CV8 Elementary Functions : *LogarithmicFunctions*

In real functions, it is known that the logarithmic function is the inverse function of the exponential function, and vice versa. The exponential function e^z has been defined as the following form

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Similar to e^x of real variable x, the following properties are also suitable for e^z:

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

$$e^{z_1-z_2} = e^{z_1}/e^{z_2}$$

$$e^{nz} = (e^z)^n \qquad n \in Z$$

$$\frac{de^z}{dz} = e^z \quad \text{(It is entire.)}$$

$$e^z \neq 0 \quad \text{for any } z$$

However, some properties are only suitable for e^z, not for e^x. For examples, we know that e^z has a pure imaginary period $2\pi i$ since $e^{z+2\pi i} = e^z$ and e^z can be negative if e^z is a real number.

Now, if $e^w = z$, where z is any nonzero complex number, then what is w? To solve this problem, we let $w = u+iv$ and $z = re^{i\Theta}$, where $-\pi<\Theta\leq\pi$. Then,

$$e^{u+iv} = e^u e^{iv} = re^{i\Theta}$$

i.e., $e^u = r$ and $v = \Theta + 2n\pi$. Obviously, $u = \ln r$ and we have

$$w = u + i\,v = \ln r + i\,(\Theta + 2n\pi) \qquad n \in Z$$

According to this expression, define the ***logarithmic function*** as

$$\begin{aligned} \log z &= \ln r + i\,(\Theta + 2n\pi) \qquad n \in Z \\ &= \ln|z| + i \arg z \end{aligned}$$

and then

$$e^{\log z} = z \qquad \text{for} \quad z \neq 0$$

Similarly, The ***principle value*** of $\log z$ is denoted as

$$\mathrm{Log}\, z = \ln r + i\,\Theta = \ln|z| + i\,\mathrm{Arg}\, z$$

Hence, $\log z = \mathrm{Log}\, z + 2n\pi i \quad n \in Z$.

Example

If $z=-1-\sqrt{3}i$, then r=2 and Θ=−2π/3.

Hence,

$$log\ z = ln\ 2 + i\left(-\frac{2\pi}{3}+2n\pi\right) = ln\ 2 + 2\left(n-\frac{1}{3}\right)\pi i$$

$$Log\ z = ln\ 2 - \frac{2}{3}\pi i\ .$$

Since $log\ z = ln\left|z\right| + i\ arg\ z$, we have

$$log\left(e^z\right) = ln\left|e^z\right| + i\ arg\left(e^z\right)$$

When z=x+iy, it can be obtained that

$$\left|e^z\right| = \left|e^{x+iy}\right| = e^x\left|e^{iy}\right| = e^x$$

$$arg\left(e^z\right) = arg\left(e^x e^{iy}\right) = arg\left(e^{iy}\right) = y + 2n\pi\ , \quad n\in Z$$

Hence,

$$\begin{aligned} log\left(e^z\right) &= ln\ e^x + i\left(y+2n\pi\right) \\ &= x + iy + 2n\pi i = z + 2n\pi i \end{aligned}$$

From the definition of *logz*, expressed as

$$log\ z = ln\ r + i\ \theta$$

we know that it is a multiple-valued function. If we let α denote any real number and restrict the value of θ as $\alpha<\theta<\alpha+2\pi$, then the logarithmic function is expressed as

$$log\ z = ln\ r + i\ \theta = u + i\ v\ \ \left(\ r>0, \alpha<\theta<\alpha+2\pi\ \right)$$

which is a single-valued function.

It is easy to check that *logz* is analytic since both $u = ln\ r$ and $v=\theta$ are continuous in the domain $r>0, \alpha<\theta<\alpha+2\pi$ and satisfy the Cauchy-Riemann conditions $ru_r = v_\theta$ and $u_\theta = -rv_r$. Furthermore, its derivative is

$$\frac{d}{dz}log\ z = e^{-i\theta}\left(u_r + iv_r\right) = e^{-i\theta}\left(\frac{1}{r}+i\,0\right) = \left(re^{i\theta}\right)^{-1} = \frac{1}{z}$$

for $\left|z\right|>0$ and $\alpha < arg\ z < \alpha + 2\pi$.

