2. Second-Order Linear Ordinary Differential Equations with Constant Coefficients

In general, the second-order linear ordinary differential equations with constant coefficients, or 2nd-order linear CODE, is described as below:

(2-1) $$L[y(t)] \equiv \ddot{y}(t) + p_1\dot{y}(t) + p_0 y(t) = q(t)$$

where $L[y(t)] \equiv \ddot{y}(t) + p_1\dot{y}(t) + p_0 y(t)$, t is an independent variable within an interval I, $q(t)$ is a given function, $y(t)$ is the unknown function to be solved, and p_1 and p_0 are constant coefficients.

The linear operator $L[y(t)] \equiv \ddot{y}(t) + p_1\dot{y}(t) + p_0 y(t)$, similar to the case of 1st-order linear CODE, allows us to decompose the solution $y(t)$ into two parts as below:

(2-2) $$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the homogeneous solution satisfying

(2-3) $$\ddot{y}_h(t) + p_1\dot{y}_h(t) + p_0 y_h(t) = 0$$

and $y_p(t)$ is a particular solution obtained from

(2-4) $$\ddot{y}_p(t) + p_1\dot{y}_p(t) + p_0 y_p(t) = q(t)$$

Next, let's solve $y_h(t)$ for the homogeneous equation (2-3).

Homogeneous Solutions

Same as the 1st-order linear CODE, the homogeneous solution of (2-3) can be chosen as

(2-5) $$y_h(x) = Ae^{\lambda x}$$

where A is an arbitrary constant. Apply it to (2-3) and get

(2-6) $$\left(\lambda^2 + p_1\lambda + p_0\right)Ae^{\lambda t} = 0$$

which implies

(2-7) $$\lambda^2 + p_1\lambda + p_0 = 0$$

since $e^{\lambda t} \neq 0$ for all t. Note that (2-7) is called the characteristic equation and the characteristic roots are $\lambda_1, \lambda_2 = \dfrac{-p_1 \pm \sqrt{p_1^2 - 4p_0}}{2}$. According to the numeric sign of $p_1^2 - 4p_0$, the homogeneous solution $y_h(t)$ can be classified into three cases, which are $p_1^2 - 4p_0 > 0$, $p_1^2 - 4p_0 = 0$ and $p_1^2 - 4p_0 < 0$.

For ***Case-I***: $p_1^2 - 4p_0 > 0$, the roots $\lambda_1, \lambda_2 = \dfrac{-p_1 \pm \sqrt{p_1^2 - 4p_0}}{2}$ are distinct real numbers. Since $\lambda_1 \neq \lambda_2$, we know that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are linearly independent and the homogeneous solution is formed by their linear combination and expressed as

(2-8) $$y_h(t) = y_{h1}(t) + y_{h2}(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

where $y_{h1}(t) = A_1 e^{\lambda_1 t}$ and $y_{h2}(t) = A_2 e^{\lambda_2 t}$. Note that the coefficients A_1 and A_2 are arbitrary real constants.

For ***Case-II***: $p_1^2 - 4p_0 = 0$ or $p_0 = \dfrac{p_1^2}{4}$, the roots $\lambda_1 = \lambda_2 = -p_1/2$, which are real and repeated. Hence, we can only obtain one homogeneous solution directly from the repeated roots $-p_1/2$, which is expressed as

(2-9) $$y_{h1}(t) = A_1 e^{-\frac{p_1}{2}t}$$

However, for a 2nd-order homogeneous equation, we are supposed to find two homogeneous solutions. Here, we will introduce a method commonly used to determine the homogeneous solution different to (2-9). Let's rewrite (2-3) into

(2-10) $$\ddot{y}_h(t) + p_1 \dot{y}_h(t) + \frac{p_1^2}{4} y_h(t) = \frac{d}{dt}\left(\dot{y}_h(t) + \frac{p_1}{2} y_h(t)\right) + \frac{p_1}{2}\left(\dot{y}_h(t) + \frac{p_1}{2} y_h(t)\right) = 0$$

which can be further decomposed into two 1st-order CODEs as below:

(2-11) $$\dot{y}_h(t) + \frac{p_1}{2} y_h(t) = z(t)$$

(2-12) $$\dot{z}(t) + \frac{p_1}{2} z(t) = 0$$

From (2-12), we obtain $z(t) = Ae^{-\frac{p_1}{2}t}$, and then (2-11) becomes

(2-13) $$\dot{y}_h(t) + \frac{p_1}{2} y_h(t) = Ae^{-\frac{p_1}{2}t}$$

Since (2-13) is a 1st-order linear CODE, its solution $y_h(t)$ should consist of a homogeneous solution $y_{h1}(t) = A_1 e^{-\frac{p_1}{2}t}$, and a particular solution

(2-14)
$$
\begin{aligned}
y_{h2}(t) &= \int_a^t A e^{-\frac{p_1}{2}\tau} e^{-\frac{p_1}{2}(t-\tau)} d\tau + y_{h2}(a) e^{-\frac{p_1}{2}(t-a)} \\
&= A e^{-\frac{p_1}{2}t} \int_a^t d\tau + y_{h2}(a) e^{\frac{p_1}{2}a} e^{-\frac{p_1}{2}t} = \left(A(t-a) + y_{h2}(a) e^{\frac{p_1}{2}a} \right) e^{-\frac{p_1}{2}t} \\
&= A_2 t e^{-\frac{p_1}{2}t} + A_3 e^{-\frac{p_1}{2}t}
\end{aligned}
$$

where $A_2 = A$ and $A_3 = -Aa + y_{h2}(a) e^{\frac{p_1}{2}a}$. Because only one particular solution is needed and the term $A_3 e^{-\frac{p_1}{2}t}$ has the same form of $y_{h1}(t)$, we can neglect the term $A_3 e^{-\frac{p_1}{2}t}$ and choose

(2-15)
$$y_{h2}(t) = A_2 t e^{-\frac{p_1}{2}t}$$

Therefore, the homogeneous solution for the case of $p_1^2 - 4p_0 = 0$ is

(2-16)
$$y_h(t) = y_{h1}(t) + y_{h2}(t) = A_1 e^{-\frac{p_1}{2}t} + A_2 t e^{-\frac{p_1}{2}t}$$

where A_1 and A_2 are arbitrary real constants.

