6. Special First-Order Ordinary Differential Equations

Linear Differential Equations

A 1st-order ODE is said to be linear if it can be expressed by the following form

(6-1) $$y' + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are continuous on an interval I of x. The linearity can be seen from the terms on the left-hand side, which are defined as

(6-2) $$L[y] \equiv y' + p(x)y$$

It is easy to check that $L[a_1 y_1 + a_2 y_2] = a_1 L[y_1] + a_2 L[y_2]$, i.e., the operator (6-2) satisfies the superposition principle. Hence, $L[y] \equiv y' + p(x)y$ is a linear operator and we call (6-1) a linear ODE.

To solve the 1st-order linear ODE (6-1), we can adopt the method of integrating factor by choosing $\mu = e^{\int p(x)dx}$. Then, multiply $\mu = e^{\int p(x)dx}$ to (6-1) and obtain

(6-3) $$p(x)e^{\int p(x)dx}y - q(x)e^{\int p(x)dx} + e^{\int p(x)dx}y' = 0$$

where $M = p(x)e^{\int p(x)dx}y - q(x)e^{\int p(x)dx}$ and $N = e^{\int p(x)dx}$. Since

(6-4) $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = p(x)e^{\int p(x)dx}$$

we know that (6-3) is exact and a potential function φ exists such that $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$. Hence, from $\frac{\partial \varphi}{\partial y} = N = e^{\int p(x)dx}$, we have

(6-5) $$\varphi = e^{\int p(x)dx}y + g(x)$$

which leads to

(6-6) $$\frac{\partial \varphi}{\partial x} = p(x)e^{\int p(x)dx}y + g'(x)$$

Further from $\frac{\partial \varphi}{\partial x} = M = p(x)e^{\int p(x)dx}y - q(x)e^{\int p(x)dx}$, we obtain

(6-7) $$g'(x) = -q(x)e^{\int p(x)dx}$$

Let

(6-8) $$g(x) = -\int q(x)e^{\int p(x)dx}dx$$

then the potential function is

(6-9) $$\varphi = e^{\int p(x)dx} y - \int q(x) e^{\int p(x)dx} dx$$

and the solution is

(6-10) $$e^{\int p(x)dx} y - \int q(x) e^{\int p(x)dx} dx = C$$

or explicitly expressed as

(6-11) $$y = e^{-\int p(x)dx} \int q(x) e^{\int p(x)dx} dx + C e^{-\int p(x)dx}$$

with C constant. This is also the explicit solution of $y' + p(x)y = q(x)$ because the integrating factor $\mu = e^{\int p(x)dx} \neq 0$.

In fact, the 1st-order linear ODE (6-1) can be also solved from by multiplying the integrating factor $e^{\int p(x)dx}$, i.e.,

(6-12) $$e^{\int p(x)dx} y' + p(x) e^{\int p(x)dx} y = q(x) e^{\int p(x)dx}$$

Since $\dfrac{d}{dx}\left(e^{\int p(x)dx} y\right) = e^{\int p(x)dx} y' + p(x) e^{\int p(x)dx} y$, we have

(6-13) $$\frac{d}{dx}\left(e^{\int p(x)dx} y\right) = q(x) e^{\int p(x)dx}$$

which leads to

(6-14) $$e^{\int p(x)dx} y = \int q(x) e^{\int p(x)dx} dx + C$$

with C constant. Clearly,

(6-15) $$y = e^{-\int p(x)dx} \int q(x) e^{\int p(x)dx} dx + C e^{-\int p(x)dx}$$

same as the solution shown in (6-11).

