# ***CV6 Analytic Functions*** **:** ***Derivatives and Cauchy-Riemann Equations***

Let the domain of *f*(*z*) contain a neighborhood of $z_0$, then the derivative of *f* at $z_0$ is defined as

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

The function *f* is said to be ***differentiable*** at $z_0$ when $f'(z_0)$ exists.

Example  Suppose that $f(z)=z^2$, then

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}=\lim_{\Delta z\to 0}(2z+\Delta z)=2z$$

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Example  Suppose that $f(z)=|z|^2$, then

$$f'(z)=\lim_{\Delta z\to 0}\frac{|z+\Delta z|^2-|z|^2}{\Delta z}=\lim_{\Delta z\to 0}\frac{(z+\Delta z)(\bar z+\overline{\Delta z})-z\bar z}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\left(\bar z+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z}\right)=\bar z\pm z$$

Clearly, it is not unique, unless $\bar z+z=\bar z-z$. That means $f'(z)$ exists only at $z$=0.

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Based on the definition of derivative, we can determine the following derivatives:

$$\frac{d}{dz}c=0$$

$$\frac{d}{dz}[c\cdot f(z)]=c\cdot f'(z)$$

$$\frac{d}{dz}z^n=n\cdot z^{n-1},\quad n\in N$$

$$\frac{d}{dz}z^{-n}=-n\cdot z^{-(n+1)},\quad \text{for } z\neq 0 \text{ and } n\in N.$$

Moreover, let $g(z)$, $f(z)$ and $F(z)=g(f(z))$ be differentiable, then

$$\frac{d}{dz}[f(z)+F(z)]=f'(z)+F'(z)$$

$$\frac{d}{dz}[f(z)F(z)]=f'(z)F(z)+f(z)F'(z)$$

$$\frac{d}{dz}\left[\frac{f(z)}{F(z)}\right]=\frac{f'(z)F(z)-f(z)F'(z)}{F^2(z)}$$

$$F'(z)=\frac{dg(f(z))}{dz}=\frac{dg(f(z))}{df(z)}\frac{df(z)}{dz}=g'(f(z))f'(z)$$

which are similar to the derivatives of functions in the real number system.

Example

$$\frac{d}{dz}\left(2z^2+i\right)^5=5\left(2z^2+i\right)^4(4z)=20z\left(2z^2+i\right)^4$$

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Suppose that $g(z_0)=f(z_0)=0$ and that $g'(z_0)$ and $f'(z_0)$ exist, where $g'(z_0)\neq 0$, then

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}=\lim_{z\to z_0}\frac{\left(\frac{f(z)-f(z_0)}{z-z_0}\right)}{\left(\frac{g(z)-g(z_0)}{z-z_0}\right)}$$

$$=\frac{\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}}{\lim\limits_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}}=\frac{f'(z_0)}{g'(z_0)}$$

Clearly, the L'Hopital rule is still available for complex functions.

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***Theorem*:**

Consider $f(z)=u(x,y)+i\cdot v(x,y)$. If $f'(z_0)$ exists where $z_0=x_0+i\,y_0$, then the first-order partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$ and satisfy the Cauchy-Riemann equations

$$u_x(x_0,y_0)=v_y(x_0,y_0)$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Besides, $f'(z_0)$ can be written as

$$f'(z_0) = u_x(x_0, y_0) + i \cdot v_x(x_0, y_0)$$

or

$$f'(z_0) = v_y(x_0, y_0) - i \cdot u_y(x_0, y_0)$$

***<u>Proof</u>*:**

Since $f'(z_0)$ exists, it can be evaluated as

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} \\ &\quad + i\left[\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}\right] \end{aligned}$$

for any $\Delta x \to 0$ and $\Delta y \to 0$. Assume $\Delta x >> \Delta y \to 0$, i.e., $\Delta z = \Delta x$, then $\Delta y$ is negligible and we can have

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \\ &\quad + i\left[\lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right] \\ &= u_x(x_0, y_0) + i\, v_x(x_0, y_0) \end{aligned}$$

Similarly, in case that $\Delta y >> \Delta x \to 0$, i.e., $\Delta z = i\Delta y$, then the derivative $f'(z_0)$ can be obtained as

$$\begin{aligned} f'(z_0) &= -i\left[\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}\right] \\ &\quad + \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \\ &= -i\, u_y(x_0, y_0) + v_y(x_0, y_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0) \end{aligned}$$

Clearly, the first partial derivatives $u_x(x_0, y_0)$, $u_y(x_0, y_0)$, $v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ exist and satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ at $(x_0, y_0)$.

