4. Inverse Laplace Transform

Given *F*(4-*s*), how to transform it back to $f(t)=\mathcal{L}^{-1}\{F(s)\}$? Here, we will introduce a method not to use the definition of inverse Laplace transform; instead, we will adopt partial fraction expansion to partition *F*(4-*s*) into suitable simple terms, i.e., $F(s)=\sum_{k=1}^{n}F_k(s)$. Then, match each simple term $F_k(s)$ to $f_k(t)$ and obtain the result as $f(t)=\sum_{k=1}^{n}f_k(t)$. Below use some examples to show the operation of inverse Laplace transform.

Simple poles

Consider the Laplace transform $F(s)=\dfrac{s^2+12}{s(s+2)(s+3)}$. From partial fraction expansion, we have

(4-1) $$F(s)=\frac{s^2+12}{s(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+3}$$

where the numerator can be rearranged as

(4-2) $$s^2+12=A(s+2)(s+3)+Bs(s+3)+Cs(s+2)$$

There are two methods to solve *A*, *B* and *C*. For the first method, we rewrite (4-2) as

(4-3) $$s^2+12=(A+B+C)s^2+(5A+3B+2C)s+6A$$

and compare the coefficients to get three linear equations: 6*A*=12, 5*A*+3*B*+2*C*=0 and *A*+*B*+*C*=1. Solving these equations yields *A*=2, *B*=−8 and *C*=7.

For the second method, apply *s*=0, *s*=−2 and *s*=−3 to (4-2) and solve the coefficients *A*, *B* and *C* as below:

(4-4) $$A=\left.\frac{s^2+12}{(s+2)(s+3)}\right|_{s=0}=sF(s)\big|_{s=0}=2$$

(4-5) $$B=\left.\frac{s^2+12}{s(s+3)}\right|_{s=-2}=(s+2)F(s)\big|_{s=-2}=-8$$

(4-6) $$C=\left.\frac{s^2+12}{s(s+2)}\right|_{s=-3}=(s+3)F(s)\big|_{s=-3}=7$$

Hence, from either of the above two methods we can obtain the partial fraction expansion of $F(s)$ as

(4-7) $$F(s)=\frac{2}{s}+\frac{-8}{s+2}+\frac{7}{s+3}$$

From the mapping set of Laplace transform, the inverse Laplace transform of $F(s)$ is

(4-8) $$f(t)=\mathcal{L}^{-1}\{F(s)\}=2-8e^{-2t}+7e^{-3t}$$

for $t\geq 0$, i.e., $f(t)=\left(2-8e^{-2t}+7e^{-3t}\right)u(t)$.

Repeated poles

Consider the Laplace transform $F(s)=\dfrac{s^3+2s+6}{s(s+1)^2(s+3)}$. From partial fraction expansion, we have

(4-9) $$F(s)=\frac{s^3+2s+6}{s(s+1)^2(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{D}{s+3}$$

and the numerator is

(4-10) $$s^3+2s+6=A(s+1)^2(s+3)+Bs(s+1)(s+3)+Cs(s+3)+Ds(s+1)^2$$

First, apply $s=0$, $s=-1$ and $s=-3$ to (4-10) and solve the coefficients A, C and D as below:

(4-11) $$A=sF(s)\big|_{s=0}=\left.\frac{s^3+2s+6}{(s+1)^2(s+3)}\right|_{s=0}=2$$

(4-12) $$C=(s+1)^2F(s)\big|_{s=-1}=\left.\frac{s^3+2s+6}{s(s+3)}\right|_{s=-1}=-\frac{3}{2}$$

(4-13) $$D=(s+3)F(s)\big|_{s=-3}=\left.\frac{s^3+2s+6}{s(s+1)^2}\right|_{s=-3}=\frac{9}{4}$$

To determine B, we can compare the coefficient of s^3, i.e., $A+B+D=1$. Hence, $B=1-A-D=-\dfrac{13}{4}$. The other method to determine B is based on the following process:

(4-14) $$(s+1)^2F(s)=\frac{s^3+2s+6}{s(s+3)}=(s+1)^2\left(\frac{A}{s}+\frac{D}{s+3}\right)+B(s+1)+C$$