For ***Case-III***: $p_1^2 - 4p_0 < 0$, the roots are $\lambda_1, \lambda_2 = \alpha \pm j\beta$, with $\alpha = -\frac{p_1}{2}$ and $\beta = \frac{\sqrt{4p_0 - p_1^2}}{2}$. Since $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2^* = \lambda = \alpha + j\beta$, we have

(2-17)
$$y_h(t) = B_1 e^{\lambda t} + B_2 e^{\lambda^* t}$$

where B_1 and B_2 are complex numbers. Due to the fact that $y_h(t) \in R$, we can rewrite (2-17) as

(2-18)
$$y_h(t) = A e^{\lambda t} + A^* e^{\lambda^* t}$$

where $B_1 = B_2^* = A$ and A is an arbitrary complex number. Actually, (2-18) can be also expressed as

(2-19)
$$y_h(t) = e^{\alpha t} \left(A_1 \cos \beta t + A_2 \sin \beta t \right)$$

where $A_1 = A + A^*$ and $A_2 = j(A - A^*)$ are real numbers. Hence,

(2-20)
$$
\begin{aligned}
y_h(t) &= y_{h1}(t) + y_{h2}(t) = A_1 e^{\alpha t} \cos \beta t + A_2 e^{\alpha t} \sin \beta t \\
&= e^{-\frac{p_1}{2}t} \left(A_1 \cos\left(\frac{1}{2}\sqrt{4p_0 - p_1^2}\, t \right) + A_2 \sin\left(\frac{1}{2}\sqrt{4p_0 - p_1^2}\, t \right) \right)
\end{aligned}
$$

where $y_{h1}(t) = A_1 e^{\alpha t} \cos \beta t$ and $y_{h2}(t) = A_2 e^{\alpha t} \sin \beta t$.

Nonhomogeneous Solutions

Assume $y_p(t)$ is a particular solution of (2-1) with $q(t) \neq 0$, which is called a nonhomogeneous equation, then its solution, or nonhomogeneous solution, for each of the three cases are shown as below:

Case-I: $p_1^2 - 4p_0 > 0$, λ_1 and λ_2 are real numbers and $\lambda_1 \neq \lambda_2$

(2-21) $$y(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + y_p(t)$$

Case-II: $p_1^2 - 4p_0 = 0$, $\lambda_1 = \lambda_2 = -\dfrac{p_1}{2}$

(2-22) $$y(t) = A_1 e^{-\frac{p_1}{2}t} + A_2 t e^{-\frac{p_1}{2}t} + y_p(t)$$

Case-III: $p_1^2 - 4p_0 < 0$, $\lambda_1, \lambda_2 = \alpha \pm j\beta = -\dfrac{p_1}{2} \pm j\dfrac{1}{2}\sqrt{4p_0 - p_1^2}$

(2-23) $$y(t) = e^{\alpha t}\left(A_1 \cos\beta t + A_2 \sin\beta t\right) + y_p(t)$$

Clearly, these nonhomogeneous solutions are not unique since each of them consists of two arbitrary real numbers A_1 and A_2. To achieve a unique solution, we have to include two extra conditions to determine A_1 and A_2. Next, let's focus on the initial value problems, or IVPs for short.

Initial Value Problems

Let's consider a 2nd-order linear CODE with two extra conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$, which is expressed as

(2-24) $$\ddot{y}(t) + p_1\dot{y}(t) + p_0 y(t) = q(t), \quad y(0) = y_0, \ \dot{y}(0) = v_0$$

In mathematics, since the extra conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are given at the initial point t=0, the problem to solve $y(t)$ for $t \geq 0$ is an IVP.

Consider an example of ***Case-I***. If a particular solution $y_p(t)$ has been determined, then $y(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + y_p(t)$ shown in (2-21) must satisfy the initial conditions:

(2-25) $$y(0) = A_1 + A_2 + y_p(0) = y_0$$

(2-26) $$\dot{y}(0) = \lambda_1 A_1 + \lambda_2 A_2 + \dot{y}_p(0) = v_0$$

Hence, the coefficients A_1 and A_2 can be determined from (2-25) and (2-26) and expressed as

(2-27) $$A_1 = \frac{-\lambda_2}{\lambda_1-\lambda_2}\left(y_0 - y_p(0)\right) + \frac{1}{\lambda_1-\lambda_2}\left(v_0 - \dot{y}_p(0)\right)$$

(2-28) $$A_2 = \frac{\lambda_1}{\lambda_1-\lambda_2}\left(y_0 - y_p(0)\right) - \frac{1}{\lambda_1-\lambda_2}\left(v_0 - \dot{y}_p(0)\right)$$

where $\lambda_1 \neq \lambda_2$. Clearly, the solution $y(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + y_p(t)$ is unique.

Boundary Value Problem (BVP)

Instead of the initial conditions, we can also use boundary conditions to uniquely determine the solution of a 2nd-order linear CODE.

For example, consider a 2nd-order linear CODE with boundary conditions, which is expressed as

(2-29) $$\ddot{y}(t) + p_1\dot{y}(t) + p_0y(t) = q(t), \quad y(a) = y_a, \quad y(b) = y_b$$

In mathematics, (2-29) is a kind of boundary value problem, or BVP for short, since $y(a) = y_a$ and $y(b) = y_b$ are conditions given at the boundary points *t*=*a* and *t*=*b*.

Once again, let's solve the BVP in (2-29) for ***Case-I***. Assume a particular solution $y_p(t)$ has been determined, then $y(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + y_p(t)$ in (2-21) must satisfy

(2-30) $$y(a) = A_1e^{\lambda_1 a} + A_2e^{\lambda_2 a} + y_p(a) = y_a$$

(2-31) $$y(b) = A_1e^{\lambda_1 b} + A_2e^{\lambda_2 b} + y_p(b) = y_b$$

As a result, we have

(2-32) $$A_1 = \frac{\left(y_a - y_p(a)\right)e^{-\lambda_2 a} - \left(y_b - y_p(b)\right)e^{-\lambda_2 b}}{e^{(\lambda_1-\lambda_2)a} - e^{(\lambda_1-\lambda_2)b}}$$

(2-33) $$A_2 = \frac{\left(y_a - y_p(a)\right)e^{-\lambda_1 a} - \left(y_b - y_p(b)\right)e^{-\lambda_1 b}}{e^{(\lambda_2-\lambda_1)a} - e^{(\lambda_2-\lambda_1)b}}$$

Thus, the nonhomogeneous solution $y(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + y_p(t)$ in (2-21) is unique.

