5. General First-Order Ordinary Differential Equations

Although we have learned how to solve CODE, there are a lot of engineering problems related to general ODEs, not CODEs. From now on, we will focus on the general ODEs and start with 1st-order ODEs.

There are two kinds of expression to represent a 1st-order ODE, which are described as

(5-1) $$F(x, y(x), y'(x)) = 0$$

or

(5-2) $$y'(x) = f(x, y(x))$$

where $y(x)$ is the unknown function and x is the independent variable. Here, we will focus on the form of (5-2) and simply represent it as $y' = f(x, y)$. For example, $y' = 3y + 6$ and $y' = xy^{-2} + e^{-y/2}$ are 1st-order ODEs. For the first one $y' = 3y + 6$, it is a CODE and we have learned how to solve the equation. For the second one $y' = xy^{-2} + e^{-y/2}$, since it is not a CODE, we have to learn different methods to determine the solution.

For the 1st-order ODE (5-2) defined on an interval I of x, if $\varphi(x)$ is a solution of (5-2), then it should satisfy

(5-3) $$\varphi' = f(x, \varphi), \quad \text{for } x \in I.$$

For example,

(5-4) $$\varphi = -2 + ke^{-3x}$$

is a general solution of

(5-5) $$y' = 3y + 6$$

where k is an arbitrary number.

Explicit and Implicit Solutions

The solution $y = \varphi(x)$ is called an explicit solution since $\varphi(x)$ can be directly determined by x. If a solution cannot be explicitly expressed as a function of x, then it is called an implicit solution. For example, a 1st-order ODE is given as

(5-6) $$y' = \frac{y + 2x}{e^y - x}, \quad \text{for } e^y \neq x$$

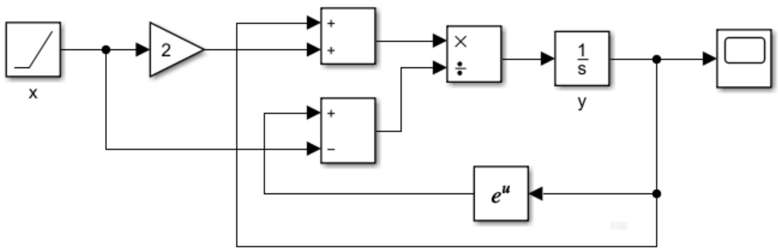
and its solution cannot be explicitly expressed as $y = \varphi(x)$. However, we can find that its solution should satisfy the following equation

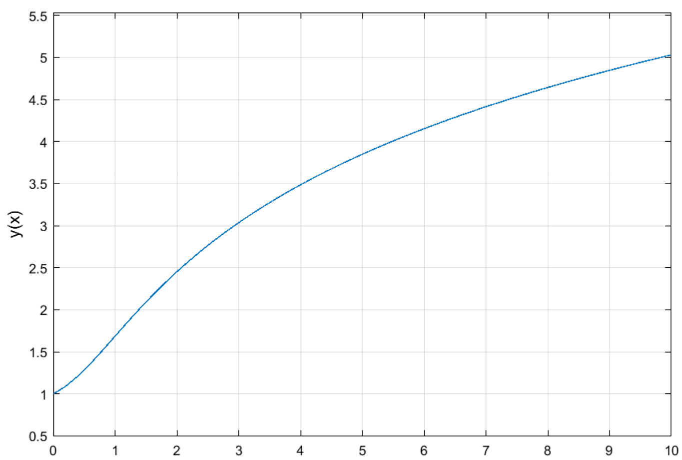
(5-7) $$xy + x^2 - e^y = k$$

where k is an arbitrary number. To verify that the $y(x)$ in (5-7) is a solution of (5-6), let's take the derivative of (5-7) with respect to x and then we will obtain that $y + xy' + 2x - e^y y' = 0$, which can be rearranged as $y' = \dfrac{y+2x}{e^y - x}$. Therefore, $y(x)$ in (5-7) is indeed a solution of (5-6). That means (5-7) is an implicit way to represent the solution $y(x)$. Hence, we call (5-7) an implicit solution.

Numeric Solutions

In addition to explicit and implicit solution, we can also use the numeric result to represent the solution. For example, (5-6) can be solved by the Matlab/Simulink for $y(0)=1$. From (5-7), we know that $k=-e=-2.71828$. The block diagram and numeric result of the solution $y(x)$ is shown below.





Next, we will introduce some ODEs with some special properties and show the way to solve them.