If $f(z)$ doest not satisfy the Cauchy-Riemann equations at $z_0$, then $f'(z_0)$ does not exist. However, when the Cauchy-Riemann equations for $f(z)$ are hold at $z_0$, there is no guarantee that $f'(z_0)$ exists.

Example Consider the function

$$f(z)=\begin{cases}\bar{z}^2/z & \text{when } z\neq 0\\ 0 & \text{when } z=0\end{cases}$$

For $z=z_0+\Delta z$, where $z_0=0$ and $\Delta z\to 0$, we have

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{f(\Delta z)-f(0)}{\Delta z}=\frac{f(\Delta z)}{\Delta z}=\frac{(\Delta x-i\Delta y)^2}{(\Delta x+i\Delta y)^2}$$

If $\Delta z=\Delta x$ or $\Delta z=i\Delta y$, then

$$\left.\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right|_{\substack{z_0=0\\ \Delta z=\Delta x}}=\left(\frac{\Delta x}{\Delta x}\right)^2=1$$

$$\left.\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right|_{\substack{z_0=0\\ \Delta z=i\Delta y}}=\left(\frac{-i\Delta y}{i\Delta y}\right)^2=1$$

If $\Delta z=\Delta x+i\Delta x$, i.e., $\Delta y=\Delta x$, then

$$\left.\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right|_{\substack{z_0=0\\ \Delta z=\Delta x+i\Delta x}}=\left(\frac{(1-i)\Delta x}{(1+i)\Delta x}\right)^2=-1$$

Clearly, $f'(0)$ doesn't exist.

Now, let's check Cauchy-Riemann equations at $z$=0. Rewrite $f(z)$ for $z\neq 0$ as

$$f(x+iy)=\frac{(x^3-3xy^2)+i(y^3-3x^2y)}{x^2+y^2}$$

where $u(x,y)=\dfrac{x^3-3xy^2}{x^2+y^2}$ and $v(x,y)=\dfrac{y^3-3x^2y}{x^2+y^2}$. Hence, the first partial derivatives at $z$=0 are

$$u_x(0,0)=1,\quad u_y(0,0)=0,\quad v_x(0,0)=0,\quad v_y(0,0)=1$$

which satisfy the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$ at (0,0), but $f'(0)$ doesn't exist.

$\boxed{\text{Example}}$ For $f(z)=|z|^2$, it is known that only $f'(0)$ exists.

Since $f(x+iy)= x^2+y^2$, we have $u(x,y)= x^2+y^2$ and $v(x,y)= 0$.

Clearly, $u_x(0,0)=2x=0$, $u_y(0,0)=2y=0$, $v_x(0,0)=0$ and $v_y(0,0)=0$, which satisfy the Cauchy-Riemann equations.

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***Theorem*:**

The function $f(z)=u(x,y)+i\cdot v(x,y)$ is defined throughout some $\varepsilon$ neighborhood of $z_0=x_0+i\,y_0$ and the first-order partial derivatives of $u$ and $v$ exist everywhere in that neighborhood. If those partial derivatives are continuous at $(x_0, y_0)$ and satisfy the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$ at $(x_0, y_0)$, then $f'(z_0)$ exists.

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$\boxed{\text{Example}}$ Consider $f(z)=e^z=e^xe^{iy}=e^x cosy+ie^x siny$.

Then $u(x,y)=e^x cosy$ and $v(x,y)=e^x siny$. We have

$$u_x(x,y)=e^x cosy \quad \text{and} \quad v_x(x,y)=e^x siny$$
$$u_y(x,y)=-e^x siny \quad \text{and} \quad v_y(x,y)=e^x cosy$$

Clearly, these derivatives are everywhere continuous and satisfy the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$ everywhere. Thus, $f'(z)=u_x+iv_x=e^x cosy+ie^x siny=e^z$. Note that $f'(z)=f(z)$.