Particular Solutions

Now, let's determine the particular solutions $y_p(t)$ corresponding to the cases of $q(t) \in \left\{1, t, t^2, sin\,\omega t, cos\,\omega t\right\}$ for a 2nd-order linear CODE shown in (2-1). That means the particular solution satisfies

(2-34) $$\ddot{y}_p(t) + p_1\dot{y}_p(t) + p_0 y_p(t) = q(t)$$

For $q(t)=1$, we can assume $y_p(t)$ is also a constant, which means $\dot{y}_p(t)=0$ and $\ddot{y}_p(t)=0$. Then, from (2-34) we have $p_0 y_p(t)=1$, i.e.,

(2-35) $$y_p(t) = \frac{1}{p_0}$$

which is chosen as the particular solution of $q(t)=1$.

For $q(t)=t$, we assume $y_p(t)=at+b$, and thus $\dot{y}_p(t)=a$ and $\ddot{y}_p(t)=0$. Substituting them into (2-34) yields

(2-36) $$p_1 a + p_0(at+b) = t$$

which results in $p_0 a = 1$ and $p_1 a + p_0 b = 0$. Hence, $a = \frac{1}{p_0}$ and $b = -\frac{p_1}{p_0^2}$, i.e.,

(2-37) $$y_p(t) = at + b = \frac{1}{p_0}t - \frac{p_1}{p_0^2}$$

which is a particular solution of $q(t)=t$.

For $q(t)=t^2$, let $y_p(t)=at^2+bt+c$, then $\dot{y}_p(t)=2at+b$ and $\ddot{y}_p(t)=2a$. From (2-34), we have

(2-38) $$2a + p_1(2at+b) + p_0(at^2+bt+c) = t^2$$

which implies $p_0 a = 1$, $2p_1 a + p_0 b = 0$ and $2a + p_1 b + p_0 c = 0$. Therefore, $a = \frac{1}{p_0}$, $b = -2\frac{p_1}{p_0^2}$ and $c = -2\frac{1}{p_0^2} + 2\frac{p_1^2}{p_0^3}$. Hence,

(2-39) $$y_p(t) = at^2 + bt + c = \frac{1}{p_0}t^2 - 2\frac{p_1}{p_0^2}t - 2\frac{1}{p_0^2} + 2\frac{p_1^2}{p_0^3}$$

which is chosen as the particular solution of $q(t)=t^2$.

For $q(t)=sin\,\omega t$ and $q(t)=cos\,\omega t$, we use $q(t)=e^{j\omega t}$ as a substitution, and assume $y_p(t)=Ae^{j\omega t}$ where A is a complex number. The derivatives of $y_p(t)$ are $\dot{y}_p(t)=(j\omega)Ae^{j\omega t}$ and $\ddot{y}_p(t)=(j\omega)^2 Ae^{j\omega t}$. Then, apply them to (2-34) to obtain

(2-40) $$(j\omega)^2 Ae^{j\omega t} + p_1(j\omega)Ae^{j\omega t} + p_0 Ae^{j\omega t} = e^{j\omega t}$$

which leads to

(2-41) $$A=\frac{1}{p_0-\omega^2+jp_1\omega}=\frac{p_0-\omega^2-jp_1\omega}{\left(p_0-\omega^2\right)^2+\left(p_1\omega\right)^2}=\frac{p_0-\omega^2-jp_1\omega}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2}$$

Hence, the particular solution of $q(t)=e^{j\omega t}$ is

(2-42) $$\begin{aligned} y_p(t) &= \frac{p_0-\omega^2-jp_1\omega}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2}e^{j\omega t} \\ &= \frac{p_0-\omega^2-jp_1\omega}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2}\left(\cos\omega t+j\sin\omega t\right) \\ &= \frac{\left(p_0-\omega^2\right)\cos\omega t+p_1\omega\sin\omega t}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2}+j\frac{\left(p_0-\omega^2\right)\sin\omega t-p_1\omega\cos\omega t}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2} \end{aligned}$$

Clearly, if $q(t)=\sin\omega t$, which is the imaginary part of $e^{j\omega t}$, then the particular solution is the imaginary part in (2-42), i.e.,

(2-43) $$y_p(t)=\frac{\left(p_0-\omega^2\right)\sin\omega t-p_1\omega\cos\omega t}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2}$$

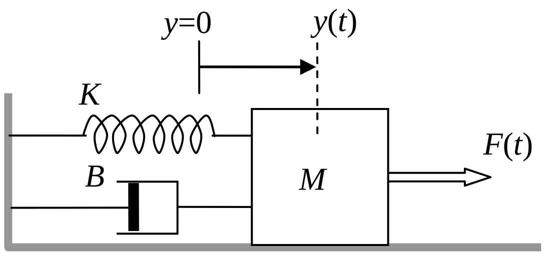
For $q(t)=\cos\omega t$, which is the real part of $e^{j\omega t}$, we have

(2-44) $$y_p(t)=\frac{\left(p_0-\omega^2\right)\cos\omega t+p_1\omega\sin\omega t}{\omega^4+\left(p_1-2p_0\right)\omega^2+p_0^2}$$

which is the real part in (2-42).

Motion of an *MBK* System

Let's consider the simplest mechanical system, called the *MBK* system, and adopt the *N*ewton's second law to derive its dynamic systems.