The solution (6-15) is not unique since constant C is arbitrary. As mentioned before, C can be uniquely determined if the initial condition $y(x_0) = y_0$ is given. Rewrite (6-15) as

(6-16) $$y(x) = e^{-\int_{x_0}^{x} p(\lambda)d\lambda} \int_{x_0}^{x} q(\tau) e^{\int_{x_0}^{\tau} p(\lambda)d\lambda} d\tau + C e^{-\int_{x_0}^{x} p(\lambda)d\lambda}$$

and then,

(6-17) $$y(x_0) = e^{-\int_{x_0}^{x_0} p(\lambda)d\lambda} \int_{x_0}^{x_0} q(\tau) e^{\int_{x_0}^{\tau} p(\alpha)d\alpha} d\tau + C e^{-\int_{x_0}^{x_0} p(\lambda)d\lambda} = C = y_0$$

In conclusion, for the following IVP

(6-18) $$y' + p(x)y = q(x), \qquad y(x_0) = y_0$$

the solution can be uniquely determined as

(6-19) $$\begin{aligned} y(x) &= e^{-\int_{x_0}^{x} p(\lambda)d\lambda} \int_{x_0}^{x} q(\tau) e^{\int_{x_0}^{\tau} p(\lambda)d\lambda} d\tau + y_0 e^{-\int_{x_0}^{x} p(\lambda)d\lambda} \\ &= \int_{x_0}^{x} q(\tau) e^{-\int_{\tau}^{x} p(\lambda)d\lambda} d\tau + y_0 e^{-\int_{x_0}^{x} p(\lambda)d\lambda} \end{aligned}$$

Next, let's take some examples of 1st-order ODE for demonstration.

Consider $y' + 2y = sin 3x$ with initial condition $y(0)=1$. Since $p(x)=2$, the integrating factor is given as

(6-20) $$\mu = e^{\int p(x)dx} = e^{2x}$$

Multiplying $\mu = e^{2x}$ yields $e^{2x}y' + 2e^{2x}y = e^{2x} sin 3x$, i.e.,

(6-21) $$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x} sin 3x$$

Hence,

(6-22) $$e^{2x}y = \int e^{2x} sin 3x dx + C = \frac{2}{13}e^{2x} sin 3x - \frac{3}{13}e^{2x} cos 3x + C$$

i.e., the solution is

(6-23) $$y = \frac{1}{13}\left(2 sin 3x - 3 cos 3x\right) + Ce^{-2x}$$

From the initial condition $y(0)=1$, we have $1 = -\frac{3}{13} + C$, or $C = \frac{16}{13}$, and then

(6-24) $$y = \frac{1}{13}\left(2 sin 3x - 3 cos 3x\right) + \frac{16}{13}e^{-2x}$$

which is the unique solution for $y(0)=1$.

Consider the other example of 1st-oreder linear ODE with initial condition, which is given as

(6-25) $$y' + \frac{2}{x}y = x^2, \qquad y(1)=2$$

where $x \neq 0$. Since $p(x) = \frac{2}{x}$ and $q(x) = x^2$, the integrating factor is

(6-26) $$\mu = e^{\int p(x)dx} = e^{2ln|x|} = x^2, \quad \text{for} \quad x \neq 0$$

Multiplying (6-25) by $\mu = x^2$ yields $x^2y' + 2xy = x^4$, i.e., $\frac{d}{dx}\left(x^2 y\right) = x^4$.

Hence, $x^2 y = \frac{1}{5}x^5 + C$ or $y = \frac{1}{5}x^3 + \frac{C}{x^2}$. From the initial condition $y(1)=2$, it can be obtained that $2 = \frac{1}{5} + C$ or $C = \frac{9}{5}$. Therefore, the solution is

(6-27) $$y = \frac{1}{5}x^3 + \frac{9}{5x^2}, \quad \text{for} \quad x \neq 0$$

which is a unique solution.

For some linear ODEs, their form may be simple but their solutions cannot be expressed in a closed form. For example,

(6-28) $$y' - xy = 1$$

which seems quite simple; however, its solution is obtained as

(6-29) $$y = e^{x^2/2}\int e^{-x^2/2}dx + Ce^{x^2/2}$$

where $\int e^{-x^2/2}dx$ cannot be written into a closed form.

