CV9 Elementary Functions : *Trigonometric and Hyperbolic Functions*

Before discuss the complex trigonometric functions, let's define the following exponential function:

$$f(z)=e^{iz}=e^{ix}\,e^{-y}=e^{-y}(\cos x+i\sin x)=u+iv$$

where

$$u_x=-e^{-y}\sin x,\quad v_y=-e^{-y}\sin x$$

$$u_y=-e^{-y}\cos x,\quad v_x=e^{-y}\cos x$$

i.e., the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$ are satisfied. That means e^{iz} is differentiable for all z or it is an entire function. Similarly, we know that e^{-iz} is also an entire function. Now, let's define two basic trigonometric functions, *sinz* and *cosz*, as below:

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}\quad \text{and}\quad \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

Obviously, both of them are also entire functions since they are linear combinations of the entire functions e^{iz} and e^{-iz}. Besides, the Euler's formula is still hold as below:

$$e^{iz}=\cos z+i\sin z$$

Based on the definition of *sinz* and *cosz*, the other four trigonimetric functions are defined as

$$\tan z=\frac{\sin z}{\cos z},\quad \cot z=\frac{\cos z}{\sin z},\quad \sec z=\frac{1}{\cos z},\quad \csc z=\frac{1}{\sin z}$$

It can be checked that identities established for real trigonometric functions are also satisfied and listed as below:

$$\sin(-z)=-\sin z,\qquad \cos(-z)=\cos z$$

$$\tan(-z)-\tan z,\qquad \sin^2 z+\cos^2 z=1$$

$$1+\tan^2 z=\sec^2 z,\qquad 1+\cot^2 z=\csc^2 z$$

$$\sin\left(z+\frac{\pi}{2}\right)=\cos z,\qquad \sin\left(z-\frac{\pi}{2}\right)=-\cos z$$

$$\sin(z+\pi)=-\sin z,\qquad \cos(z+\pi)=-\cos z$$

$$\sin(z+2\pi)=\sin z,\qquad \cos(z+2\pi)=\cos z$$

$$\sin(z_1\pm z_2)=\sin z_1\cos z_2\pm\cos z_1\sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

Moreover, the derivatives of trigonometric functions are

$$\frac{d}{dz}\sin z = \cos z, \qquad \frac{d}{dz}\cos z = -\sin z,$$
$$\frac{d}{dz}\tan z = \sec^2 z, \qquad \frac{d}{dz}\cot z = -\csc^2 z,$$
$$\frac{d}{dz}\sec z = \sec z \tan z, \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

In addition, when y is any real number, the hyperbolic functions are

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

which result in

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i \sinh y$$

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y$$

Therefore,

$$\sin z = \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$$
$$= \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$$
$$= \cos x \cosh y - i \sin x \sinh y$$

It is also easy to check that

$$\cosh^2 y - \sinh^2 y = 1$$

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y$$
$$= \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$
$$= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + \sinh^2 y$$

Clearly, $|\sin z| \ge |\sin x|$ and $|\cos z| \ge |\cos x|$.

Now, let's further define the complex hyperbolic functions as below:

$$sinh\ z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad cosh\ z = \frac{e^z + e^{-z}}{2}$$

Since e^z and e^{-z} are entire, *sinh z* and *cosh z* are entire. Similar to the real trigonometric functions, we define

$$tanh\ z = \frac{sinh\ z}{cosh\ z},\quad coth\ z = \frac{cosh\ z}{sinh\ z},\quad sech\ z = \frac{1}{cosh\ z},\quad csch\ z = \frac{1}{sinh\ z}$$

which satisfy the following identities:

$$cosh^2 z - sinh^2 z = 1 \qquad 1 - tanh^2 z = sech^2 z$$

$$coth^2 z - 1 = csch^2 z \qquad sinh(-z) = -sinh\ z$$

$$cosh(-z) = cosh\ z \qquad tanh(-z) - tanh\ z$$

$$sinh(z_1 \pm z_2) = sinh\ z_1\ cosh\ z_2 \pm cosh\ z_1\ sinh\ z_2$$

$$cosh(z_1 \pm z_2) = cosh\ z_1\ cosh\ z_2 \pm sinh\ z_1\ sinh\ z_2$$

$$tanh(z_1 \pm z_2) = \frac{tanh\ z_1 \pm tanh\ z_2}{1 \pm tanh\ z_1\ tanh\ z_2}$$