In engineering, most of the systems are constructed by mechanical components such as dampers and springs. The simplest one is shown in the figure, called the mass-damper-spring system or *MBK* system in brief, where *M* is the mass of the moving object, *B* is the damping coefficient of the damper and *K* is the stiffness of the spring. Let *F*(*t*) be an extra force exerted on the object at time *t* and assume *y*(*t*) is the resulted deviation of the spring referring to its unforced status y=0. Then, there are two forces reacted to restrain the motion of the object, expressed as

(2-45) $$F_B(t) = -By'(t)$$

(2-46) $$F_K(t) = -Ky(t)$$

where $F_B(t)$ is caused by the damper and $F_K(t)$ is the force from the spring. According to the Newton's second law of motion, we have

(2-47) $$F(t) + F_B(t) + F_K(t) = My''(t)$$

which can be written as

(2-48) $$My'' + By' + Ky = F(t)$$

By monic process, (2-48) is changed into the normal form as

(2-49) $$y'' + a_1 y' + a_0 y = f(t)$$

where $a_1 = \frac{B}{M} > 0$, $a_0 = \frac{K}{M} > 0$ and $f(t) = \frac{F(t)}{M}$. Clearly, the dynamic model of an *MBK* system is represented by a 2nd-order linear OCDE. If the initial conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are further included, then

(2-50) $$y'' + a_1 y' + a_0 y = f(t), \quad y(0) = y_0, \ \dot{y}(0) = v_0$$

which is an IVP and possessed of two characteristic roots λ_1 and λ_2. Next, let's discuss the dynamic motion of the *MBK* system.

Unforced Overdamped Motion

First, let's discuss the case of distinct real λ_1 and λ_2, or call unforced and overdamped motion, under different initial conditions. Consider an *MBK* system with B=1.25, K=0.25, and M=1 and suppose that there is no driving force, i.e., $f(t)$=0. Hence, the *MBK* system performs an unforced motion and is described as

(2-51) $$y'' + a_1 y' + a_0 y = 0$$

where $a_1 = \frac{B}{M} = 1.25$ and $a_0 = \frac{K}{M} = 0.25$. Its characteristic equation is

(2-52) $$\lambda^2 + 1.25\lambda + 0.25 = (\lambda + 0.25)(\lambda + 1) = 0$$

and the characteristic roots are $\lambda_1 = -0.25$ and $\lambda_2 = -1$. Hence, the homogeneous solution is

(2-53) $$y(t) = A_1 e^{-0.25t} + A_2 e^{-1t}$$

where A_1 and A_2 depend on the initial conditions $y(0) = y_0$ and $y'(0) = v_0$.

To display the dynamic behavior under different initial conditions, the so-called phase plane method has been often used as the tool in mathematics. A phase plane uses $y(t)$ and $y'(t)$ as the horizontal axis and vertical axis, and a curve with an arrow on the phase plane is called a trajectory, which is related to specific initial conditions $y(0)=y_0$ and $y'(0)=v_0$. That means a trajectory shows the dynamic behavior of $y(t)$ along the direction of arrow as t is increasing. A set of trajectories is called a portrait of $y(t)$. To display the dynamic behavior of (2-51), let's choose 12 initial conditions as below:

Curve	1	2	3	4	5	6	7	8	9	10	11	12
y_0	0	–4	–8	–8	–8	–8	8	8	8	8	4	0
v_0	8	8	8	6	4	2	–2	–4	–6	–8	–8	–8

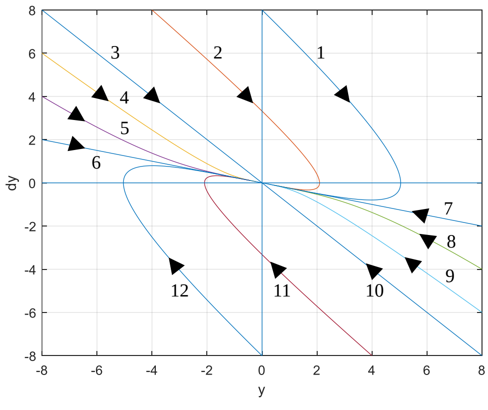
and use the command ode45 of Matlab to simulate all these conditions.

To run Matlab, we have to transform (2-51) into a set of state equations by defining state variables as $y_1(t)=y(t)$ and $y_2(t)=y'(t)$. Then, (2-51) can be represented as

(2-54)
$$\begin{cases} y_1' = y_2 \\ y_2' = -a_0 y_1 - a_1 y_2 \end{cases}$$

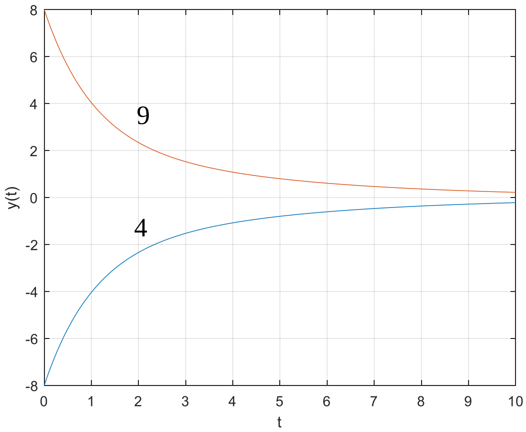
which will be applied in the simulation program listed below.

