3. Laplace Transform

Up to now, we have learned the method to solve 1st-order and 2nd-order CODEs. However, how to solve CODEs of higher order? Here, we will introduce a method usually used in engineering, which is called the Laplace transform named after its discoverer Pierre-Simon Laplace.

Definition of Laplace Transform

First, let's present the definition of Laplace transform. Consider a piecewise continuous function $f(t)$ for $t \geq 0$; in general, we assume $f(t)=0$ for $t<0$. Define the Laplace transform of $f(t)$ as

(3-1) $$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

where $t=0^-$ is a negative infinitesimal value and the term st in e^{-st} is dimensionless. For convenience, we also denote the Lapalce transform as

(3-2) $$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

to emphasize that the Laplace transform $\mathcal{L}\{f(t)\}=F(s)$ is a function of s. Note that $t \in [0,\infty]$ is a nonnegative real number and s is a complex number.

In engineering, since $f(t)$ often represents a function of time t in *sec*, the dimensionless of st implies that s is a frequency in sec^{-1}. Most importantly, $f(t)$ and $F(s)$ form a pair of the Laplace transform, where $f(t)$ is in the time-domain and $F(s)$ is in the frequency-domain.

In mathematics, the complex variable $s=\sigma+j\omega$ consists of the real part $Re(s)=\sigma$ and imaginary part $Im(s)=\omega$. Besides, $s=\sigma+j\omega$ is also known as a point on the complex plane, or s-plane, with axes σ and $j\omega$.

Convergence of Laplace Transform

The Laplace transform of $f(t)$ in (3-2) converges absolutely if the integral satisfies the following condition

(3-3) $$\int_{0^-}^{\infty} \left| f(t) e^{-st} \right| dt < \infty$$

which also implies

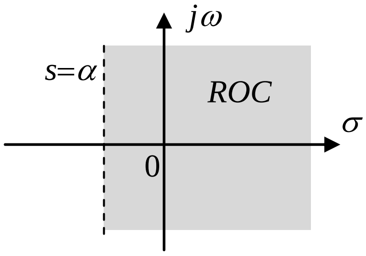
(3-4) $$\int_{0^-}^{\infty}\left|f(t)e^{-(\sigma+j\omega)t}\right|dt=\int_{0^-}^{\infty}e^{-\sigma t}\left|e^{-j\omega t}\right|\left|f(t)\right|dt<\infty$$

Since $\left|e^{-j\omega t}\right|=\left|\cos\omega t+j\sin\omega t\right|=\sqrt{\cos^2\omega t+\sin^2\omega t}=1$, we know that (3-3) is equivalent to

(3-5) $$\int_{0^-}^{\infty}e^{-\sigma t}\left|f(t)\right|dt<\infty$$

Hence, the Laplace transform is a kind of absolutely convergence. If the Laplace transform of *f*(*t*) exists, then there exists a region related to $Re(s)=\sigma$ on the *s*-plane, in which the *condition* (3-5) is satisfied. Such region is known as the region of convergence, usually denoted as *ROC* for short.

Due to the fact that $\left|F(s)\right|<\infty$ in the *ROC*, the Laplace transform of *f*(*t*) is formally expressed as



(3-6) $$F(s)=\int_{0^-}^{\infty}f(t)e^{-st}dt,\qquad Re(s)>\alpha$$

where $Re(s)>\alpha$ is the *ROC* as shown in the figure.

In electrical engineering, fortunately, most of the practical signals satisfy the condition (3-5) and their Laplace transforms exist in some specified *ROC*s. For simplicity and without loss of generality, we often neglect the *ROC*, i.e., $Re(s)>\alpha$, in (3-6) and only write *F*(*s*) to represent the Laplace transform, just like the form in (3-2).

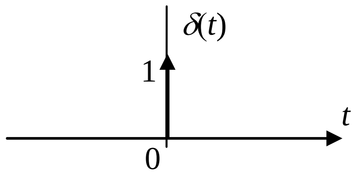
Dirac Delta Function

The lower bound $t=0^-$ in the integral is mainly used to include any discontinuity of *f*(*t*) occurring at *t*=0, such as the singularity function $\delta(t)$, named as Dirac delta function, or simply called delta function.

The delta function is shown in the figure, which is an ideal function subject to the following conditions:

(3-7) $$\delta(t)=0,\quad \text{for}\quad t\neq 0$$

(3-8) $$\int_{-\infty}^{\infty}\delta(t)dt=1$$



Clearly, the delta function has an infinite discontinuity at *t*=0 and the number ‘1’ marked on the arrow denotes that the area ‘under’ the arrow standing at *t*=0 is 1. In engineering, the delta function is usually called the unit impulse function since it is just like an impulse occurring at the moment t=0.

When the impulse happens at $t=t_0$, not t=0, the conditions (3-7) and (3-8) should be modified into the following expression:

(3-9) $$\delta(t-t_0)=0, \quad \text{for} \quad t \neq t_0$$

(3-10) $$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$$

In addition, there is an important property shown as

(3-11) $$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

which is called the sifting property to sift $f(t_0)$ from $f(t)$.