Separable Variables

In general, a 1st-order ODE with variables separable is shown as the following form

(5-8) $$y' = g(x)h(y)$$

If $h(y) \neq 0$, then it can be also expressed as a differential form

(5-9) $$\frac{dy}{h(y)} = g(x)dx$$

It is obvious that the variables *x* and *y* are totally separated. Take integration on both sides, and get

(5-10) $$\int \frac{dy}{h(y)} = \int g(x)dx + k$$

where *k* is a constant. Note that (5-10) is a general implicit solution of *y*(*x*).

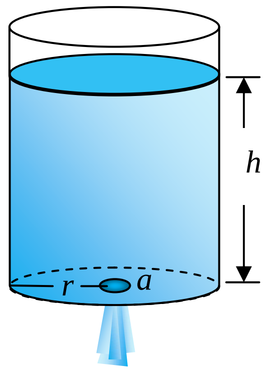
For example, consider $e^{x+y}y' = \frac{1}{y}$ with *y*(0)=1, which can be rearranged as $y' = e^{-x}\left(\frac{1}{y}e^{-y}\right)$. Obviously, it has separable variables with $g(x) = e^{-x}$ and $h(y) = \frac{1}{y}e^{-y}$. From (5-9), we have $ye^{y}dy = e^{-x}dx$ and take integration on both sides to get $\int ye^{y}dy = \int e^{-x}dx + k$, which results in a general implicit solution shown as

(5-11) $$(y-1)e^{y} = -e^{-x} + k$$

From the initial condition *y*(0)=1, we have $k = 1$, i.e., *y* satisfies

(5-12) $$(y-1)e^{y} = -e^{-x} + 1$$

which is an implicit solution for *y*(5-0)=1.



Next, let's introduce an application concerning separable variables, which is a cylindrical water tank with radius *r* shown on the right. The water level is *h* and we want to discharge the water through a drain hole at the bottom. If the cross-sectional area of the hole is *a*, how long it will take to empty the tank?

The volume of water discharged from the drain hole is *a*·*dx*, where *dx* is the distance that the water leaves down from the hole. Since the volume of discharged water is equal to the decreased volume $-(\pi r^2)dh$ of the water on the top, we have $adx = -\pi r^2 dh$ or

(5-13) $$-\pi r^2 \frac{dh}{dt} = a\frac{dx}{dt} = av = a\sqrt{2gh}$$

where $v = \dfrac{dx}{dt}$ is the velocity of the discharged water from the drain hole, and according to the Torricelli's theorem, the velocity is $v = \sqrt{2gh}$. Hence,

(5-14) $$\frac{dh}{\sqrt{2gh}} = -\frac{a}{\pi r^2}dt$$

which clearly has separable variables. Further, taking integral on both sides yields $\sqrt{\dfrac{2h}{g}} = -\dfrac{a}{\pi r^2}t + k$, where k is a constant. After rearrangement, the water level is obtained as

(5-15) $$h = \frac{g}{2}\left(\frac{a}{\pi r^2}t - k\right)^2$$

Let h_0 be the initial water level at t=0, then $k = \sqrt{2h_0/g}$, i.e.,

(5-16) $$h(t) = \frac{g}{2}\left(\frac{a}{\pi r^2}t - \sqrt{\frac{2h_0}{g}}\right)^2$$

If the tank is empty at t=t_f, then $h(t_f) = \dfrac{a}{\pi r^2}t_f - \sqrt{\dfrac{2h_0}{g}} = 0$, or $t_f = \dfrac{\pi r^2}{a}\sqrt{\dfrac{2h_0}{g}}$.

Therefore, it takes the time $t_f = \dfrac{\pi r^2}{a}\sqrt{\dfrac{2h_0}{g}}$ to completely discharged the water tank from the initial water level $h(0) = h_0$.

Exactness and Potential Function

In general, a 1st-order ODE $y' = f(x,y)$ can be rewritten into the following form

(5-17) $$M(x,y) + N(x,y)y' = 0$$

or

(5-18) $$M(x,y)dx + N(x,y)dy = 0$$

where $M(x,y)$, $N(x,y)$, $\partial M/\partial y$ and $\partial N/\partial x$ are all continuous within a rectangle region *S* in the *x*-*y* plane. An interesting thing happens if a function $\varphi(x,y)$ satisfies $\frac{\partial \varphi}{\partial x} = M(x,y)$ and $\frac{\partial \varphi}{\partial y} = N(x,y)$, then (5-17) and (5-18) can be expressed as

(5-19) $$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0$$

and

(5-20) $$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0$$

Taking the derivative of $\varphi(x,y)$ with respect to *x* gets

(5-21) $$\frac{d}{dx}\varphi(x,y) = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0$$

which also implies the differential of $\varphi(x,y)$ is

(5-22) $$d\varphi(x,y(x)) = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0$$

Therefore, (5-22) is equivalent to

(5-23) $$\varphi(x,y) = C$$

where *C* is a constant. That means the solution *y*(*x*) can be implicitly represented by $\varphi(x,y) = C$. Here, the function $\varphi(x,y)$ is usually called a potential function. Besides, (5-17) is said to be exact within a rectangle region *S* in the *x*-*y* plane.