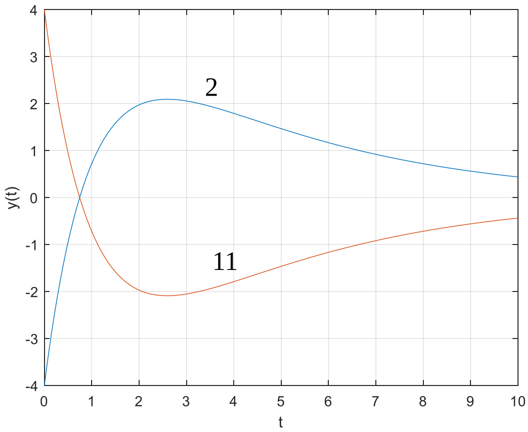
```
==================================
  Create m-file: MBKCase1.m
  function dy=MBKCase1(t,y)
  dy=zeros(2,1);    % a column vector
  dy(1)=y(2);
  dy(2)=-0.25*y(1)-1.25*y(2);
==================================
  Create m-file: phaseplane1.m
  y0(1)=0; dy0(1)=8; y0(2)=-4; dy0(2)=8;
  y0(3)=-8; dy0(3)=8; y0(4)=-8; dy0(4)=6;
  y0(5)=-8; dy0(5)=4; y0(6)=-8; dy0(6)=2;
  y0(7)=-y0(6); dy0(7)=-dy0(6);
  y0(8)=-y0(5); dy0(8)=-dy0(5);
  y0(9)=-y0(4); dy0(9)=-dy0(4);
  y0(10)=-y0(3); dy0(10)=-dy0(3);
  y0(11)=-y0(2); dy0(11)=-dy0(2);
  y0(12)=-y0(1); dy0(12)=-dy0(1);
  figure(1)
  for i=1:12
    [t,y]=ode45(@MBKCase1,[0:0.01:10], [y0(i) dy0(i)])
    plot(y(:,1),y(:,2))
    hold on
  end
  grid; xlabel('y'); ylabel('dy');
  line([0 0],[8 -8]), line([-8 8],[0 0])    % two axes
```



```
% Plot Curve 4,9 and Curve 2,11
figure(2)
[t,y4]=ode45(@MBKCase1,[0:0.01:10], [y0(4) dy0(4)])
[t,y9]=ode45(@MBKCase1,[0:0.01:10], [y0(9) dy0(9)])
plot(t,y4(:,1),t,y9(:,1))
grid; xlabel('t'); ylabel('y(t)');
figure(3)
[t,y2]=ode45(@MBKCase1,[0:0.01:10], [y0(2) dy0(2)])
[t,y11]=ode45(@MBKCase1,[0:0.01:10], [y0(11) dy0(11)])
plot(t,y2(:,1),t,y11(:,1))
grid; xlabel('t'); ylabel('y(t)');
================================
```

After the simulation of 12 initial conditions, the portrait of $y(t)$ is obtained and shown in the phase plane. Note that Curve 3 and Curve 10 belong to the same straight line $y'-\lambda_2 y=0$ with slop $\lambda_2=-1$ and Curve 6 and Curve 7 belong to the same straight line $y'-\lambda_1 y=0$ with slop $\lambda_1=-0.25$. Most importantly, the origin $y(t)=0$ for all t is also a solution and called the equivalent point. From the portrait, it is clear that there is no intersection between any two of the trajectories, which implies that all the trajectories approach the equivalent point $y(t)=0$, but cannot reach it.





Besides, any curve bounded by these two straight lines has the property $y(t)>0$ or $y(t)<0$, such as the solution of Curve 4 and 9. On the other hand, any curve outside these two straight lines, such as Curve 2 and 11, has the property that $y(t)$ first reaches its maximum or minimum at $y'=0$ and then converges to 0.

Critical Damping Motion

Next, let's discuss the case of $\lambda_1=\lambda_2$, or call critical damping motion, under different initial conditions. Consider the *MBK* system with B=4, K=4, and M=1 and without driving force, i.e., $f(t)$=0. Hence, the *MBK* system performs an unforced motion and is described as

(2-55) $$y'' + a_1 y' + a_0 y = 0$$

where $a_1 = \dfrac{B}{M} = 4$ and $a_0 = \dfrac{K}{M} = 4$. Its characteristic equation is

(2-56) $$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$$

which has a repeated root $\lambda = -2$. The homogeneous solution is

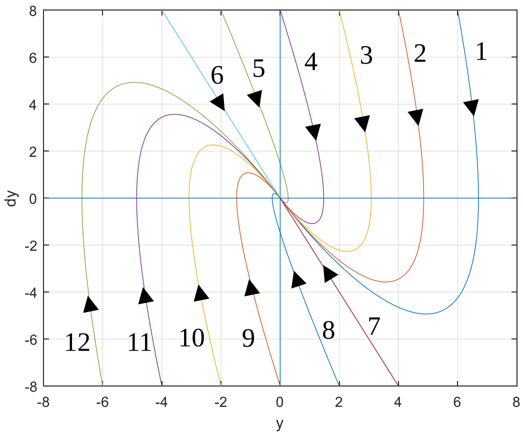
(2-57) $$y(t) = A_1 e^{-2t} + A_2 t e^{-2t}$$

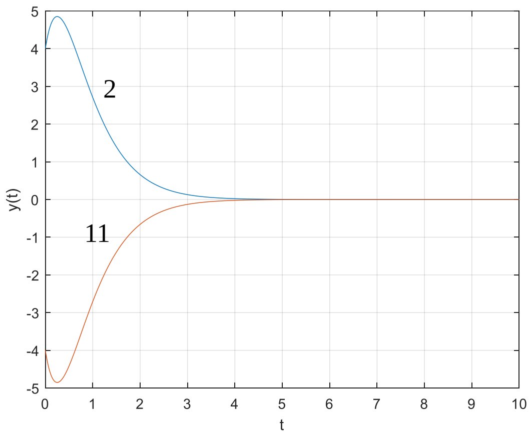
where A_1 and A_2 are determined by the initial conditions $y(0) = y_0$ and $y'(0) = v_0$. Again, below uses 12 initial conditions to display the dynamic behavior of (2-55):

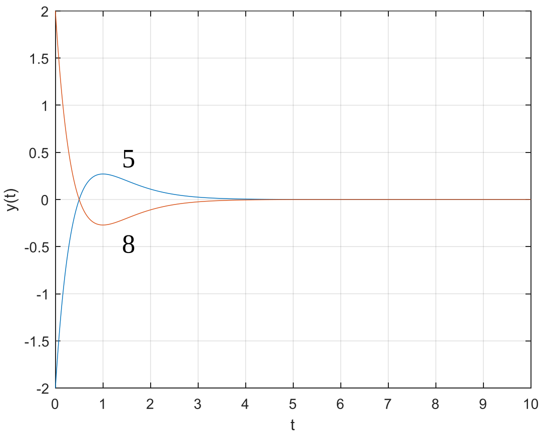
Curve	1	2	3	4	5	6	7	8	9	10	11	12
y_0	6	4	2	0	–2	–4	4	2	0	–2	–4	–6
v_0	8	8	8	8	8	8	–8	–8	–8	–8	–8	–8