Homogeneous Equations

A 1st-order ODE $y' = f(x,y)$ can be also expressed as the following form $M(x,y) + N(x,y)y' = 0$, or

(6-30) $$y' = f(x,y) = -\frac{M(x,y)}{N(x,y)}$$

If both $M(x,y)$ and $N(x,y)$ are homogeneous functions, which have the same degree n, i.e., $M(\lambda x,\lambda y) = \lambda^n M(x,y)$ and $N(\lambda x,\lambda y) = \lambda^n N(x,y)$ where λ is a factor. For example, the polynomial $M(x,y) = x^2 + 7xy + 5y^2$ contains there terms of degree 2, i.e., it is a homogeneous function of degree 2. Hence, $M(\lambda x,\lambda y) = \lambda^2 x^2 + 7\lambda^2 xy + 5\lambda^2 y^2 = \lambda^2 M(x,y)$. Now, let's change (6-30) into

(6-31) $$y' = -\frac{M(x,y)}{N(x,y)} = -\frac{\lambda^n M(x,y)}{\lambda^n N(x,y)} = -\frac{M(\lambda x,\lambda y)}{N(\lambda x,\lambda y)} = f(\lambda x,\lambda y)$$

Let $\lambda = \frac{1}{x}$, then $y' = f\left(1,\frac{y}{x}\right) = -\frac{M(1,y/x)}{N(1,y/x)}$. For simplicity, it is often simply written as

(6-32) $$y' = f\left(\frac{y}{x}\right)$$

and called a homogeneous equation for $x \neq 0$.

Note that the term "homogeneous equation" used here means the equation is formed by a homogeneous function, which is different to the "homogeneous equation" of an ODE without $q(t)$ on the right hand side.

Let's define $u = \frac{y}{x}$, then $y = ux$ and $y' = u + xu'$. Hence, (6-32) can be changed into $u + xu' = f(u)$, or

(6-33) $$\frac{du}{f(u) - u} = \frac{dx}{x}$$

which is an ODE with separable variables. After integrating, we can obtain u and the solution is $y=ux$.

Consider the 1st-order ODE $xy' = \frac{y^2}{x} - 2y$ for $x \neq 0$, which can be further rewritten as

(6-34) $$y' = \frac{y^2 - 2xy}{x^2} = -\frac{M(x,y)}{N(x,y)}$$

with $M(x,y) = -y^2 + 2xy$ and $N(x,y) = x^2$. Both $M(x,y)$ and $N(x,y)$ are homogeneous functions of order 2 and then

(6-35) $$y' = \left(\frac{y}{x}\right)^2 - 2\frac{y}{x} = f\left(\frac{y}{x}\right)$$

Define $u = \frac{y}{x}$, then $y = ux$ and $y' = u + xu'$. Hence, $u + xu' = u^2 - 2u$, i.e., $xu' = u^2 - 3u$, or

(6-36) $$\frac{du}{u^2 - 3u} = \frac{dx}{x}$$

After integration, we have $ln|u-3| - ln|u| - ln|x^3| = 3C_1$ with C_1 constant. That means $\frac{u-3}{ux^3} = \pm e^{3C_1}$ or

(6-37) $$y = \frac{3x}{1 - Cx^3}$$

where $C = \pm e^{3C_1}$ and $x \neq 0$.

Bernoulli Equations

A 1st-order ODE is called Bernoulli equation if it is expressed as the following form

(6-38) $$y' + p(x)y = r(x)y^{\alpha}$$

in which α is a real number.

The Bernoulli equation (6-38) is possessed of separable variables if $\alpha = 1$ and is linear if $\alpha = 0$. For the case of $\alpha \neq 1$, (6-38) can be transformed to a linear equation by defining

(6-39) $$v = y^{1-\alpha}$$

Its derivative is $v' = (1-\alpha)y^{-\alpha}y'$ and from (6-38) we have

(6-40) $$v' = (1-\alpha)y^{-\alpha}\left(r(x)y^{\alpha} - p(x)y\right) = (1-\alpha)(r(x) - p(x)v)$$

or

(6-41) $$v' + (1-\alpha)p(x)v = (1-\alpha)r(x)$$

which is linear and can be solved by the methods introduced before.