A ***branch*** of a multiple-valued function $f(z)$ is any single-valued function $F(z)$ that is analytic in some domain at each point z of which the value $F(z)$ is one of the values $f(z)$. For example, the single-valued function

$$\log z = \ln r + i\,\theta \quad \left(r > 0, \alpha < \theta < \alpha + 2\pi \right)$$

is a branch of the multiple-valued function $\log z = \ln r + i\,\theta$ and the function for α=0

$$\mathrm{Log}\, z = \ln r + i\,\Theta \qquad \left(r > 0, -\pi < \Theta < \pi \right)$$

is called the ***principle branch***.

A ***branch cut*** is a portion of a line or curve that is introduced to define a branch $F(z)$ of a multiple-valued function $f(z)$. Points on the branch cut for $F(z)$ are singular points of $F(z)$, and any point that is common to all branch cuts of $f(z)$ is called a ***branch point***.

Example

The origin z=0 and the ray θ=α make up the branch cut for the branch

$$\log z = \ln r + i\,\theta \quad \left(r > 0, \alpha < \theta < \alpha + 2\pi \right).$$

Clearly, the origin is a branch point since it is common to all the branch cuts of the multiple-valued function $\log z = \ln r + i\,\theta$.

Example

Show that

(a) $\log\left(i^2\right) = 2\log i$ when $\log z = \ln r + i\,\theta \quad \left(r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4} \right)$

(b) $\log\left(i^2\right) \neq 2\log i$ when $\log z = \ln r + i\,\theta \quad \left(r > 0, \dfrac{3\pi}{4} < \theta < \dfrac{11\pi}{4} \right)$

***<u>Solution</u>*:**

Since $\log z = \ln r + i\left(\Theta + 2n\pi\right)\left(r > 0, -\pi < \Theta < \pi \right)$, we have

(a) For $r > 0, \dfrac{\pi}{4} < \theta < \dfrac{9\pi}{4}$, it can be obtained that

$$\log\left(i^2\right) = \ln 1 + i\left(\pi\right) = i\pi \quad \text{and} \quad \log\left(i\right) = \ln 1 + i\left(\frac{\pi}{2}\right) = i\frac{\pi}{2}$$

It is clear that $\log\left(i^2\right) = 2\log i$.

(b) For $r>0, \frac{3\pi}{4}<\theta<\frac{11\pi}{4}$, it can be obtained that

$$log\left(i^2\right)=ln\,1+i\left(\pi\right)=i\,\pi \quad \text{and} \quad log\left(i\right)=ln\,1+i\left(\frac{5\pi}{2}\right)=i\,\frac{5\pi}{2}$$

It is clear that $log\left(i^2\right)\neq 2\,log\ i$.

Below introduce some identities involving logarithmic function. Let z_1 and z_2 denote any two nonzero complex numbers. Since

$$arg\left(z_1z_2\right)=arg\ z_1+arg\ z_2 \quad \text{and} \quad ln\left|z_1z_2\right|=ln\left|z_1\right|+ln\left|z_2\right|,$$

we have

$$\begin{aligned} &ln\left|z_1z_2\right|+i\ arg\left(z_1z_2\right) \\ &\quad=\left(ln\left|z_1\right|+ln\left|z_2\right|\right)+i\left(arg\ z_1+arg\ z_2\right) \\ &\quad=\left[ln\left|z_1\right|+i\left(arg\ z_1\right)\right]+\left[ln\left|z_2\right|+i\left(arg\ z_2\right)\right] \end{aligned} .$$

Hence, it is true that

$$log\left(z_1z_2\right)=log\ z_1+log\ z_2$$

under the statement :

"If values of two of the three logarithms are specified, then there is a value of the third logarithm such that the equation is hold."

Example

Write $z_1 = z_2 = -1$ and thus $z_1z_2 = 1$. If $log\ z_1 = \pi\, i$ and $log\ z_2 = -\pi\, i$ are specified, the equation

$$log\left(z_1z_2\right)=log\ z_1+log\ z_2$$

is satisfied when $log(z_1z_2)=0$ is chosen. However, it is not true for principle values, i.e.,

$$Log\left(z_1z_2\right)\neq Log\ z_1+Log\ z_2$$

because $Log(z_1z_2)=0$ and $Log\ z_1 + Log\ z_2 = 2\pi\, i$.