The Matlab simulation program is shown below:

```
==================================
Create m-file: MBKCase2.m
function dy=MBKCase2(t,y)
dy=zeros(2,1);   % a column vector
dy(1)=y(2);
dy(2)=-0.25*y(1)-1.25*y(2);
==================================
Create m-file: phaseplane2.m
y0(1)=6; dy0(1)=8; y0(2)=4; dy0(2)=8;
y0(3)=2; dy0(3)=8; y0(4)=0; dy0(4)=8;
y0(5)=-2; dy0(5)=8; y0(6)=-4; dy0(6)=8;
y0(7)=-y0(6); dy0(7)=-dy0(6);
y0(8)=-y0(5); dy0(8)=-dy0(5);
y0(9)=-y0(4); dy0(9)=-dy0(4);
y0(10)=-y0(3); dy0(10)=-dy0(3);
y0(11)=-y0(2); dy0(11)=-dy0(2);
y0(12)=-y0(1); dy0(12)=-dy0(1);
figure(1)
for i=1:12
   [t,y]=ode45(@MBKCase2,[0:0.01:10], [y0(i) dy0(i)])
   plot(y(:,1),y(:,2))
   hold on
end
grid; xlabel('y'); ylabel('dy');
line([0 0],[8 -8]), line([-8 8],[0 0])    % two axes
% Plot Curve 4,9 and Curve 1,12
figure(2)
[t,y4]=ode45(@MBKCase2,[0:0.01:10], [y0(4) dy0(4)])
[t,y9]=ode45(@MBKCase2,[0:0.01:10], [y0(9) dy0(9)])
plot(t,y4(:,1),t,y9(:,1))
grid; xlabel('t'); ylabel('y(t)');
figure(3)
[t,y1]=ode45(@MBKCase2,[0:0.01:10], [y0(1) dy0(1)])
[t,y12]=ode45(@MBKCase2,[0:0.01:10], [y0(12) dy0(12)])
plot(t,y1(:,1),t,y12(:,1))
grid; xlabel('t'); ylabel('y(t)');
==================================
```







The resulted portrait of $y(t)$ is shown in the phase plane, where Curve 6 and Curve 7 belong to the same straight line with slop $\lambda = -2$. All the other curves have the property that $y(t)$ first reaches its maximum or minimum at $y' = 0$ and then converges to 0, such as $y(t)$ of Curve 2 and 11 and $y(t)$ of Curve 5 and 8.

Unforced Underdamped Motion

Next, let's discuss the case of $\lambda_1, \lambda_2 = \alpha \pm j\beta$, or call underdamped motion, under different initial conditions. Consider the *MBK* system with *B*=0.2, *K*=0.5, and *M*=1 and suppose that there is no driving force, i.e., *f*(*t*)=0. Hence, the *MBK* system performs an unforced motion and is described as

(2-58) $$y'' + a_1 y' + a_0 y = 0$$

where $a_1 = \dfrac{B}{M} = 0.2$ and $a_0 = \dfrac{K}{M} = 0.5$. Its characteristic equation is

(2-59) $$\lambda^2 + 0.2\lambda + 0.5 = (\lambda + 0.1 + j0.7)(\lambda + 0.1 - j0.7) = 0$$

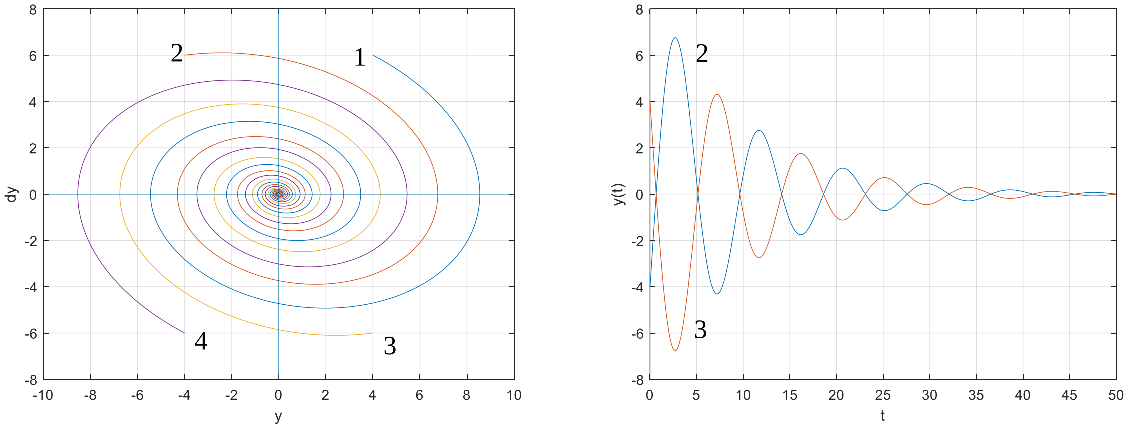
with roots $\lambda = -0.1 \pm j0.7$. The homogeneous solution is

(2-60) $$y(t) = (A_1 \cos t + A_2 \sin t)e^{-t}$$

where A_1 and A_2 are related to the initial conditions $y(0) = y_0$ and $y'(0) = v_0$. To display the dynamic behavior of (2-58), let's choose 4 initial conditions as below:

Curve	1	2	3	4
y_0	4	–4	4	–4
v_0	6	6	–6	–6

The resulted portrait of $y(t)$ is shown in the phase plane and all the curves are spirally converged to the origin. From $y(t)$ of Curve 2 and 3, both are oscillating and damped in peak values.