Consider $y' + xy = 2xy^2$, which is a Bernoulli equation with $\alpha = 2$, where $p(x) = x$ and $r(x) = 2x$. Let $v = y^{1-\alpha} = y^{-1}$, then

(6-42) $$v' = -y^{-2}y' = -y^{-2}\left(2xy^2 - xy\right) = -2x + xv$$

or

(6-43) $$v' - xv = -2x$$

which is a linear ODE. Then, choose $\mu = e^{\int(-x)dx} = e^{-x^2/2}$ as the integrating factor whose derivative is $\mu' = -xe^{-x^2/2} = -x\mu$. Then, multiplying μ into (6-43) gets $\mu v' - \mu xv = -2\mu x$ or $\mu v' + \mu' v = -2\mu x$. Hence,

(6-44) $$d(\mu v) = (\mu v' + \mu' v)dx = -2\mu x dx$$

which results in

(6-45) $$\mu v = \int(-2\mu x)dx + C = -\int e^{-x^2/2}dx^2 + C = 2e^{-x^2/2} + C$$

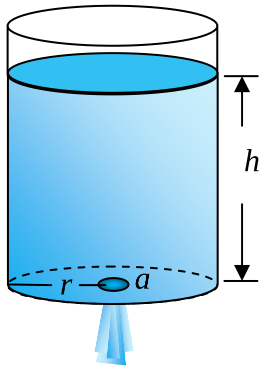
Hence, $e^{-x^2/2}y^{-1} = 2e^{-x^2/2} + C$ or

(6-46) $$y = \frac{1}{2 + Ce^{x^2/2}}$$

which is the explicit solution of $y' + xy = 2xy^2$.

For the example of discharging water through a drain hole at the bottom of a tank, the related ODE is

(6-47) $$\frac{dh}{dt} = -\frac{a}{\pi r^2}\sqrt{2gh}$$



or

(6-48) $$h' = -\frac{a\sqrt{2g}}{\pi r^2} h^{\frac{1}{2}}$$

which is a Bernoulli equation with $\alpha = \frac{1}{2}$,

$p(x) = 0$ and $r(x) = -\frac{a\sqrt{2g}}{\pi r^2}$. Let $v = h^{1-\alpha} = h^{\frac{1}{2}}$, then

(6-49) $$v' = \frac{1}{2} h^{-\frac{1}{2}} h' = \frac{1}{2} h^{-\frac{1}{2}} \left(-\frac{a\sqrt{2g}}{\pi r^2} h^{\frac{1}{2}} \right) = -\sqrt{\frac{g}{2}} \frac{a}{\pi r^2}$$

Hence, $v(t) = -\sqrt{\frac{g}{2}} \frac{a}{\pi r^2} t + C$, i.e.,

(6-50) $$h(t) = \left(-\sqrt{\frac{g}{2}} \frac{a}{\pi r^2} t + C \right)^2 = \frac{g}{2} \left(\frac{a}{\pi r^2} t - k \right)^2$$

where $k = C\sqrt{2/g}$ is an arbitrary constant.

Riccati Equations

A 1st-order ODE is called Riccati equation if it can be expressed as the following form

(6-51) $$y' = p(x)y^2 + q(x)y + r(x)$$

which is not a linear equation. Assume $Y(x)$ is a solution of (6-51) and define

(6-52) $$y = Y(x) + \frac{1}{z}$$

Then, the derivative of y is

(6-53) $$y' = Y'(x) - \frac{1}{z^2} z'$$

Substituing (6-52) and (6-53) into (6-51) obtains

(6-54) $$\begin{aligned} Y' - \frac{1}{z^2} z' &= p(x)\left(Y + \frac{1}{z} \right)^2 + q(x)\left(Y + \frac{1}{z} \right) + r(x) \\ &= p(x)Y^2 + q(x)Y + r(x) + p(x)\left(2\frac{Y}{z} + \frac{1}{z^2} \right) + \frac{1}{z} q(x) \end{aligned}$$

Since $Y(x)$ is a solution of (6-51), we know that $Y' = p(x)Y^2 + q(x)Y + r(x)$.

