CV7 Analytic Functions : *Analytic Functions*

A function $f(z)$ is analytic in an open set if it has a derivative at each point in that set. Hence, it can be said that a function $f(z)$ is analytic at a point z_0 if it is analytic throughout some neighborhood of z_0. Besides, if a function $f(z)$ is analytic in a set S which is not open, then we can find an open set containing S in which $f(z)$ is analytic.

Example

$f(z)=1/z$ is analytic at each nonzero point in the finite plane. But, $f(z)=|z|^2$ is not analytic at any point since its derivative exists only at $z=0$ and not throughout any neighborhood of it.

An entire function is a function which is analytic at each point in ***the entire finite plane***. For example, any polynomial is an entire function. If $f(z)$ fails to be analytic at z_0 but is analytic at some points in every neighborhood of z_0, then z_0 is called a singular point, or singularity, of f(z). For example, $f(z)=1/z$ has a singular point at $z=0$, but $f(z)=|z|^2$ has no singular points since it is nowhere analytic.

If two functions are analytic in a domain D, then their sum and product are both analytic in D, and their quotient is analytic in D if the function in the denominator does not vanish at any point in D. In addition, a composite of two analytic functions is also analytic.
Any linear combination $c_1f_1(z)+ c_2f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

***Theorem*:**
If $f'(z)=0$ in a domain D, then $f(z)$ must be constant throughout the domain.

Example

The function $f(z)=\dfrac{z^3+4}{(z^2-3)(z^2+1)}$ is analytic throughout the z plane except for the singular points $z=\pm\sqrt{3}$ and $z=\pm i$.

Example

The function $f(z) = \cosh x \cos y + i \sinh x \sin y$ is analytic everywhere since

$$u_x = \sinh x \cos y = v_y$$

$$u_y = -\cosh x \sin y = -v_x .$$

It is clear that $f(z)$ is entire.

***Theorem*:**

The function $f(z) = u(x,y) + i\,v(x,y)$ and its conjugate $\overline{f(z)} = u(x,y) - i\,v(x,y)$ are both analytic in a given domain *D*. Then, $f(z)$ must be constant throughout *D*.

***Proof*:**

Since $f(z) = u(x,y) + i\,v(x,y)$ and its conjugate are analytic, they must satisfy the Cauchy-Riemann equations as below:

For $f(z)$, we have $u_x = v_y$ and $u_y = -v_x$

For $\overline{f(z)}$, we have $u_x = -v_y$ and $u_y = v_x$

These equations result in $u_x = v_x = 0$, i.e., $f'(z)=0$ in *D*, then $f(z)$ must be constant throughout *D*.

A real-valued function *H* of two real variables *x* and *y* is said to be ***harmonic*** in a given domain of *xy* plane if, throughout the domain, it has continuous derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x,y) + H_{yy}(x,y) = 0$$

known as **Laplace's equation**. *For example, the electrostatic potential V(x,y) in a region free of charges is harmonic.*

A real-valued function *H* of two real variables *r* and θ in the polar form is said to be harmonic in a given domain if, throughout the domain, it has continuous derivatives of the first and second order and satisfies the partial differential equation

$$r^2 H_{rr}(r,\theta)+rH_r(r,\theta)+H_{\theta\theta}(x,y)=0$$

known as **the polar form of Laplace's equation**.

***Theorem*:**

If a function $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

***Proof*:**

Since $f(z)=u(x,y)+i\,v(x,y)$ is analytic, the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$ must be true. Further taking derivative with respect to x and y, we have

$$u_{xx}=v_{xy} \quad \text{and} \quad u_{xy}=-v_{xx}$$

$$u_{xy}=v_{yy} \quad \text{and} \quad u_{yy}=-v_{xy}$$

which result in $u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$.

Clearly, both u and v are harmonic in D.

Other proof for the above theorem could be obtained directly from the following:

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u+i\,v=0$$

based on the truth that

$$\frac{\partial f(z)}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z)=0\,.$$

That means

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u+i\,v=\left(u_{xx}+u_{yy}\right)+i\left(v_{xx}+v_{yy}\right)=0$$

Clearly, $u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$.

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equations

$u_x = v_y$ and $u_y = -v_x$ throughout *D*, *v* is said to be a harmonic conjugate of *u*.

Theorem:

A function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain *D* if and only if *v* is harmonic conjugate of *u*.