2 Forced Motion

Next, let's discuss the case of forced motion, under specified initial conditions. Consider the *MBK* system with *B*=1.25, *K*=0.25, and *M*=1 and the extra force is $f(t) = cos\,0.4\pi t$. Hence, the system performs a forced motion and is described as

(2-61) $$y'' + a_1 y' + a_0 y = f(t)$$

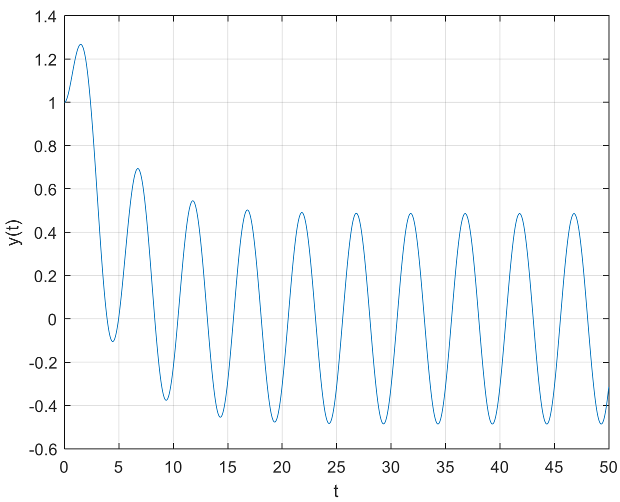
where $a_1 = \dfrac{B}{M} = 1.25$ and $a_0 = \dfrac{K}{M} = 0.25$. Its characteristic equation is

(2-62) $$\lambda^2 + 1.25\lambda + 0.25 = (\lambda + 1)(\lambda + 0.25) = 0$$

whose roots are −1 and −0.25. If the initial conditions are $y(0) = 1$ and $y'(0) = 0$, then the solution is

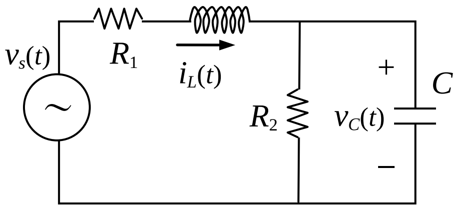
(2-63) $$y(t) = 0.1815e^{-t} + 0.1311e^{-0.25t} + 0.484\,cos(0.4\pi t - 130°)$$

and its simulation result is plotted in the figure. It can be seen that $y(t)$ only depends on the extra force $f(t) = cos\,0.4\pi t$ as $t \to \infty$, nothing to do with the initial conditions.



RLC Circuit

Consider an *RLC* circuit which contains a voltage source $v_s(t)$. The initial conditions are $v_C(0)=v_{C0}$ and $i_L(0)=i_{L0}$. It is known that the component equations of a capacitor and an inductor are



(2-64) $$i_C(t)=C\frac{dv_C(t)}{dt}=Cv_C'(t)$$

(2-65) $$v_L(t)=L\frac{di_L(t)}{dt}=Li_L'(t)$$

From the Kirchhoff's voltage law, we have

(2-66) $$v_s(t)=R_1i_L(t)+v_L(t)+v_C(t)=R_1i_L(t)+L\,i_L'(t)+v_C(t)$$

and from the Kirchhoff's current law, we have

(2-67) $$i_L(t)=Cv_C'(t)+\frac{1}{R_2}v_C(t)$$

The above two equations can be rewritten as

(2-68) $$\begin{cases} v_C'(t)=-\dfrac{1}{R_2C}v_C(t)+\dfrac{1}{C}i_L(t) \\ i_L'(t)=-\dfrac{1}{L}v_C(t)-\dfrac{R_1}{L}i_L(t)+\dfrac{1}{L}v_s(t) \end{cases}$$

or as the following matrix form

(2-69) $$\begin{bmatrix} v_C'(t) \\ i_L'(t) \end{bmatrix}=\begin{bmatrix} -\dfrac{1}{R_2C} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R_1}{L} \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}+\begin{bmatrix} 0 \\ \dfrac{1}{L}v_s(t) \end{bmatrix}$$

Further derive the characteristic polynomial of the circuit as below:

(2-70) $$\left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}-\begin{bmatrix} -\dfrac{1}{R_2C} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R_1}{L} \end{bmatrix}\right|=\begin{vmatrix} \lambda+\dfrac{1}{R_2C} & -\dfrac{1}{C} \\ \dfrac{1}{L} & \lambda+\dfrac{R_1}{L} \end{vmatrix}$$
$$=\lambda^2+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\lambda+\left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}$$

If the output is $y(t)=v_C(t)$, then based on the Cayley-Hamilton theory and the characteristic polynomial (2-70), the dynamic equation of the *RLC* circuit can be obtained as below:

(2-71)
$$\ddot{y}(t)+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\dot{y}(t)+\left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}y(t)$$
$$=\ddot{v}_C(t)+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\dot{v}_C(t)+\left(1+\frac{R_1}{R_2}\right)\frac{1}{LC}v_C(t)=\frac{1}{LC}v_s(t)$$

and the initial conditions are

(2-72)
$$\begin{cases} y(0)=v_C(0)=v_{C0} \\ \dot{y}(0)=\dot{v}_C(0)=-\dfrac{1}{R_2C}v_C(0)+\dfrac{1}{C}i_L(0)=-\dfrac{1}{R_2C}v_{C0}+\dfrac{1}{C}i_{L0} \end{cases}$$

Now, consider the following case: the component parameters are $R_1=9$, $R_2=7$, $C=3$ and $L=20$, the input voltage source is $v_s(t)=3$, and the initial conditions are $v_C(0)=0.2$ and $i_L(0)=0$. Then, the dynamic equation shown in (2-68) is

(2-73)
$$\begin{cases} v_C'(t)=-\dfrac{1}{21}v_C(t)+\dfrac{1}{3}i_L(t) \\ i_L'(t)=-\dfrac{1}{20}v_C(t)-\dfrac{9}{20}i_L(t)+\dfrac{1}{20}v_s(t) \end{cases}$$

with initial conditions $v_C(0)=0.2$ and $i_L(0)=0$. In addition, the dynamic equation can be also represented by (2-71) with initial conditions shown in (2-72), i.e.,

(2-74)
$$\ddot{y}(t)+\frac{209}{420}\dot{y}(t)+\frac{4}{105}y(t)=\frac{1}{60}v_s(t)$$

with $y(0)=0.2$ and $y'(0)=-\dfrac{1}{105}$. The characteristic equation is

(2-75)
$$\begin{vmatrix} \lambda+\dfrac{1}{21} & -\dfrac{1}{3} \\ \dfrac{1}{20} & \lambda+\dfrac{9}{20} \end{vmatrix}=\lambda^2+\frac{209}{420}\lambda+\frac{4}{105}=0$$

with roots $\lambda_1=-0.0945$ and $\lambda_2=-0.4031\approx 4.2656\lambda_1$.