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In addition to the coordinate $(x,y)$, let's consider the Cauchy-Riemann equation in polar coordinate $(r,\theta)$. Since

$$z=x+iy=r\,e^{i\theta}$$

where $x=r\cos\theta$ and $y=r\sin\theta$, we have $\dfrac{\partial x}{\partial r}=\cos\theta$, $\dfrac{\partial y}{\partial r}=\sin\theta$, $\dfrac{\partial x}{\partial\theta}=-r\sin\theta$, and $\dfrac{\partial y}{\partial\theta}=r\cos\theta$. Therefore,

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \iff u_r=u_x\cos\theta+u_y\sin\theta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \iff u_\theta=-r\,u_x \sin\theta+r\,u_y\cos\theta$$

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r} \iff v_r=v_x\cos\theta+v_y\sin\theta$$

$$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} \iff v_\theta=-r\,v_x \sin\theta+r\,v_y\cos\theta.$$

From the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$, we have

$$u_r=u_x\cos\theta-v_x\sin\theta,\quad u_\theta=-r\left(u_x\sin\theta+v_x\cos\theta\right)$$

$$v_r=u_x\sin\theta+v_x\cos\theta,\quad v_\theta=r\left(u_x\cos\theta-v_x\sin\theta\right).$$

Clearly, the Cauchy-Riemann equations become

$$r\,u_r=v_\theta \quad \text{and} \quad u_\theta=-r\,v_r\,.$$

***Theorem***: The function $f(z)=u(r,\theta)+i\cdot v(r,\theta)$ is defined throughout some $\varepsilon$ neighborhood of a nonzero point $z_0=r_0\,e^{i\theta_0}$ and the first-order partial derivatives of $u$ and $v$ exist everywhere in that neighborhood. If those partial derivatives are continuous at ($r_0$, $\theta_0$) and satisfy the Cauchy-Riemann equations $r\,u_r=v_\theta$ and $u_\theta=-r\,v_r$ at ($r_0$, $\theta_0$), then $f'(z_0)=e^{-i\theta}(u_r+i\,v_r)$ exists.

Example Consider $f(z)=\dfrac{1}{z}=r^{-1}(\cos\theta-i\sin\theta)\ \ (z\neq 0)$.

Then $u(r,\theta)=r^{-1}\cos\theta$ and $v(r,\theta)=-r^{-1}\sin\theta$. We have

$$u_r(r,\theta)=-r^{-2}\cos\theta \quad \text{and} \quad v_r(r,\theta)=r^{-2}\sin\theta$$
$$u_\theta(r,\theta)=-r^{-1}\sin\theta \quad \text{and} \quad v_\theta(r,\theta)=-r^{-1}\cos\theta$$

Clearly, these derivatives are continuous at nonzero points and satisfy the Cauchy-Riemann equations $r\,u_r=v_\theta$ and $u_\theta=-r\,v_r$ at nonzero points. Thus,

$$f'(z)=e^{-i\theta}\left(-r^{-2}\cos\theta+i\,r^{-2}\sin\theta\right)=-\frac{1}{z^2}.$$

Example Consider $f(z)=\sqrt[3]{r}\,e^{i\theta/3}=z^{1/3}$. Then

$$u(r,\theta)=\sqrt[3]{r}\cos(\theta/3) \quad \text{and} \quad v(r,\theta)=\sqrt[3]{r}\sin(\theta/3)$$

We have

$$r\,u_r = \frac{1}{3}\sqrt[3]{r}\,cos(\theta/3) = v_\theta$$

$$u_\theta = -\frac{1}{3}\sqrt[3]{r}\,sin(\theta/3) = -r\,v_r$$

Clearly, these derivatives are continuous at nonzero points and satisfy the Cauchy-Riemann equations at nonzero points.

Thus,

$$\begin{aligned} f'(z) &= e^{-i\theta}\left(\frac{1}{3\left(\sqrt[3]{r}\right)^2}cos(\theta/3) + i\frac{1}{3\left(\sqrt[3]{r}\right)^2}sin(\theta/3)\right) \\ &= \frac{1}{3}z^{-2/3} \end{aligned}$$

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Since $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we have $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$ and $\frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$. The partial derivative becomes

$$\begin{aligned} \frac{\partial f(z)}{\partial \bar{z}} &= \frac{\partial f(z)}{\partial x}\left(\frac{\partial x}{\partial \bar{z}}\right) + \frac{\partial f(z)}{\partial y}\left(\frac{\partial y}{\partial \bar{z}}\right) \\ &= \frac{\partial f(z)}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f(z)}{\partial y}\left(\frac{i}{2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f(z) \end{aligned}$$

and thus, the partial operator with respect to $\bar{z}$ can be defined as

$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Let $f(z) = u(x,y) + i\cdot v(x,y)$, then

$$\begin{aligned} \frac{\partial f(z)}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x,y) + i\,v(x,y)) \\ &= \frac{1}{2}\left[(u_x - v_y) + i\,(u_y + v_x)\right] \end{aligned}.$$

Clearly, if $f(z) = u(x,y) + i\cdot v(x,y)$ satisfies Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, then $\frac{\partial f(z)}{\partial \bar{z}} = 0$.