Hence, (6-54) is further simplified as $-\frac{1}{z^2}z' = p(x)\left(2\frac{Y}{z}+\frac{1}{z^2}\right)+\frac{1}{z}q(x)$, or

(6-55) $$z' + \left(q(x)+2p(x)Y(x)\right)z = -p(x)$$

which is a linear ODE and can be solved by the methods introduced before.

Consider $y' = xy^2 - \left(2x+\frac{1}{x}\right)y + x + \frac{1}{x}$ for x>0, which is a Riccati equation with $p(x)=x$, $q(x)=-\left(2x+\frac{1}{x}\right)$ and $r(x)=x+\frac{1}{x}$. It can be found that $Y(x)=1$ is a solution; hence, from (6-52) we let

(6-56) $$y = Y(x)+\frac{1}{z} = 1+\frac{1}{z}$$

and from (6-55), we achieve the linear equation as

(6-57) $$z' - \frac{1}{x}z = -x$$

Further choose the integrating factor as

(6-58) $$\mu = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-ln\,x} = x^{-1}$$

whose derivative is $\mu' = -x^{-2} = -\frac{\mu}{x}$. After multiplying μ into (6-57), it can be obtained that $\mu z' - \frac{1}{x}z\mu = -x\mu$ or $\frac{d}{dx}(\mu z) = -1$. Clearly, the result can be found as $\mu z = -x + C$ with C constant. Hence,

(6-59) $$z = -x^2 + Cx$$

and then the solution in (6-56) is

(6-60) $$y = 1+\frac{1}{z} = \frac{-x^2+Cx+1}{-x^2+Cx}$$

which is an explicit solution.

Further consider a free falling object with mass m. If it encounters a quadratic friction, then the dynamic model can be described as

(6-61) $$mv'(t) = mg - \beta v^2(t)$$

where $v(t)$ is the velocity and $-\beta v^2(t)$ is the quadratic friction. It can be rewritten as

(6-62) $$v' = -\frac{\beta}{m}v^2 + g = p(t)v^2 + q(t)v + r(t)$$

which is a Riccati equation with $p(t) = -\frac{\beta}{m}$, $q(t) = 0$ and $r(t) = g$. First, find a solution, which is $Y(t) = \sqrt{\frac{mg}{\beta}}$ and let

(6-63) $$v = Y(x) + \frac{1}{z} = \sqrt{\frac{mg}{\beta}} + \frac{1}{z}$$

From (6-55), we have the following linear ODE

(6-64) $$z' - 2\sqrt{\frac{\beta g}{m}}z = \frac{\beta}{m}$$

and the solution is

(6-65) $$z = ke^{2\sqrt{\frac{\beta g}{m}}t} - \frac{1}{2}\sqrt{\frac{\beta}{mg}}$$

Hence,

(6-66) $$v = \sqrt{\frac{mg}{\beta}} + \left(ke^{2\sqrt{\frac{\beta g}{m}}t} - \frac{1}{2}\sqrt{\frac{\beta}{mg}}\right)^{-1}$$

which is an explicit solution. If the initial velocity is zero, i.e., $v(0) = 0$. Then, from (6-66), we obtain $\sqrt{\frac{mg}{\beta}} + \left(k - \frac{1}{2}\sqrt{\frac{\beta}{mg}}\right)^{-1} = 0$, or $k = -\frac{1}{2}\sqrt{\frac{\beta}{mg}}$ and the solution in (6-65) becomes

(6-67) $$v(t) = \sqrt{\frac{mg}{\beta}}\left(\frac{e^{\sqrt{\frac{\beta g}{m}}t} - e^{-\sqrt{\frac{\beta g}{m}}t}}{e^{\sqrt{\frac{\beta g}{m}}t} + e^{-\sqrt{\frac{\beta g}{m}}t}}\right) = \sqrt{\frac{mg}{\beta}}\,tanh\sqrt{\frac{\beta g}{m}}t$$

As $t \to \infty$, we have the terminal velocity $v(\infty) = \sqrt{\frac{mg}{\beta}}$.