The existence of *ROC* in the Laplace transform implies that $f(t)$ can be uniquely determined from $F(s)$ by the inverse Laplace transform, which is expressed as

(3-12) $$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

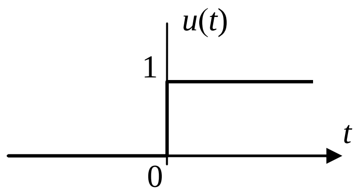
The inverse Laplace transform is a topic in the course "Complex Variable", not in this course. Later, we will not determine $f(t)$ from $F(s)$ based on (3-12). Instead, we previously develop a mapping set of the pair $\mathcal{L}\{f(t)\} = F(s)$ for some functions $f(t)$ often used in engineering. When $F(s)$ is given, we just determine $f(t)$ by checking the mapping set.

There is one important concept concerning the lower bound $t=0^-$ in the integral of (3-1), which defines the Laplace transform. We have explained that the lower bound $t=0^-$ is required if the delta function exists at t=0. On the other hand, if a function $f(t)$ without any area "standing" at t=0, then its Laplace transform can be simply defined as

(3-13) $$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

with lower bound t=0, not $t=0^-$. Since the integral in (3-13) neglects $f(t)$ for $t<0$, it can be expressed as

(3-14) $$\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t)u(t)\}$$



where $u(t)$ is depicted in the figure and defined as

(3-15) $$u(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$

In mathematics, $u(t)$ is also a singularity function and called the unit step function.

Mapping Set of Laplace Transform

Next, let's show the mapping set of the pair $\mathcal{L}\{f(t)\}=F(s)$ for some functions $f(t)$ commonly used in engineering, such as $\delta(t)$, $u(t)$, $r(t)$, e^{-at}, $e^{-j\omega_0 t}$, $cos\,\omega_0 t$, $sin\,\omega_0 t$, and t^n. The mapping set is listed as below:.

(3-16) $$\mathcal{L}\{\delta(t)\}=1$$

(3-17) $$\mathcal{L}\{u(t)\}=\frac{1}{s}$$

(3-18) $$\mathcal{L}\{r(t)\}=\frac{1}{s^2}$$

(3-19) $$\mathcal{L}\{e^{-at}\}=\frac{1}{s+a},\ (a\in\Re)$$

(3-20) $$\mathcal{L}\{e^{-j\omega_0 t}\}=\frac{1}{s+j\omega_0},\qquad (\omega_0\in\Re)$$

(3-21) $$\mathcal{L}\{cos\,\omega_0 t\}=\frac{s}{s^2+\omega_0^2}$$

(3-22) $$\mathcal{L}\{sin\,\omega_0 t\}=\frac{\omega_0}{s^2+\omega_0^2}$$

(3-23) $$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$$

Now, let's calculate all these Laplace transforms one by one. First, for the delta function $\delta(t)$ in (3-16), we have

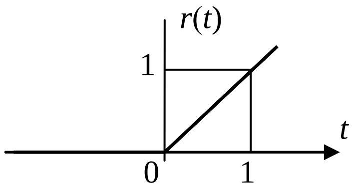
(3-24) $$\mathcal{L}\{\delta(t)\}=\int_{0^-}^{\infty}\delta(t)e^{-st}dt=e^{-s\cdot 0}=1$$

and the *ROC* is the whole *s*-plane. As for the unit step function in (3-17), its Laplace transform is

(3-25) $$\mathcal{L}\{u(t)\}=\int_{0}^{\infty}1\cdot e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_{t=0}^{\infty}=\lim_{t\to\infty}\frac{-1}{s}e^{-st}+\frac{1}{s}$$

where $\lim_{t\to\infty}\frac{-1}{s}e^{-st}=\lim_{t\to\infty}\frac{-1}{\sigma+j\omega}e^{-\sigma t}e^{-j\omega t}$. Since $\left|e^{-j\omega t}\right|=1$, if $Re(s)=\sigma>0$, we have $\lim_{t\to\infty}\frac{-1}{s}e^{-st}=0$. Hence, (3-26) $$\mathcal{L}\{u(t)\}=\frac{1}{s}, \quad \text{for } Re(s)>0$$

where $Re(s)>0$ is the *ROC*. For the ramp function in (3-18), which is depicted in the figure and defined as



(3-27) $$r(t)=\begin{cases} t, & \text{for } t\geq 0 \\ 0, & \text{for } t<0 \end{cases}$$

we can calculate its Laplace transform as

(3-28) $$\begin{aligned}\mathcal{L}\{r(t)\}&=\int_0^\infty te^{-st}dt=-\frac{1}{s}\int tde^{-st}\\ &=-\frac{1}{s}te^{-st}\Big|_{t=0}^{\infty}+\frac{1}{s}\int_0^\infty e^{-st}dt=\frac{1}{s}\int_0^\infty e^{-st}dt=\frac{1}{s^2}\end{aligned}$$

If $Re(s)=\sigma>0$, then $-\frac{1}{s}te^{-st}\Big|_{t=0}^{\infty}=0$. Thus,

(3-29) $$\mathcal{L}\{r(t)\}=\frac{1}{s^2}, \quad \text{for } Re(s)>0$$

where $Re(s)>0$ is the *ROC*.