Example

Consider $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$. Since these are the real and imaginary components of the entire function $f(z) = z^2$, we know that *v* is a harmonic conjugate of *u* throughout the plane. However, *u* is not a harmonic conjugate of *v*.

Consider $F(z) = v(x,y) + iu(x,y) = U(x,y) + i\,V(x,y)$, where $U(x,y)=2xy$ and $V(x,y)=x^2-y^2$. Then, it is easy to check that

$$U_x = -V_y \neq V_y \quad \text{and} \quad U_y = V_x \neq -V_x$$

i.e., $F(z)$ does not satisfy the Cauchy-Riemann equations. Therefore, $F(z) = v(x,y) + iu(x,y)$ is not analytic. In other words, $u(x,y)$ is not a harmonic conjugate of $v(x,y)$.

Example

Given $u(x,y)=y^3-3x^2y$, please determine its harmonic conjugate $v(x,y)$ and their corresponding function

$$f(z) = u(x,y) + i\,v(x,y).$$

<u>Solution</u>:

The following Cauchy-Riemann equations are true:

$$v_y = u_x = -6x \quad \text{and} \quad v_x = -u_y = 3x^2 - 3y^2$$

The first equation results in

$$v(x,y) = -6xy + k(x)$$

and taking partial derivative with respect to *x* yields

$$v_x = -6y + \frac{\partial k(x)}{\partial x} = 3x^2 - 3y^2$$

$$\Rightarrow \frac{\partial k(x)}{\partial x} = 3x^2 - 3y^2 + 6y$$

We have

$$k(x) = x^3 - 3xy^2 + 6xy + C$$

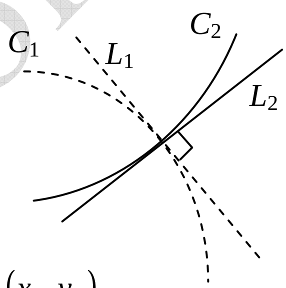
Therefore,

$$v(x,y) = x^3 - 3xy^2 + C$$

The corresponding equation is

$$\begin{aligned} f(z) &= u(x,y) + i\,v(x,y) \\ &= \left(y^3 - 3x^2 y\right) + i\left(x^3 - 3xy^2 + C\right) \end{aligned}$$

which is, of course, analytic.



A function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in the domain *D*. There are two families of level curves

$$C_1 : \; u(x,y) = \alpha_1$$

$$C_2 : \; v(x,y) = \alpha_2$$

where α_1 and α_2 are arbitrary real constants. Let the point (x_0, y_0), corresponding to $z_0 = x_0 + i\,y_0$ in the domain *D*, be an intersect of two particular level curves $u(x,y) = \alpha_1' \in C_1$ and $v(x,y) = \alpha_2' \in C_2$. If L_1 and L_2 are the lines respectively tangent to $u(x,y) = \alpha_1'$ and $v(x,y) = \alpha_2'$ at (x_0, y_0), then L_1 and L_2 are perpendicular to each other. It can be also stated that the two families of level curves C_1 and C_2 are orthogonal.

To explain the above statement, let's consider two level curves $u(x,y) = \alpha_1'$ and $v(x,y) = \alpha_2'$. Then, we have

$$du(x,y) = u_x dx + u_y dy = 0$$

$$dv(x,y) = v_x dx + v_y dy = 0$$

At the intersect (x_0, y_0), the slope of L_1 and L_2 are

$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y} \quad \text{and} \quad m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}.$$

from the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, we have

$$m_1 m_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = \frac{u_x}{u_y}\frac{v_x}{v_y} = -1$$

Evidently, L_1 and L_2 are perpendicular at their intersect (x_0, y_0).

***Lemma*:**

Suppose that

(i) $f(z)$ is analytic throughout a domain D;

(ii) $f(z)=0$ at each point z of a domain or line segment in D.

Then $f(z) \equiv 0$ in D; i.e., $f(z)$ is identically equal to zero throughout the whole D.

Suppose that two functions f and g are analytic in the same domain D and that $f(z)=g(z)$ at each point z of some domain or line segment contained in D. The difference $h(z)=f(z)-g(z)$ is also analytic in D, and $h(z)=0$ throughout the subdomain or along the line segment. According to the above theorem, then $h(z)=0$ thorughout D; i.e., $f(z)=g(z)$ at each point z in D.

***Theorem*:**

A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.

