+4=(\lambda-2)^2=0$ with repeated root $\lambda=2$. Hence,

(7-62) $$z(t)=c_1e^{2t}+c_2te^{2t}$$

From $x=e^t$, we have $t=ln\,x$ and then

(7-63) $$y(x)=z(ln\,x)=c_1e^{2\,ln\,x}+c_2(ln\,x)e^{2\,ln\,x}=c_1x^2+c_2x^2\,ln\,x$$

which is the solution of $x^2y''-3xy'+4y=0$ for x>0.

Consider the 2nd-order ODE $x^2y''-xy'+2y=0$ for x>0, with initial conditions $y(1)=1$ and $y'(1)=0$. Let $x=e^t$ and $y(x)=z(t)$, the differential equation is transformed to

(7-64) $$z''(t)-2z'(t)+2z(t)=0$$

whose characteristic equation is $\lambda^2-2\lambda+2=(\lambda-1+j)(\lambda-1-j)=0$ with roots $\lambda=1\pm j$. Hence,

(7-65) $$z(t)=c_1e^t\,cos\,t+c_2e^t\,sin\,t$$

From $x=e^t$, we have $t=ln\,x$ and then

(7-66) $$\begin{aligned}y(x)&=z(ln\,x)=c_1e^{ln\,x}\,cos(ln\,x)+c_2e^{ln\,x}\,sin(ln\,x)\\&=c_1x\,cos(ln\,x)+c_2x\,sin(ln\,x)\end{aligned}$$

According to the initial conditions $y(1)=1$ and $y'(1)=0$, we have $c_1=1$ and $c_2=-1$. Hence, $y(x)=x\,cos(ln\,x)-x\,sin(ln\,x)$ for x>0.

Linear Nonhomogeneous Equations

After the homogeneous solution y_h is solved, the general solution of the linear nonhomogeneous equation

(7-67) $$y''+p(x)y'+q(x)y=r(x)$$

can be expressed as $y=y_h+y_p$, where y_p is a particular solution. Here, we will introduce the method to determine a particular solution, which is similar to the order-reduction method used to find the second homogeneous solution.

Let y_1 and y_2 be the independent homogeneous solutions of (7-67). We assume the particular solution is related to y_1 and y_2 as below:

(7-68) $$y_p=\mu_1(x)y_1+\mu_2(x)y_2$$

in which $\mu_1(x)$ and $\mu_2(x)$ are two differentiable functions. Then,

(7-69) $$y_p' = \mu_1' y_1 + \mu_1 y_1' + \mu_2' y_2 + \mu_2 y_2'$$

Further assume

(7-70) $$\mu_1' y_1 + \mu_2' y_2 = 0$$

then (7-69) becomes

(7-71) $$y_p' = \mu_1 y_1' + \mu_2 y_2'$$

Then, the second derivative of y_p is

(7-72) $$y_p'' = \mu_1' y_1' + \mu_1 y_1'' + \mu_2' y_2' + \mu_2 y_2''$$

Substitute these terms into (7-67) and obtain

(7-73) $$\mu_1' y_1' + \mu_2' y_2' + \mu_1 (y_1'' + p y_1' + q y_1) + \mu_2 (y_2'' + p y_2' + q y_2) = r$$

Since $y_1'' + p y_1' + q y_1 = 0$ and $y_2'' + p y_2' + q y_2 = 0$, (7-73) can be reduced as

(7-74) $$\mu_1' y_1' + \mu_2' y_2' = r$$

Solve (7-70) and (7-74), and achieve

(7-75) $$\mu_1' = -\frac{r y_2}{W} \quad \text{and} \quad \mu_2' = \frac{r y_1}{W}$$

where $W = y_1 y_2' - y_1' y_2$ is the Wronskian of y_1 and y_2. After determine μ_1 and μ_2 from (7-75), we have a particular solution as shown in (7-68).

For example, if we want to find the solution for $y'' + 4y = sec\, x$, then first solve the homogeneous solutions, which are $y_1 = cos\, 2x$ and $y_2 = sin\, 2x$. The Wronslian is $W = y_1 y_2' - y_1' y_2 = 2$ and from (7-75) we have

(7-76) $$\mu_1' = -\frac{sec\, x\, sin\, 2x}{2} \quad \text{and} \quad \mu_2' = \frac{sec\, x\, cos\, 2x}{2}$$

Then,

(7-77) $$\mu_1 = -\int \frac{sec\, x\, sin\, 2x}{2} dx = -\int sin\, x dx = cos\, x$$

(7-78) $$\mu_2 = \int \frac{sec\, x\, cos\, 2x}{2} dx = \int \left(cos\, x - \frac{1}{2} sec\, x \right) dx = sin\, x - \frac{1}{2} ln\left| sec\, x + tan\, x \right|$$

From (7-68), a particular solution can be obtained as

(7-79) $$y_p = u y_1 + v y_2 = cos\, x\, cos\, 2x + sin\, 2x \left(sin\, x - \frac{1}{2} ln\left| sec\, x + tan\, x \right| \right)$$

Hence, the general solution is

(7-80)
$$y = c_1 \cos x + c_2 \sin x + \cos x \cos 2x + \sin 2x\left(\sin x - \frac{1}{2} \ln\left|\sec x + \tan x\right| \right)$$

Next, find the general solution for $y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1$ for x>0. Two of the independent homogeneous solutions are $y_1 = x$ and $y_2 = x^4$. The Wronslian is $W = y_1 y_2' - y_1' y_2 = 3x^4$ and from (7-75) we have

(7-81)
$$\mu_1' = -\frac{x^4\left(x^2+1\right)}{3x^4} \quad \text{and} \quad \mu_2' = \frac{x\left(x^2+1\right)}{3x^4}$$

Then,

(7-82)
$$\mu_1 = -\int \frac{x^2+1}{3} dx = -\frac{1}{9}x^3 - \frac{1}{3}x$$

(7-83)
$$\mu_2 = \int \frac{1}{3}\left(x^{-1} + x^{-3}\right) dx = \frac{1}{3} \ln x - \frac{1}{6} x^{-2}$$

From (7-68), a particular solution can be achieved as

(7-84)
$$y_p = u y_1 + v y_2 = -\frac{1}{9}x^4 - \frac{1}{2}x^2 + \frac{1}{3}x^4 \ln x$$

Hence, the general solution is

(7-85)
$$y = c_1 x + c_2 x^4 - \frac{1}{9}x^4 - \frac{1}{2}x^2 + \frac{1}{3}x^4 \ln x$$