Consider the exponential function e^{-at} in (3-19) where a is real. Its Laplace transform is calculated as below:

(3-30) $$\mathcal{L}\{e^{-at}\}=\int_0^\infty e^{-at}e^{-st}dt=\int_0^\infty e^{-(s+a)t}dt=-\frac{1}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty}$$

If $Re(s)>-a$, then $-\frac{1}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty}=\frac{1}{s+a}$. Therefore, it can be obtained that

(3-31) $$\mathcal{L}\{e^{-at}\}=\frac{1}{s+a}, \quad \text{for } Re(s)>-a$$

where $Re(s)>-a$ is the *ROC*. Similarly, for the exponential function $e^{-j\omega_0 t}$ in (3-20) with ω_0 real, its Laplace transform is

(3-32) $$\begin{aligned}\mathcal{L}\{e^{-j\omega_0 t}\}&=\int_0^\infty e^{-j\omega_0 t}e^{-st}dt=\int_0^\infty e^{-(s+j\omega_0)t}dt\\ &=-\frac{1}{s+j\omega_0}e^{-(s+j\omega_0)t}\Big|_{t=0}^{\infty}\end{aligned}$$

If $Re(s)>0$, then $-\frac{1}{s+j\omega_0}e^{-(s+j\omega_0)t}\Big|_{t=0}^{\infty}=\frac{1}{s+j\omega_0}$. Hence, we have

(3-33) $$\mathcal{L}\{e^{-j\omega_0 t}\}=\frac{1}{s+j\omega_0}, \quad \text{for } Re(s)>0$$

where the *ROC* is $Re(s)>0$.

The Laplace transforms of trigonometric functions in (3-21) and (3-22) can be derived from (3-33) as below

(3-34) $$\mathcal{L}\left\{e^{-j\omega_0 t}\right\} = \mathcal{L}\{\cos\omega_0 t\} - j\mathcal{L}\{\sin\omega_0 t\} = \frac{1}{s+j\omega_0} = \frac{s-j\omega_0}{s^2+\omega_0^2}$$

Hence,

(3-35) $$\mathcal{L}\{\cos\omega_0 t\} = \frac{s}{s^2+\omega_0^2}$$

(3-36) $$\mathcal{L}\{\sin\omega_0 t\} = \frac{\omega_0}{s^2+\omega_0^2}$$

where the *ROC* is $Re(s) > 0$, same as that of $\mathcal{L}\left\{e^{-j\omega_0 t}\right\}$.

Actually, (3-21) and (3-22) can be also derived by directly calculating their Laplace transforms as below:

(3-37) $$\begin{aligned}\mathcal{L}\{\sin\omega_0 t\} &= \int_0^\infty \sin\omega_0 t\, e^{-st} dt = -\frac{1}{s}\int \sin\omega_0 t\, de^{-st} \\ &= -\frac{1}{s}\left(\sin\omega_0 t\, e^{-st}\Big|_{t=0}^{\infty} - \int e^{-st} d\sin\omega_0 t\right)\end{aligned}$$

Obviously, if $Re(s)=\sigma>0$ then

(3-38) $$\begin{aligned}\mathcal{L}\{\sin\omega_0 t\} &= \frac{1}{s}\int e^{-st} d\sin\omega_0 t \\ &= \frac{\omega_0}{s}\int_{0-}^\infty \cos\omega_0 t\, e^{-st}\, dt = \frac{\omega_0}{s}\mathcal{L}\{\cos\omega_0 t\}\end{aligned}$$

Similarly, we have

(3-39) $$\begin{aligned}\mathcal{L}\{\cos\omega_0 t\} &= \int_{0^-}^\infty \cos\omega_0 t\, e^{-st}\, dt = -\frac{1}{s}\int \cos\omega_0 t\, de^{-st} \\ &= -\frac{1}{s}\left(\cos\omega_0 t\, e^{-st}\Big|_{t=0}^{\infty} - \int e^{-st} d\cos\omega_0 t\right) \\ &= -\frac{1}{s}\left(-1 + \omega_0\int_0^\infty \sin\omega_0 t\, e^{-st} dt\right) \\ &= \frac{1}{s} - \frac{\omega_0}{s}\mathcal{L}\{\sin\omega_0 t\} = \frac{1}{s} - \frac{\omega_0^2}{s^2}\mathcal{L}\{\cos\omega_0 t\}\end{aligned}$$

From (3-38) and (3-39), it can be obtained that $\mathcal{L}\{\cos\omega_0 t\} = \dfrac{s}{s^2+\omega_0^2}$ and

$\mathcal{L}\{\sin\omega_0 t\} = \dfrac{\omega_0}{s^2+\omega_0^2}$, as expected.