Taking derivative obtains

(4-15)
$$\frac{d}{ds}\left((s+1)^2 F(s)\right)=\frac{d}{ds}\left(\frac{s^3+2s+6}{s(s+3)}\right)$$
$$=2(s+1)\left(\frac{A}{s}+\frac{D}{s+3}\right)+(s+1)^2\frac{d}{ds}\left(\frac{A}{s}+\frac{D}{s+3}\right)+B$$

Let s=–1, then

(4-16)
$$B=\frac{d}{ds}\left((s+1)^2 F(s)\right)\Big|_{s=-1}=\frac{d}{ds}\left(\frac{s^3+2s+6}{s(s+3)}\right)\Bigg|_{s=-1}$$
$$=\frac{(3s^2+2)(s^2+3s)-(s^3+2s+6)(2s+3)}{(s^2+3s)^2}\Bigg|_{s=-1}=-\frac{13}{4}$$

which is the same as the one obtained from the comparison of the coefficient of s^3. Hence,

(4-17)
$$F(s)=\frac{s^3+2s+6}{s(s+1)^2(s+3)}=\frac{2}{s}+\frac{-13/4}{s+1}+\frac{-3/2}{(s+1)^2}+\frac{9/4}{s+3}$$

and the inverse Laplace transform is

(4-18)
$$f(t)=\mathcal{L}^{-1}\{F(s)\}=2-\frac{13}{4}e^{-t}-\frac{3}{2}te^{-t}+\frac{9}{4}e^{-3t}$$

for $t\geq 0$, i.e., $f(t)=\left(2-\frac{13}{4}e^{-t}-\frac{3}{2}te^{-t}+\frac{9}{4}e^{-3t}\right)u(t)$.

Complex poles

Consider the Laplace transform $F(s)=\frac{10}{(s+1)(s^2+4s+13)}$. From partial fraction expansion, we have

(4-19)
$$F(s)=\frac{10}{(s+1)(s^2+4s+13)}=\frac{A}{s+1}+\frac{B(s+2)+3C}{(s+2)^2+3^2}$$

where A can be determined by

(4-20)
$$A=(s+1)F(s)\big|_{s=-1}=\frac{10}{s^2+4s+13}\Big|_{s=-1}=1$$

Then, choose s=–2, (4-19) becomes

(4-21)
$$F(-2)=-\frac{10}{9}=-A+\frac{C}{3}=-1+\frac{C}{3}$$

which results in $C=-\frac{1}{3}$. Finally, apply s=0 to (4-19) and obtain

(4-22) $$F(0)=\frac{10}{13}=A+\frac{2B+3C}{13}=1+\frac{2B}{13}-\frac{1}{13}$$

which results in $B=-1$. Hence,

(4-23) $$F(s)=\frac{10}{(s+1)(s^2+4s+13)}=\frac{1}{s+1}-\frac{s+2}{(s+2)^2+3^2}-\frac{1}{3}\frac{3}{(s+2)^2+3^2}$$

and the inverse Laplace transform is

(4-24) $$f(t)=\mathcal{L}^{-1}\{F(s)\}=e^{-t}-e^{-2t}\cos 3t-\frac{1}{3}e^{-2t}\sin 3t$$

for $t\geq 0$, i.e., $f(t)=\left(e^{-t}-e^{-2t}\cos 3t-\frac{1}{3}e^{-2t}\sin 3t\right)u(t)$.

Application to the IVP of Linear CODEs

We have mentioned that the Laplace transform is one of the techniques to solve initial value problems (IVPs) of a CODE whose order is higher than two. Here, we will take some examples for demonstration, and you will see that the linear CODEs are no longer solved by complicated integration. Instead, they will be solved in an algebraic way based on the Laplace transform and inverse Laplace transform.