In conclusion, if $M(x,y) + N(x,y)y' = 0$ is exact, then there exists an implicit solution $\varphi(x,y) = C$, where $\frac{\partial \varphi}{\partial x} = M(x,y)$ and $\frac{\partial \varphi}{\partial y} = N(x,y)$. Also, from the truth of

(5-24) $$\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = \frac{\partial^2 \varphi}{\partial x \partial y}$$

we know that

(5-25) $$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Now, one question is raised: Can we declare that if (5-25) is true, then

$M(x,y)+N(x,y)y'=0$ is exact? The answer is YES! Let's explain it below.

Assume both $\frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial N(x,y)}{\partial x}$ are continuous on a rectangle region *S*. Under the condition $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$ as shown in (5-25), we choose an arbitrary point (x_0,y_0) in *S* and define

(5-26) $$\varphi(x,y)=\int_{x_0}^{x}M(s,y_0)ds+\int_{y_0}^{y}N(x,t)dt$$

for any point (*x*,*y*) in *S*. Then, we have

(5-27) $$\begin{aligned}\frac{\partial\varphi}{\partial x}&=\frac{\partial}{\partial x}\int_{x_0}^{x}M(s,y_0)ds+\frac{\partial}{\partial x}\int_{y_0}^{y}N(x,t)dt\\&=M(x,y_0)+\int_{y_0}^{y}\frac{\partial N(x,t)}{\partial x}dt=M(x,y_0)+\int_{y_0}^{y}\frac{\partial M(x,t)}{\partial t}dt\\&=M(x,y_0)+M(x,y)-M(x,y_0)=M(x,y)\end{aligned}$$

(5-28) $$\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial y}\int_{y_0}^{y}N(x,t)dt=N(x,y)$$

Obviously, if $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$, then there exists a function $\varphi(x,y)$ as shown in (5-26), which satisfies $\frac{\partial\varphi}{\partial x}=M(x,y)$ and $\frac{\partial\varphi}{\partial y}=N(x,y)$. That means $M(x,y)+N(x,y)y'=0$ is exact.

For example, consider the 1st-order ODE $x(1+3y)+(x^2y-x)y'=0$, where $M=x(1+3y)$ and $N=x^2y-x$. Since $\frac{\partial M}{\partial y}=3x$ is not equal to $\frac{\partial N}{\partial x}=2xy-1$, we know that $x(1+3y)+(x^2y-x)y'=0$ is not exact and we cannot solve it by choosing a potential function.

Next, consider $y'=-\frac{xy^2-1}{x^2y+e^{-y}}$. To solve it, we first write it into the following form

(5-29) $$M+Ny'=xy^2-1+(x^2y+e^{-y})y'=0$$

where $M=xy^2-1$ and $N=x^2y+e^{-y}$. Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=2xy$, we know that (5-29) is exact. Hence, from $\frac{\partial \varphi}{\partial x} = M(x,y)$, we have

(5-30) $$\varphi = \int Mdx = \frac{1}{2}x^2y^2 - x + h(y)$$

and from $\frac{\partial \varphi}{\partial y} = N(x,y)$, we have

(5-31) $$\frac{\partial \varphi}{\partial y} = x^2y + \frac{d}{dy}h(y) = x^2y + e^{-y}$$

i.e., $\frac{d}{dy}h(y) = e^{-y}$ or $h(y) = -e^{-y}$. Therefore, the potential function in (5-30) is expressed as

(5-32) $$\varphi = \frac{1}{2}x^2y^2 - x - e^{-y}$$

That means the implicit solution is $\varphi = C$ or

(5-33) $$\frac{1}{2}x^2y^2 - x - e^{-y} = C$$

where C is a constant.