Finally, let's consider the function t^n in (3-23), whose Laplace transform is

(3-40)
$$\mathcal{L}\{t^n\}=\int_0^{\infty} t^n e^{-st}dt=-\frac{1}{s}\int t^n de^{-st}=-\frac{1}{s}\left(t^n e^{-st}\Big|_{t=0}^{\infty}-\int e^{-st}dt^n\right)$$
$$=\frac{1}{s}\int e^{-st}dt^n=\frac{n}{s}\int_{0^-}^{\infty} e^{-st}t^{n-1}dt=\frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

where $Re(s)=\sigma>0$. Hence, we have

(3-41)
$$\mathcal{L}\{t^n\}=\frac{n}{s}\mathcal{L}\{t^{n-1}\}=\frac{n(n-1)}{s^2}\mathcal{L}\{t^{n-2}\}=\frac{n(n-1)(n-2)}{s^3}\mathcal{L}\{t^{n-3}\}$$
$$=\cdots=\frac{n(n-1)\cdots(n-(k-1))}{s^k}\mathcal{L}\{t^{n-k}\}=\frac{n!}{s^k(n-k)!}\mathcal{L}\{t^{n-k}\}$$

For $k=n$,

(3-42)
$$\mathcal{L}\{t^n\}=\frac{n!}{s^n(n-n)!}\mathcal{L}\{t^{n-n}\}=\frac{n!}{s^n}\mathcal{L}\{1\}=\frac{n!}{s^n}\mathcal{L}\{u(t)\}=\frac{n!}{s^{n+1}}$$

where the *ROC* is $Re(s)=\sigma>0$.

Next, let's discuss some important properties of the Laplace transform. By the use of these properties, we can determine the transform pair $\mathcal{L}\{f(t)\}=F(s)$ of some functions $f(t)$ not in the mapping set from (3-16) to (3-23).

Linearity

If $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, i.e., then

(3-43)
$$\mathcal{L}\{af_1(t)+bf_2(t)\}=\int_{0^-}^{\infty}(af_1(t)+bf_2(t))e^{-st}dt$$
$$=a\int_{0^-}^{\infty} f_1(t)e^{-st}dt+b\int_{0^-}^{\infty} f_2(t)e^{-st}dt$$
$$=aF_1(s)+bF_2(s)$$

Scaling property

If $F(s)$ is the Laplace transforms of $f(t)$, then with $a>0$ the Laplace transform of $f(at)$ is

(3-44)
$$\mathcal{L}\{f(at)\}=\int_{0^-}^{\infty} f(at)e^{-st}dt=\int_{0^-}^{\infty} f(\tau)e^{-s\left(\frac{\tau}{a}\right)}d\left(\frac{\tau}{a}\right)$$
$$=\frac{1}{a}\int_{0^-}^{\infty} f(\tau)e^{-\left(\frac{s}{a}\right)\tau}d\tau=\frac{1}{a}F\left(\frac{s}{a}\right)$$

Shifting property in variable *t*

If $F(s)$ is the Laplace transforms of $f(t)$, then with $\tau\geq 0$ the Laplace transform of $f(t-\tau)u(t-\tau)$ is

(3-45)
$$\begin{aligned}\mathcal{L}\{f(t-\tau)u(t-\tau)\} &= \int_{0^-}^{\infty} f(t-\tau)u(t-\tau)e^{-st}dt \\ &= \int_{\tau^-}^{\infty} f(t-\tau)u(t-\tau)e^{-st}dt \\ &= \int_{\tau^-}^{\infty} f(t-\tau)e^{-st}dt\end{aligned}$$

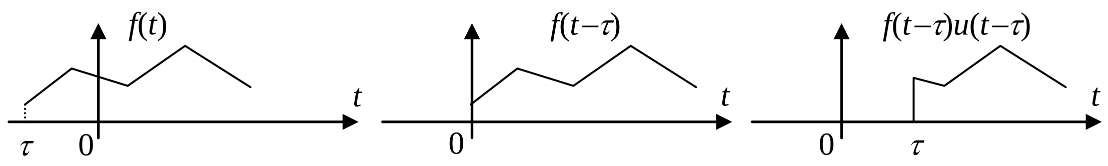
Choose $v=t-\tau$, then $dv=dt$, $t=v+\tau$ and

(3-46)
$$\begin{aligned}\int_{\tau^-}^{\infty} f(t-\tau)e^{-st}dt &= \int_{0^-}^{\infty} f(v)e^{-s(v+\tau)}dv \\ &= e^{-s\tau}\int_{0^-}^{\infty} f(v)e^{-sv}dv = e^{-s\tau}F(s)\end{aligned}$$

Hence,

(3-47)
$$\mathcal{L}\{f(t-\tau)u(t-\tau)\} = e^{-s\tau}F(s)$$

which is the shifting property of Laplace transform in variable *t*.