There are two ways to simulate the *RLC* circuit, based on (2-73) or (2-74). If we use (2-73), then the Matlab simulation program can be written as below:

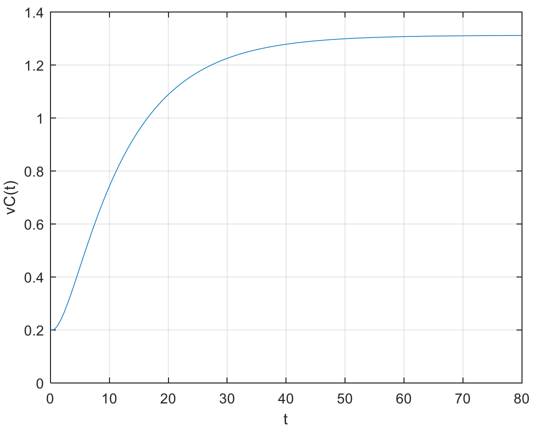
```
==================================
   Create m-file: RLC1.m
   function dy=RLC1(t,y)
   R1=9; R2=7; C=3; L=20; Vs=3;
   dy=zeros(2,1);    % a column vector
   dy(1)=-1/R2/C*y(1)+1/C*y(2);
   dy(2)=-1/L*y(1)-R1/L*y(2)+1/L*Vs;
==================================
   Create m-file: RLC1sol.m
   vC0=0.2; iL0=0;
```

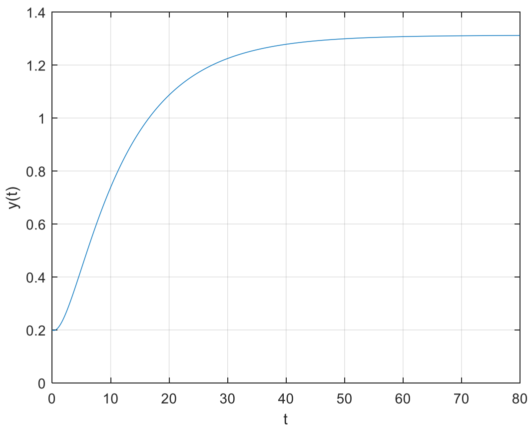
```
    [t,y]=ode45(@RLC1,[0:0.01:80], [vC0 iL0])
    plot(t,y(:,1))
    grid; xlabel('t'); ylabel('vC(t)');
==================================
```

On the other hand, if (2-74) is adopted, we can write the Matlab simulation program as below:

```
==================================
    Create m-file: RLC2.m
    function dy=RLC2(t,y)
    R1=9; R2=7; C=3; L=20; Vs=3;
    dy=zeros(2,1);    % a column vector
    dy(1)=y(2);
    dy(2)=-1*(1+R1/R2)/L/C*y(1)-(1/R2/C+R1/L)*y(2)+1/L/C*Vs;

==================================================
    Create m-file: RLC2sol.m
    y0=0.2; dy0=-1/105;
    [t,y]=ode45(@RLC1,[0:0.01:80], [y0 dy0])
    plot(t,y(:,1))
    grid; xlabel('t'); ylabel('y(t)');
==================================
```





The simulation results of $v_C(t)$ based on (2-73) and $y(t)$ based on (2-74) are shown in the above figures, left and right respectively. It is easy to observe that both are the same as expected.

Finally, let's introduce an important property called the dominant characteristic root. It is known that the characteristic roots of (2-74) are $\lambda_1 = -0.0945$ and $\lambda_2 = -0.4031 \approx 4.2656\lambda_1$. Since $|\lambda_2|$ is more than four times of $|\lambda_1|$, it implies that $e^{\lambda_2 t}$ will be vanished much faster than $e^{\lambda_1 t}$ as t increases. Hence, λ_1 is called the dominant characteristic root. Most importantly, without voltage source $v_s(t) = 0$, (2-74) can be approximately modeled as a 1st-order CODE whose characteristic root is

λ_1. To verify this property, let's choose $v_s(t)=0$ and rewrite (2-74) as

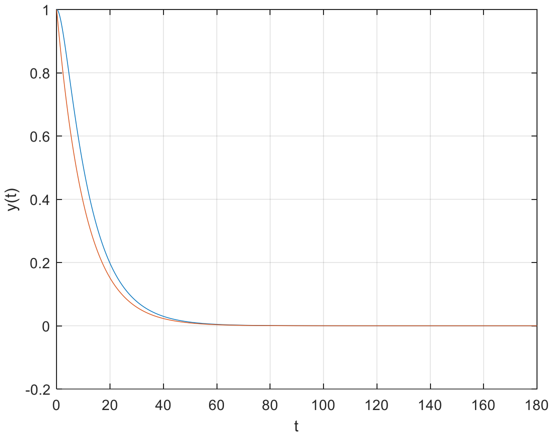
(2-76) $$\ddot{y}_1(t)+a_1 y_1(t)+a_0 y_1(t)=0, \quad y_1(0)=1, \ y_1'(0)=0$$

where $a_1=\dfrac{209}{420}$ and $a_0=\dfrac{4}{105}$. The approximate system is

(2-77) $$\dot{y}_2(t)-\lambda_1 y_2(t)=0, \quad y_2(0)=1$$

where the initial conditions $y_2(0)=y_1(0)=1$ and $\dot{y}_2(0)\neq\dot{y}_1(0)$.

```
=================================
Create m-file: y1.m
function dy=y1(t,y)
a1=209/420; a0=4/105;
dy=zeros(2,1);    % a column vector
dy(1)=y(2);
dy(2)=-a0*y(1)-a1*y(2);
=============================
Create m-file: y2.m
function dy=y2(t,y)
lamda=-0.0945;
dy=lamda*y
=============================
Create m-file: y1y2sol.m
[t,yy1]=ode45(@y1,[0:0.01:80], [1 0])
[t,yy2]=ode45(@y2,[0:0.01:80], [1])
plot(t, yy1(:,1),t,yy2)
grid; xlabel('t'); ylabel('y(t)');
===============================
```



From the figure, $y_1(t)$ is almost equal to $y_2(t)$ after $4\tau\approx 40$, where the time constant is $\tau=1/|\lambda_1|=10.582$ of (1-77). We know that λ_1 indeed can be treated as a dominant characteristic root of (1-76).