***Theorem*:** (*Reflection Principle*)

Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then

$$\overline{f(z)} = f(\bar{z})$$

for each point z in the domain if and only if $f(x)$ is real for each point x on the segment.

***<u>Proof</u>*:**

Let $f(z) = u(x,y) + i\,v(x,y)$ be analytic, then the Cauchy- Riemann equations are satisfied, i.e., $u_x = v_y$ and $u_y = -v_x$. Define $F(z) = \overline{f(\bar{z})} = U(x,y) + i\,V(x,y)$. Then, from the truth that $\overline{f(\bar{z})} = u(x,-y) - i\,v(x,-y)$ we obtain $U(x,y) = u(x,t)$ and

$V(x,y) = -v(x,t)$ with $t=-y$. Besides,

$$U_x = u_x,\ U_y = \frac{\partial u(x,t)}{\partial t}\frac{\partial t}{\partial y} = -u_t,$$

$$V_x = -v_x,\ V_y = -\frac{\partial v(x,t)}{\partial t}\frac{\partial t}{\partial y} = v_t.$$

It is easy to check that $U_x = V_y$ and $U_y = -V_x$, i.e., *U* and *V* satisfy the Cauchy-Riemann equations. Thus, *F*(*z*) is an analytic function. Now, let's show that if *f*(*x*) is real for each point *x* on the segment contained in *D*, then $\overline{f(z)} = f(\bar{z})$ for each point *z* in the domain *D*. Since $f(z) = u(x,y) + i\,v(x,y)$, if *f*(*x*) is real then

$$f(x) = u(x,0) \text{ and } v(x,0) = 0.$$

Hence, $F(x) = U(x,0) + i\,V(x,0) = u(x,0) = f(x)$, which implies that $F(z) = f(z)$ at each point *x* on the segment. It is known that an analytic function defined on a domain *D* is uniquely determined by its values along any line segment lying in *D*. Thus $F(z) = f(z)$ holds throughout *D*. Therefore, $\overline{f(\bar{z})} = f(z)$ or $\overline{f(z)} = f(\bar{z})$.

Next, let's show that if $\overline{f(z)} = f(\bar{z})$ for each point *z* in the domain *D*, then *f*(*x*) is real for each point *x* on the segment contained in *D*.

Let *z*=*x*, then $\overline{f(z)} = f(\bar{z})$ implies $\overline{f(x)} = f(x)$. Clearly, *f*(*x*) is real.

This proves the theorem of reflection principle.

Example

Based on the reflection principle, it is truth that

$$\overline{z+1} = \bar{z}+1,\ \overline{z^2} = \bar{z}^2, \text{ and } \overline{e^z} = e^{\bar{z}}$$

since $x+1$, x^2, and e^x are real. Similarly, $\overline{z+i} \neq \bar{z}+i$ and $\overline{i\,z^2} \neq i\,\bar{z}^2$ since $x+i$ and $i\,x^2$ are not real.

If *f*(*z*) is analytic and not constant throughout a domain *D*, then it cannot be constant throughout any neighborhood lying in *D*. If *f*(*z*)=ω and ω is a constant throughout a neighborhood in *D*, then *f*(*z*) must be the constant ω throughout the domain D.

P7-1 In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points.

(a) $f(z)=\dfrac{2z+1}{z(z^2+1)}$, (b) $f(z)=\dfrac{z^3+i}{z^2-3z+2}$

(c) $f(z)=\dfrac{z^2+1}{(z+2)(z^2+2z+2)}$

P7-2 Verify that $g(z)=ln\,r+i\theta$, $r>0$ and $0<\theta<2\pi$, is analytic in the indicated domain of definition, with derivative $g'(z)=\dfrac{1}{z}$. Then show that the composite function $G(z)=g(z^2+1)$ is analytic for $x>0$ and $y>0$ with the derivative $G'(z)=\dfrac{2z}{z^2+1}$.

P7-3 In a domain D, a function v is a harmonic conjugate of u and also that u is a harmonic conjugate of v. Show how it follows that both $u(x,y)$ and $v(x,y)$ must be constant throughout D.

P7-4 Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$ when

(a) $u(x,y)=2x(1-y)$; (b) $u(x,y)=2x-x^3+3xy^2$

(c) $u(x,y)=sinh\,x\,sin\,y$; (d) $u(x,y)=y/(x^2+y^2)$

P7-5 Show that $\overline{f(z)}=-f(\bar{z})$ if and only if $f(x)$ is pure imaginary.