Note that $f(t-\tau)u(t-\tau)\neq f(t-\tau)$ if $f(t)\neq 0$ for $t<0$, as shown in the above figure, which implies that

(3-48)
$$\mathcal{L}\{f(t-\tau)u(t-\tau)\}\neq \mathcal{L}\{f(t-\tau)\}$$

For example, let's consider the difference between the Laplace transforms $\mathcal{L}\{\cos\omega_0(t-t_0)\cdot u(t-t_0)\}$ and $\mathcal{L}\{\cos\omega_0(t-t_0)\}$. From (3-35) and (3-47), we have

(3-49)
$$\mathcal{L}\{\cos\omega_0(t-t_0)\cdot u(t-t_0)\} = e^{-st_0}\frac{s}{s^2+\omega_0^2}$$

For the Laplace transform $\mathcal{L}\{\cos\omega_0(t-t_0)\}$, it is expressed as

(3-50)
$$\mathcal{L}\{\cos\omega_0(t-t_0)\} = \int_{0^-}^{\infty}\cos\omega_0(t-t_0)e^{-st}dt$$

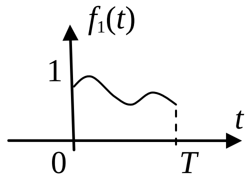
Let $v=t-t_0$, then

(3-51)
$$\begin{aligned}&\int_{0^-}^{\infty} \cos\omega_0(t-t_0)e^{-st}dt\\&=\int_{-t_0^-}^{\infty} \cos\omega_0(v)e^{-s(v+t_0)}dv=e^{-st_0}\int_{-t_0^-}^{\infty} \cos\omega_0(v)e^{-sv}dv\\&=e^{-st_0}\int_{-t_0^-}^{0^-} \cos\omega_0(v)e^{-sv}dv+e^{-st_0}\int_{0^-}^{\infty} \cos\omega_0(v)e^{-sv}dv\\&=e^{-st_0}\int_{-t_0^-}^{0^-} \cos\omega_0(v)e^{-sv}dv+e^{-st_0}\frac{s}{s^2+\omega_0^2}\end{aligned}$$

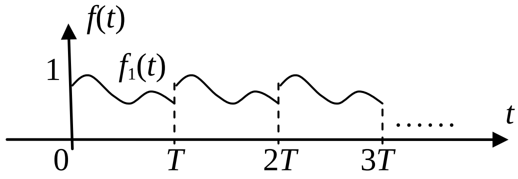
Compared to (3-49), it is clear that

(3-52) $$\mathcal{L}\{\cos\omega_0(t-t_0)\cdot u(t-t_0)\}\neq\mathcal{L}\{\cos\omega_0(t-t_0)\}$$

Laplace Transform of Periodic Functions



Next, let's introduce the case of periodic functions. Let $f_1(t)$ be a finite duration function as shown in the figure, which is zero for $t<0$ and $t\geq T$. Consider a periodic function $f(t)$ for $t>0$ with period T and $f(t)=f_1(t)$ for $0\leq t<T$. The periodic function is shown in the figure and expressed as



(3-53) $$f(t)=\sum_{k=0}^{\infty} f_1(t-kT)$$

Then, the Laplace transform is

(3-54) $$\mathcal{L}\{f(t)\}=\mathcal{L}\left\{\sum_{k=0}^{\infty} f_1(t-kT)\right\}=\sum_{k=0}^{\infty}\mathcal{L}\{f_1(t-kT)\}$$

where

(3-55)
$$\begin{aligned}\mathcal{L}\{f_1(t-kT)\}&=\mathcal{L}\{f_1(t-kT)u(t-kT)\}\\&=e^{-skT}\mathcal{L}\{f_1(t)\}\end{aligned}$$

If $F(s)=\mathcal{L}\{f(t)\}$ and $F_1(s)=\mathcal{L}\{f_1(t)\}$, then

(3-56)
$$\begin{aligned}F(s)&=\sum_{k=0}^{\infty}e^{-skT}F_1(s)\\&=\lim_{n\to\infty}\left(1+e^{-skT}+e^{-s2kT}+\cdots+e^{-snkT}\right)F_1(s)\\&=\lim_{n\to\infty}\frac{1-e^{-s(n+1)T}}{1-e^{-sT}}F_1(s)\end{aligned}$$

Since $\lim_{n\to\infty}\frac{1-e^{-s(n+1)T}}{1-e^{-sT}}=\frac{1}{1-e^{-sT}}$ for $Re(s)>0$, we have

(3-57) $$F(s)=\frac{1}{1-e^{-sT}}F_1(s),\quad \text{for } Re(s)>0$$

where $Re(s)>0$ is the *ROC*.