For the first example, let's consider an IVP of a 2nd-order CODE, which is expressed as

(4-25) $$y''+6y'+5y=f(t), \qquad y(0)=a \text{ and } y'(0)=b$$

Based on the Laplace transform of derivatives, we have

(4-26) $$s^2Y(s)-sy(0)-y'(0)+6(sY(s)-y(0))+5Y(s)=F(s)$$

where $F(s)$ and $Y(s)$ are the Laplace transforms of $f(t)$ and $y(t)$. Since $y(0)=a$ and $y'(0)=b$, we rewrite (4-26) into

(4-27) $$s^2Y(s)-sa-b+6(sY(s)-a)+5Y(s)=F(s)$$

and rearrange it as

(4-28) $$Y(s)=\frac{1}{s^2+6s+5}F(s)+\frac{(s+6)a+b}{s^2+6s+5}$$

If $f(t)=\delta(t)$ and a=b=0, then

(4-29) $$Y(s)=\frac{1}{s^2+6s+5}\mathcal{L}\{\delta(t)\}=\frac{1}{s^2+6s+5}$$

i.e.,

(4-30) $$Y(s)=\frac{1}{s^2+6s+5}=\frac{1}{4}\left(\frac{1}{s+1}-\frac{1}{s+5}\right)$$

Hence,

(4-31) $$y(t)=\frac{1}{4}\left(e^{-t}-e^{-5t}\right)u(t)$$

If $f(t)=cos\,3t$, $y(0)=a=0$ and $y'(0)=b=1$, (4-28) becomes

(4-32) $$s^2Y(s)-1+6sY(s)+5Y(s)=\mathcal{L}\{cos\,3t\}=\frac{s}{s^2+3^2}$$

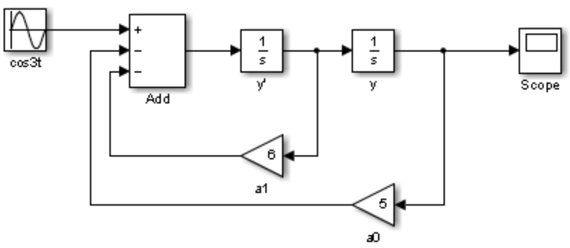
Hence,

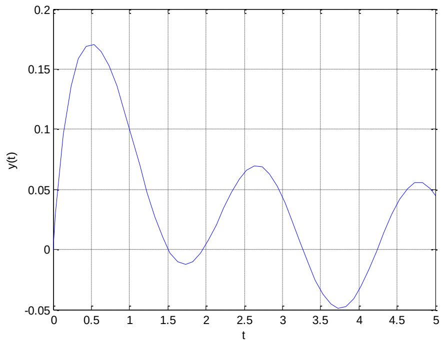
(4-33) $$Y(s)=\frac{s^2+s+9}{\left(s^2+3^2\right)\left(s^2+6s+5\right)}=\frac{A}{s+1}+\frac{B}{s+5}+\frac{Cs+3D}{s^2+3^2}$$

where $A=0.225$, $B=-0.2132$, $C=-0.0118$ and $D=0.0529$. The inverse Laplace transform is

(4-34) $$y(t)=\left(0.225e^{-t}-0.2132e^{-5t}-0.0118\,cos\,3t+0.0529\,sin\,3t\right)u(t)$$

Based on numeric analysis, we can solve (4-25) by the use of numeric tools Matlab/Simulink as well. The model of (4-25) is built in the figure on the left, where the block denoted as $\frac{1}{s}$ is an integrator, same as the Laplace transform of integral. The numeric results are plotted in the figure on the right.





For a 3rd-order CODE, we can still solve it by the Laplace transform and obtain the numeric results by Matlab/Simulink. Consider

(4-35) $$y'''+7y''+17y'+15y=15,\quad y(0)=0,\, y'(0)=1,\, y''(0)=0$$

Taking the Laplace transform yields

(4-36) $$s^3Y(s)-s+7\left(s^2Y(s)-1\right)+17sY(s)+15Y(s)=\frac{15}{s}$$

i.e.,

(4-37) $$Y(s)=\frac{s^2+7s+15}{s(s+3)(s^2+4s+5)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C(s+2)+D}{(s+2)^2+1}$$

where $A=1$, $B=-0.5$, $C=-0.5$ and $D=-1.5$. The inverse Laplace transform is

(4-38) $$y(t)=\left(1-0.5e^{-3t}-0.5e^{-2t}\cos t-1.5e^{-2t}\sin t\right)u(t)$$

Based on numeric analysis, we can solve (4-35) by Matlab/Simulink with model built in the figure on the left. The numeric results are plotted in the figure on the right.