Further, we take $e^x \cos y + x - (e^x \sin y + 2)y' = 0$ as the example, where $M = e^x \cos y + x$ and $N = -(e^x \sin y + 2)$. Hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -e^x \sin y$, i.e., the ODE is exact. From $\frac{\partial \varphi}{\partial y} = N(x,y)$, it can be obtained that

(5-34) $$\varphi = \int Ndy = e^x \cos y - 2y + g(x)$$

From $\frac{\partial \varphi}{\partial x} = M(x,y) = e^x \cos y + x$, we have

(5-35) $$\frac{\partial \varphi}{\partial x} = e^x \cos y + g'(x) = e^x \cos y + x$$

which leads to $g'(x) = x$ or $g(x) = \frac{1}{2}x^2$. The potential function in (5-34) is then expressed as

(5-36) $$\varphi = e^x \cos y - 2y + \frac{1}{2}x^2$$

and the implicit solution is $\varphi = e^x \cos y - 2y + \frac{1}{2}x^2 = C$, where C is a constant.

Integrating Factor for Exactness

Most of the ODEs are not exact. However, some of them can be modified into exact equations by multiplying a nonzero function $\mu(x,y)$, which is called an integrating factor. For example,

(5-37) $$2y^2-3xy+\left(6xy-3x^2\right)y'=0$$

where $M=2y^2-3xy$ and $N=6xy-3x^2$. Then, we have $\frac{\partial M}{\partial y}=4y-3x$ and $\frac{\partial N}{\partial x}=6y-6x$. Clearly, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ which means (5-37) is not exact. However, if we multiply a function $\mu(x,y)=y$ into (5-37), then

(5-38) $$2y^3-3xy^2+\left(6xy^2-3x^2y\right)y'=0,\quad \text{for}\quad y\neq 0$$

where $M=2y^3-3xy^2$ and $N=6xy^2-3x^2y$. Then, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=6y^2-6xy$, which means (5-38) is exact. Hence, the potential function can be obtained as $\varphi=2xy^3-\frac{3}{2}x^2y^2$, and the implicit solution for (5-38) is shown as

(5-39) $$\varphi=2xy^3-\frac{3}{2}x^2y^2=C,\qquad \text{for}\quad y\neq 0$$

with C constant. However, y=0 is also a solution of (5-37) and can be included in (5-39). Hence, the implicit solution of (5-37) is

(5-40) $$2xy^3-\frac{3}{2}x^2y^2=C$$

which contain the solution y=0, different to (5-39).

Next, let's introduce some different methods to determine integrating factors for exactness.

Consider $x-2xy-y'=0$, which is not exact since $\frac{\partial M}{\partial y}=-2x$ and $\frac{\partial N}{\partial x}=0$ are different. Multiply an integrating factor μ to obtain

(5-41) $$\mu(x-2xy)-\mu y'=0$$

where $M=\mu(x-2xy)$ and $N=-\mu$. The exactness is guaranteed if the condition $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ is satisfied, i.e.,

(5-42) $$(x-2xy)\frac{\partial \mu}{\partial y}-2\mu x=-\frac{\partial \mu}{\partial x}$$

If we choose $\mu \equiv \mu(x)$, then $\frac{\partial \mu}{\partial y}=0$ and (5-42) becomes $-2\mu x=-\frac{d\mu}{dx}$. Hence,

(5-43) $$\frac{d\mu}{\mu}=2xdx$$

which has separable variables. Further taking integration yields $ln|\mu|=x^2$ or $\mu=\pm e^{x^2}$. Now, select $\mu=e^{x^2}$ as the integrating factor and express (5-41) as

(5-44) $$e^{x^2}(x-2xy)-e^{x^2}y'=0$$

where $\frac{\partial \varphi}{\partial x}=M=e^{x^2}(x-2xy)$ and $\frac{\partial \varphi}{\partial y}=N=-e^{x^2}$. From $\frac{\partial \varphi}{\partial y}=-e^{x^2}$, it can be obtained that

(5-45) $$\varphi=-e^{x^2}y+g(x)$$

and from $\frac{\partial \varphi}{\partial x}=e^{x^2}(x-2xy)$ we have

(5-46) $$\frac{\partial \varphi}{\partial x}=-2e^{x^2}xy+g'(x)=e^{x^2}(x-2xy)$$

Clearly, $g'(x)=xe^{x^2}$ or $g(x)=\frac{1}{2}e^{x^2}$. Then, the potential function in (5-45) is

(5-47) $$\varphi=-e^{x^2}y+\frac{1}{2}e^{x^2}=e^{x^2}\left(\frac{1}{2}-y\right)$$

and the solution is $\varphi=e^{x^2}\left(\frac{1}{2}-y\right)=C$ or explicitly expressed as

(5-48) $$y=\frac{1}{2}-Ce^{-x^2}$$

where C is a constant. Since $\mu=e^{x^2}\neq 0$, (5-48) is also the explicit solution of the original ODE $x-2xy-y'=0$, which is not exact.