Shifting property in variable *s*

Similarly, there is a shifting property in variable *s*. Let's check the Laplace transform of $e^{-at}f(t)$. From the definition, we have

(3-58) $$\mathcal{L}\{e^{-at}f(t)\}=\int_{0^-}^{\infty}e^{-at}f(t)e^{-st}dt=\int_{0^-}^{\infty}f(t)e^{-(s+a)t}dt=F(s+a)$$

which is the shifting property of Laplace transform in variable *s*.

Derivative of variable *t*

If $F(s)$ is the Laplace transforms of a differentiable function $f(t)$, then what is the Laplace transform of $f'(t)$? From the definition, we have

(3-59) $$\begin{aligned}\mathcal{L}\{f'(t)\}&=\int_{0^-}^{\infty}f'(t)e^{-st}dt=\int e^{-st}df(t)=e^{-st}f(t)\Big|_{t=0^-}^{\infty}-\int f(t)de^{-st}\\&=\lim_{t\to\infty}e^{-st}f(t)-f(0^-)+s\int f(t)e^{-st}dt\\&=\lim_{t\to\infty}e^{-st}f(t)-f(0^-)+sF(s)\end{aligned}$$

Assume the *ROC* exists such that $\lim_{t\to\infty}e^{-st}f(t)=0$, then

(3-60) $$\mathcal{L}\{f'(t)\}=sF(s)-f(0^-)$$

Following the same procedure, we can obtain the Laplace transform of the second derivative as

(3-61) $$\mathcal{L}\{f''(t)\}=s\mathcal{L}\{f'(t)\}-f'(0^-)=s^2F(s)-sf(0^-)-f'(0^-)$$

Continuing the procedure, the Laplace transform of n^{th} derivative can be derived as

(3-62) $$\begin{aligned}\mathcal{L}\{f^{(n)}(t)\}=&\,s^nF(s)-s^{n-1}f(0^-)\\&-s^{n-2}f'(0^-)-\cdots-sf^{(n-2)}(0^-)-f^{(n-1)}(0^-)\end{aligned}$$

which will be used to solve IVPs of n^{th} order CODEs.

Let's use $\frac{d}{dt}\cos\omega_0 t=-\omega_0\sin\omega_0 t$ as an example. Then, taking Laplace transform leads to $\mathcal{L}\{\sin\omega_0 t\} = -\dfrac{1}{\omega_0}\mathcal{L}\left\{\dfrac{d}{dt}\cos\omega_0 t\right\}$. Based on (3-60), we have

(3-63)
$$\begin{aligned}\mathcal{L}\{\sin\omega_0 t\} &= -\frac{1}{\omega_0}\left(s\mathcal{L}\{\cos\omega_0 t\} - \cos\omega_0 t\big|_{t=0^-}\right)\\ &= -\frac{1}{\omega_0}\left(\frac{s^2}{s^2+\omega_0^2} - 1\right) = \frac{\omega_0}{s^2+\omega_0^2}\end{aligned}$$

which is the same expression shown in (3-22).

Integral of variable *t*

If $F(s) = \mathcal{L}\{f(t)\}$, then what is the Laplace transform of $\int_0^t f(\tau)d\tau$? Let's define $g(t) = \int_0^t f(\tau)d\tau$, then $g'(t) = f(t)$ and $g(0) = 0$. From (3-60), we have

(3-64)
$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}$$

i.e.,

(3-65)
$$\mathcal{L}\{f(t)\} = F(s) = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}$$

Hence,

(3-66)
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

It is clear that multiplying $1/s$ to *F*(*s*) in *s*-domain is similar to taking the integral of *f*(*t*) in *t*-domain. Since the term $1/s$ is just like an integration operator, Matlab/Simulink adopt the symbol $1/s$ to represent an integrator.

Derivative of variable *s*

If $F(s)$ is the Laplace transforms of $f(t)$, i.e., $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$. Then, taking the derivative of *F*(*s*) yields

(3-67)
$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} f(t)\left(\frac{\partial}{\partial s}e^{-st}\right)dt\\ &= \int_{0^-}^{\infty} f(t)\left(-te^{-st}\right)dt = -\int_{0^-}^{\infty}\left(tf(t)\right)e^{-st}dt = -\mathcal{L}\{tf(t)\}\end{aligned}$$

Hence,

(3-68)
$$\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$$

which, in a repeated manner, leads to

(3-69) $$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

For example, since $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$, we have

(3-70) $$\mathcal{L}\{te^{-at}\} = -\frac{d}{ds}\mathcal{L}\{e^{-at}\} = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$$

which is derived from (3-68).

Initial Value Theorem and Final Value Theorem

Consider $f(t)$ without any singularity function at t=0, then from (3-60) we have

(3-71) $$sF(s) - f(0) = \mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$

Hence,

(3-72) $$\lim_{s\to\infty}(sF(s) - f(0)) = \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = 0$$

where the integral vanishes due to the attenuation of e^{-st} as $s\to\infty$. That means

(3-73) $$f(0) = \lim_{s\to\infty} sF(s)$$

which is the so-called initial value theorem. Similarly, from (3-71) we have

(3-74) $$\lim_{s\to 0}(sF(s) - f(0)) = \int_{0^-}^\infty f'(t)dt = f(\infty) - f(0)$$

Hence,

(3-75) $$f(\infty) = \lim_{s\to 0} sF(s)$$

which is the so-called final value theorem.