The derivatives of hyperbolic functions are

$$\frac{d}{dz} sinh\ z = cosh\ z, \qquad \frac{d}{dz} cosh\ z = sinh\ z$$

$$\frac{d}{dz} tanh\ z = sech^2 z, \qquad \frac{d}{dz} coth\ z = -csch^2 z$$

$$\frac{d}{dz} sech\ z = -sech\ z\ tanh\ z \qquad \frac{d}{dz} csch\ z = -csch\ z\ coth\ z$$

and the relations between trigonometric and hyperbolic functions are

$$sin\ iz = i\ sinh\ z \qquad cos\ iz = cosh\ z \qquad tan\ iz = i\ tanh\ z$$

$$sinh\ iz = i\ sin\ z \qquad cosh\ iz = cos\ z \qquad tanh\ iz = i\ tan\ z$$

It is also easy to check that

$$\begin{aligned} sinh\ z &= \frac{e^z - e^{-z}}{2} = \frac{1}{2}e^x e^{iy} - \frac{1}{2}e^{-x}e^{-iy} \\ &= \frac{1}{2}(cosh\ x + sinh\ x)(cos\ y + i\ sin\ y) \\ &\qquad - \frac{1}{2}(cosh\ x - sinh\ x)(cos\ y - i\ sin\ y) \\ &= sinh\ x\ cos\ y + i\ cosh\ x\ sin\ y \end{aligned}$$

$$\begin{aligned}cosh\, z &= \frac{e^z+e^{-z}}{2} = \frac{1}{2}e^x e^{iy} + \frac{1}{2}e^{-x}e^{-iy} \\ &= \frac{1}{2}(cosh\, x + sinh\, x)(cos\, y + i\, sin\, y) \\ &\qquad + \frac{1}{2}(cosh\, x - sinh\, x)(cos\, y - i\, sin\, y) \\ &= cosh\, x\, cos\, y + i\, sinh\, x\, sin\, y\end{aligned}$$

and thus from $cosh^2 x - sinh^2 x = 1$, we have

$$\begin{aligned}|sinh\, z|^2 &= sinh^2 x\, cos^2 y + cosh^2 x\, sin^2 y \\ &= sinh^2 x\,(1 - sin^2 y) + cosh^2 x\, sin^2 y \\ &= sinh^2 x + (cosh^2 x - sinh^2 x) sin^2 y \\ &= sinh^2 x + sin^2 y\end{aligned}$$

$$\begin{aligned}|cosh\, z|^2 &= cosh^2 x\, cos^2 y + sinh^2 x\,(1 - cos^2 y) \\ &= sinh^2 x + (cosh^2 x - sinh^2 x) cos^2 y \\ &= sinh^2 x + cos^2 y\end{aligned}$$

Clearly, $|sinh\, z| \ge |sinh\, x|$ and $|cosh\, z| \ge |sinh\, x|$.

In order to define the inverse sin function $sin^{-1} z$, we write $w = sin^{-1} z$ when $z = sin\, w$. Hence,

$$z = sin\, w = \frac{e^{iw} - e^{-iw}}{2i} \Rightarrow \left(e^{iw}\right)^2 - 2i\, z\left(e^{iw}\right) - 1 = 0 .$$

It results in $e^{iw} = i\, z + \left(1 - z^2\right)^{1/2}$ where $\left(1 - z^2\right)^{1/2}$ is a multiple-valued function. Taking logarithms of each side, we arrive at the expression

$$w = -i\, log\left[i\, z + \left(1 - z^2\right)^{1/2}\right]$$

i.e.,

$$sin^{-1} z = -i\, log\left[i\, z + \left(1 - z^2\right)^{1/2}\right]$$

which is a multiple-valued function with infinitely many values at z.