Next, let's find the expression of z^n where $n\in Z$. It is known that the following equations are true:

$$z^n=r^ne^{i\,n\theta}$$

$$e^{n\,log\,z}=e^{n\left[ln\,r+i\theta\right]}=e^{n\,ln\,r}e^{i\theta}=r^ne^{i\,n\theta}$$

i.e., we have

$$z^n = e^{n\,log\,z} \quad \text{for } z \neq 0 \text{ and } n \in Z\,.$$

Based on the above discussion, some identities are listed below:

$$log(z_1 z_2) = log\ z_1 + log\ z_2$$

$$log(z_1/z_2) = log\ z_1 - log\ z_2$$

$$z^n = e^{n\,log\,z} \quad \text{for } z \neq 0 \text{ and } n \in Z$$

$$z^{1/m} = exp\left(\frac{1}{m} log\ z\right) \quad \text{for } z \neq 0 \text{ and } m \in N$$

which are similar to the functions in real numbers.

With the complex logarithmic functions, we can define the following function with complex exponent

$$z^c = e^{c\,log\,z}$$

where $z \neq 0$, the exponent $c \in C$ and $log\ z$ denotes the multiple-valued logarithmic function. This provides a consistent definition of

$$z^n = e^{n\,log\,z} \quad \text{and} \quad z^{1/m} = exp\left(\frac{1}{m} log\ z\right).$$

Example

$$i^{-2i} = exp(-2\,i\,log\ i) = exp\left(-2\,i\left(i\left(\frac{\pi}{2} + 2\,n\,\pi\right)\right)\right)$$

$$= exp\left(2\left(\frac{\pi}{2} + 2\,n\,\pi\right)\right) = e^{(4n+1)\pi} \qquad n \in Z$$

In addition, similar to the truth of $1/e^z = e^{-z}$, it is easy to achieve the following relation:

$$z^{-c} = e^{-c\,log\,z} = 1/e^{c\,log\,z} = 1/z^c\,.$$

Furthermore, from the truth that the branch

$$log\ z = ln\,r + i\,\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

is single-valued and analytic in the indicated domain, we have

$$z^c = e^{c\,log\,z}$$

is also single-valued and analytic in the same domain. Besides, The derivative of such a branch is

$$\frac{d}{dz}z^c = \frac{d}{dz}e^{c\log z} = \frac{c}{z}e^{c\log z} = \frac{c}{z}z^c = c\,z^{c-1}$$

which includes the principle branch.

Example

The principle value of $(-i)^i$ is

$$\begin{aligned} P.V.(-i)^i &= exp(i\,Log\,(-i)) \\ &= exp\left(i\left(i\left(-\frac{\pi}{2}\right)\right)\right) = exp\left(\frac{\pi}{2}\right) \end{aligned}.$$

On the other hand, the exponential function with base *c*, where *c* is a nonzero complex constant, is written as

$$c^z = e^{z\log c}$$

When a value of *logc* is specified, c^z is an entire function of *z*. Its derivative is derived as

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log c} = log\,c\left(e^{z\log c}\right) = c^z log\,c$$

which is similar to the expression of functions in real numbers.

P8-1

Show that

(a) $exp(2 \pm 3\pi i) = -e^2$,

(b) $exp\left(\frac{2+\pi i}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$,

(c) $exp(z+\pi i) = -exp\,z$.

P8-2

Show that the function $f(z) = exp\,\bar{z}$ is not analytic anywhere.

P8-3

Find all values of *z* such that

(a) $e^z = -2$; (b) $e^z = 1+\sqrt{3}i$; (c) $exp(2z-1) = 1$.

P8-4

For n=0, ±1, ±2, …., verify that

(a) $log\ e = 1 + 2n\pi i$; (b) $log\ i = \left(2n + \frac{1}{2}\right)\pi i$;

(c) $log\left(-1+\sqrt{3}i\right) = ln\,2 + 2\left(n + \frac{1}{3}\right)\pi i$.

P8-5

Show that

(a) $Log\,(1+i)^2 = 2Log(1+i)$,

(b) $Log\,(-1+i)^2 \neq 2Log(-1+i)$.

P8-6

Find all roots of the equation $log\ z = i\pi/2$.

P8-7

Show that $Log(z_1 z_2) = Log\ z_1 + Log\ z_2$ if $Re\,z_1 > 0$ and $Re\,z_2 > 0$.

P8-8

Verify that $log(z_1/z_2) = log\ z_1 - log\ z_2$.

P8-9

Find the principal value of (a) i^i; (b) $\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right]^{3\pi i}$; (c) $(1-i)^{4i}$

P8-10

Show that $\left(-1+\sqrt{3}i\right)^{3/2} = \pm 2\sqrt{2}$.

P8-11

Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.