However, the final value theorem (3-75) is only suitable for a function whose value is finite or 0 when $t\to\infty$. That means its Laplace transform must satisfy the following conditions:

I. All the nonzero poles of $F(s)$ must have negative real parts.

II. $F(s)$ cannot have more than one pole at s=0.

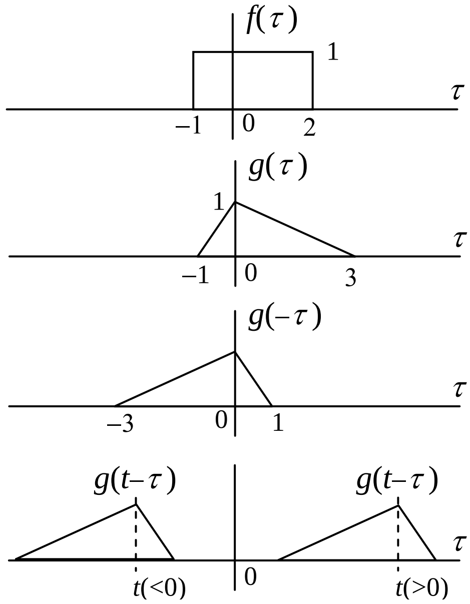
These conditions must be checked first when appying the final value theorem.

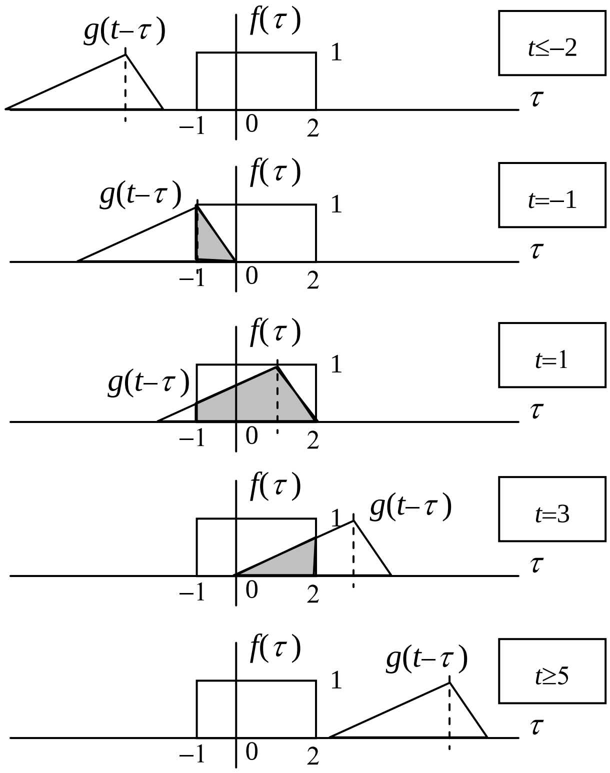
Convolution

In general, the convolution of $f(t)$ with $g(t)$ is denoted as $f(t)*g(t)$ and defined by

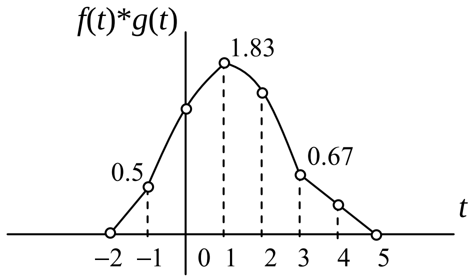
(3-76) $$f(t)*g(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

An example is depicted in the following figures.





For convenience, $f(\tau)$ is chosen as a rectangular function where $f(\tau)=1$ for $-1<\tau<2$. In addition, the process to get $g(t-\tau)$ is also shown there below $g(\tau)$. First, flip $g(\tau)$ with respect to the axis $\tau=0$ to get $g(-\tau)$. Then, shift $g(-\tau)$ by t to get $g(-(\tau-t))$ or $g(t-\tau)$.



Then, we start from $t\le -2$, which results in $f(\tau)g(t-\tau)=0$, i.e., $f(t)*g(t)=0$ for $t\le -2$. Further evaluate $f(t)*g(t)$ at $t=-1$, 1, 3, 5. In each case, the value of $f(t)*g(t)$ is exactly equal to the area overlapped by $f(\tau)$ and $g(t-\tau)$ due to $f(\tau)=1$ for $-1<\tau<2$. The overlapped areas are 0.5 at $t=-1$, 1.83 at $t=1$ and 0.67 at $t=3$. These values are shown on the curve of $f(t)*g(t)$ in the figure.