Example

$$\begin{aligned}sin^{-1}(-i) &= -i\, log\left[i\,(-i) + \left(1 - (-i)^2\right)^{1/2}\right] \\ &= -i\, log\left[1 + (2)^{1/2}\right] = -i\, log\left[1 \pm \sqrt{2}\right]\end{aligned}$$

where

$$log\left[1 + \sqrt{2}\right] = ln\left(1 + \sqrt{2}\right) + 2n\pi i \qquad \text{for } n \in Z$$

$$log\left[1-\sqrt{2}\right] = ln\left(\sqrt{2}-1\right)+(2n+1)\pi i$$
$$= -ln\left(1+\sqrt{2}\right)+(2n+1)\pi i$$

Both can be combined together as the following form:

$$log\left[1\pm\sqrt{2}\right] = (-1)^n\, ln\left(1+\sqrt{2}\right)+n\pi i\,.$$

Hence,

$$sin^{-1}(-i) = i\,(-1)^{n+1}\, ln\left[1+\sqrt{2}\right]+n\pi\,.$$

Finally, the inverse trigonometric functions are calculated and listed as below:

$$sin^{-1}z = -i\, log\left[i\,z+\left(1-z^2\right)^{1/2}\right],$$

$$cos^{-1}z = -i\, log\left[z+\left(z^2-1\right)^{1/2}\right]$$

$$tan^{-1}z = \frac{i}{2}\, log\,\frac{i+z}{i-z}, \qquad cot^{-1}z = \frac{i}{2}\, log\,\frac{z-i}{z+i}$$

$$sec^{-1}z = -i\, log\left[\frac{1+\left(1-z^2\right)^{1/2}}{z}\right],$$

$$csc^{-1}z = -i\, log\left[\frac{i+\left(z^2-1\right)^{1/2}}{z}\right]$$

Similarly, the inverse hyperbolic functions are

$$sinh^{-1}z = log\left[z+\left(z^2+1\right)^{1/2}\right] \qquad cosh^{-1}z = log\left[z+\left(z^2-1\right)^{1/2}\right]$$

$$tanh^{-1}z = \frac{1}{2}\, log\,\frac{1+z}{1-z}, \qquad coth^{-1}z = \frac{1}{2}\, log\,\frac{z+1}{z-1}$$

$$sech^{-1}z = log\left[\frac{1+\left(1-z^2\right)^{1/2}}{z}\right] \qquad csch^{-1}z = log\left[\frac{1+\left(z^2+1\right)^{1/2}}{z}\right]$$

and the derivatives of $sin^{-1}z$, $cos^{-1}z$ and $tan^{-1}z$ are

$$\frac{d}{dz}\, sin^{-1}z = \frac{1}{\left(1-z^2\right)^{1/2}}, \qquad \frac{d}{dz}\, cos^{-1}z = \frac{-1}{\left(1-z^2\right)^{1/2}}$$

$$\frac{d}{dz}\, tan^{-1}z = \frac{1}{1+z^2}$$

P9-1

Use the reflection principuple to show that, for all z,

(a) $\overline{\sin z} = \sin \bar{z}$; (b) $\overline{\cos z} = \cos \bar{z}$.

P9-2

Find all the roots of the equation $\cos z = 2$.

P9-3

Use the reflection principuple to show that, for all z,

(a) $\overline{\sinh z} = \sinh \bar{z}$; (b) $\overline{\cosh z} = \cosh \bar{z}$.

P9-4

Find all the roots of the equation $\cosh z = -2$.

P9-5

Find all the values of

(a) $\tan^{-1}(2i)$; (b) $\tan^{-1}(1+i)$; (c) $\cosh^{-1}(-1)$; (d) $\tanh^{-1}0$

P9-6

Solve the equation $\sin z = 2$ for z.

P9-7

Derive $\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$.

P9-8

Derive $\frac{d}{dz} \tan^{-1} z = \frac{1}{1+z^2}$.