Further consider $3y^2+2xy+(3xy+x^2)y'=0$ as an example, which is not exact since $\frac{\partial M}{\partial y}=6y+2x$ and $\frac{\partial N}{\partial x}=3y+2x$ are different. Choose μ as the integrating factor, i.e.,

(5-49) $$\mu\left(3y^2+2xy\right)+\mu\left(3xy+x^2\right)y'=0$$

where $M=\mu\left(3y^2+2xy\right)$ and $N=\mu\left(3xy+x^2\right)$. The exactness requires that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, which is equivalent to

(5-50) $$\left(3y^2+2xy\right)\frac{\partial \mu}{\partial y}+\mu(6y+2x)=\left(3xy+x^2\right)\frac{\partial \mu}{\partial x}+\mu(3y+2x)$$

After rearrangement, it is written as

(5-51) $$\left(3y^2+2xy\right)\frac{\partial \mu}{\partial y}-\left(3xy+x^2\right)\frac{\partial \mu}{\partial x}+3\mu y=0$$

For simplicity, we may choose $\mu\equiv\mu(x)$ or $\mu\equiv\mu(y)$, but it is still difficult for us to solve (5-51). Instead, let's try $\mu=x^a y^b$, then (5-51) becomes

(5-52) $$bx^a y^{b-1}\left(3y^2+2xy\right)-ax^{a-1}y^b\left(3xy+x^2\right)+3x^a y^{b+1}=0$$

Further multiply it with $x^{-a}y^{-b}$ on both sides to obtain

(5-53) $$3(b-a+1)xy^2+(2b-a)x^2y=0$$

for $x\neq 0$ and $y\neq 0$. Hence, $b-a+1=0$ and $2b-a=0$. It can be found that *a*=2 and *b*=1, i.e., $\mu=x^2y$. Now, we rewrite (5-49) as

(5-54) $$\left(3x^2y^3+2x^3y^2\right)+\left(3x^3y^2+x^4y\right)y'=0$$

where $M=\frac{\partial \varphi}{\partial x}=3x^2y^3+2x^3y^2$ and $N=\frac{\partial \varphi}{\partial y}=3x^3y^2+x^4y$. Then, taking the integration for $\frac{\partial \varphi}{\partial y}=3x^3y^2+x^4y$ yields

(5-55) $$\varphi=x^3y^3+\frac{1}{2}x^4y^2+g(x)$$

From $\frac{\partial \varphi}{\partial x}=3x^2y^3+2x^3y^2$, we have

(5-56) $$\frac{\partial \varphi}{\partial x}=3x^2y^3+2x^3y^2+g'(x)=3x^2y^3+2x^3y^2$$

which implies $g'(x)=0$. For simplicity, let $g(x)=0$, then the potential function in (5-55) is

(5-57) $$\varphi = x^3 y^3 + \frac{1}{2} x^4 y^2$$

and the implicit solution is

(5-58) $$x^3 y^3 + \frac{1}{2} x^4 y^2 = C$$

with C constant.

The method of integrating factor can be also used to solve the ODE with separable variables, i.e., $y' = g(x)h(y)$ or $g(x)h(y) - y' = 0$. Choose the integrating factor as $\mu = \frac{1}{h(y)}$, then

(5-59) $$g(x) - \frac{1}{h(y)} y' = 0$$

where $M(x) = g(x)$ and $N(y) = -\frac{1}{h(y)}$. It is easy to check that

(5-60) $$\frac{\partial M(x)}{\partial y} = \frac{\partial N(y)}{\partial x} = 0$$

which means (5-59) is exact. Then, from $\frac{\partial \varphi}{\partial y} = N(y) = -\frac{1}{h(y)}$, we hve

(5-61) $$\varphi = -\int \frac{1}{h(y)} dy + r(x)$$

From $\frac{\partial \varphi}{\partial x} = M = g(x)$, it can be obtained that

(5-62) $$\frac{\partial \varphi}{\partial x} = r'(x) = g(x)$$

which implies $r(x) = \int g(x) dx$. Then, the potential function in (5-61) is

(5-63) $$\varphi = -\int \frac{1}{h(y)} dy + \int g(x) dx$$

and the implicit solution is

(5-64) $$-\int \frac{1}{h(y)} dy + \int g(x) dx = C$$

with C constant.