From (3-76), we know that the convolution $f(t)*g(t)$ is also a function of t, so we can write it as

(3-77) $$q(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

If $\alpha = t - \tau$, we have

(3-78) $$\begin{aligned} q(t) &= \int_{\infty}^{-\infty} f(t-\alpha) g(\alpha)(-d\alpha) \\ &= \int_{-\infty}^{\infty} g(\alpha) f(t-\alpha) d\alpha = g(t) * f(t) \end{aligned}$$

i.e., the convolution satisfies the commutative property:

(3-79) $$q(t) = f(t) * g(t) = g(t) * f(t)$$

If the convolution is shifted by a in variable t, we have

(3-80) $$q(t-a) = \int_{-\infty}^{\infty} f(\tau) g(t-a-\tau) d\tau = f(t) * g(t-a)$$

Hence,

(3-81) $$q(t-a) = f(t) * g(t-a) \neq f(t-a) * g(t-a)$$

which means a shifting of $f(t)$ or $g(t)$, not both, in variable t will result in the same amount of shifting of their convolution in variable t.

Now, let's discuss the Laplace transform of $f(t) * g(t)$. One thing to emphasize is that we will only focus on the causal functions $f(t)$ and $g(t)$, i.e., $f(t)=0$ for $t<0$ and $g(t)=0$ for $t<0$. Their convolution is then given as

(3-82) $$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

Define $F(s) = \int_0^{\infty} f(\lambda) e^{-s\lambda} d\lambda$ and $G(s) = \int_0^{\infty} g(t) e^{-st} dt$, then

(3-83) $$F(s)G(s) = \int_0^{\infty} f(\lambda) G(s) e^{-s\lambda} d\lambda$$

Since

(3-84) $$G(s) e^{-s\lambda} = \int_0^{\infty} g(t-\lambda) u(t-\lambda) e^{-st} dt$$

we have

(3-85) $$\begin{aligned} F(s)G(s) &= \int_0^{\infty} f(\lambda) \left(\int_0^{\infty} g(t-\lambda) u(t-\lambda) e^{-st} dt \right) d\lambda \\ &= \int_0^{\infty} \left(\int_0^{\infty} f(\lambda) g(t-\lambda) u(t-\lambda) d\lambda \right) e^{-st} dt \\ &= \int_0^{\infty} \left(\int_0^{t} f(\lambda) g(t-\lambda) d\lambda \right) e^{-st} dt = \mathcal{L}\{ f(t) * g(t) \} \end{aligned}$$

i.e.,

(3-86) $$\mathcal{L}\{f(t)*g(t)\}=F(s)G(s)$$

For example, let $f_1(t)=e^{-2t}u(t)$ and $f_2(t)=\cos 3t\cdot u(t)$, then

(3-87)
$$\begin{aligned}
f_1(t)*f_2(t)&=\int_0^t e^{-2\tau}\cos 3(t-\tau)d\tau=-\frac{1}{3}\int e^{-2\tau}d\sin 3(t-\tau)\\
&=-\frac{1}{3}e^{-2\tau}\sin 3(t-\tau)\Big|_{\tau=0}^{t}-\frac{2}{3}\int_0^t e^{-2\tau}\sin 3(t-\tau)d\tau\\
&=\frac{1}{3}\sin 3t-\frac{2}{3}\int_0^t e^{-2\tau}\sin 3(t-\tau)d\tau\\
&=\frac{1}{3}\sin 3t-\frac{2}{9}\int e^{-2\tau}d\cos 3(t-\tau)\\
&=\frac{1}{3}\sin 3t-\frac{2}{9}e^{-2\tau}\cos 3(t-\tau)\Big|_{\tau=0}^{t}-\frac{4}{9}\int_0^t e^{-2\tau}\cos 3(t-\tau)d\tau\\
&=\frac{1}{3}\sin 3t-\frac{2}{9}e^{-2t}+\frac{2}{9}\cos 3t-\frac{4}{9}f_1(t)*f_2(t)
\end{aligned}$$

Hence,

(3-88) $$f_1(t)*f_2(t)=\frac{3}{13}\sin 3t+\frac{2}{13}\cos 3t-\frac{2}{13}e^{-2t}$$

whose Laplace transform is

(3-89)
$$\begin{aligned}
\mathcal{L}\{f_1(t)*f_2(t)\}&=\frac{3}{13}\left(\frac{3}{s^2+9}\right)+\frac{2}{13}\left(\frac{s}{s^2+9}\right)-\frac{2}{13}\left(\frac{1}{s+2}\right)\\
&=\frac{1}{13}\left(\frac{2s+9}{s^2+9}-\frac{2}{s+2}\right)=\frac{s}{(s^2+9)(s+2)}
\end{aligned}$$

Based on (3-86), we can directly obtain the Laplace transform as

(3-90) $$\mathcal{L}\{f_1(t)*f_2(t)\}=F_1(s)F_2(s)=\left(\frac{1}{s+2}\right)\left(\frac{s}{s^2+9}\right)=\frac{s}{(s^2+9)(s+2)}$$

Both (3-89) and (3-90) are the same.