Since $z\bar{z} = r^2$, taking derivative with respect to $\bar{z}$ yields

$z = \frac{\partial r^2}{\partial \bar{z}} = 2r\frac{\partial r}{\partial \bar{z}}$. Then, $\frac{\partial r}{\partial \bar{z}} = \frac{z}{2r} = \frac{1}{2}e^{i\theta}$.

Further taking derivative of $\bar{z} = re^{-i\theta}$ with respect to $\bar{z}$ results in

$1=\frac{\partial r}{\partial \bar{z}}e^{-i\theta}-i\,re^{-i\theta}\frac{\partial\theta}{\partial \bar{z}}=\frac{1}{2}-i\,re^{-i\theta}\frac{\partial\theta}{\partial \bar{z}}$ and then $\frac{\partial\theta}{\partial \bar{z}}=i\frac{1}{2r}e^{i\theta}$. The partial derivative becomes

$$\begin{aligned}\frac{\partial f(z)}{\partial \bar{z}}&=\frac{\partial f(z)}{\partial r}\left(\frac{\partial r}{\partial \bar{z}}\right)+\frac{\partial f(z)}{\partial \theta}\left(\frac{\partial \theta}{\partial \bar{z}}\right)\\&=\frac{\partial f(z)}{\partial r}\left(\frac{1}{2}e^{i\theta}\right)+\frac{\partial f(z)}{\partial \theta}\left(i\frac{1}{2r}e^{i\theta}\right)\\&=\frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r}+i\frac{\partial}{r\partial\theta}\right)f(z)\end{aligned}$$

and thus, the partial operator with respect to $\bar{z}$ can be defined as

$\frac{\partial}{\partial \bar{z}}=\frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r}+i\frac{\partial}{r\partial\theta}\right)$. Let $f(z)=u(r,\theta)+i\cdot v(r,\theta)$, then

$$\begin{aligned}\frac{\partial f(z)}{\partial \bar{z}}&=\frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r}+i\frac{\partial}{r\partial\theta}\right)(u(r,\theta)+i\,v(r,\theta))\\&=\frac{e^{i\theta}}{2r}[(ru_r-v_\theta)+i\,(u_\theta+rv_r)]\end{aligned}.$$

Clearly, if $f(z)=u(r,\theta)+i\cdot v(r,\theta)$ satisfies Cauchy-Riemann equations $r\,u_r=v_\theta$ and $u_\theta=-r\,v_r$, then $\frac{\partial f(z)}{\partial \bar{z}}=0$.

***P6-1***

Show that the derivative of $f(z)=\frac{1}{z}$ when z≠0 is $f'(z)=-\frac{1}{z^2}$.

***P6-2***

Find $f'(z)$ when

(a) $f(z)=3z^2-2z+4$; (b) $f(z)=(1-4z^2)^3$

(c) $f(z)=\frac{z-1}{2z+1}$ $\left(z\neq-\frac{1}{2}\right)$; (d) $f(z)=\frac{(1+z^2)^4}{z^2}$ $(z\neq 0)$

***P6-3***

Show that $f'(z)$ doesn't exist at any point if

(a) $f(z)=\bar{z}$; (b) $f(z)=z-\bar{z}$;

(c) $f(z)=2x+ixy^2$; (d) $f(z)=e^xe^{-iy}$

**_P6-4_**

Determine where $f'(z)$ exist and find its value when

(a) $f(z)=\dfrac{1}{z}$; (b) $f(z)=x^2+iy^2$; (c) $f(z)=z\cdot Im\,z$

***Ans*:** (a) $f'(z)=-\dfrac{1}{z^2}\ (z\neq 0)$, (b) $f'(x+ix)=2x$, (c) $f'(0)=0$

**_P6-5_**

Show that each of these functions is differentiable in the indicated domain of definition, and then find $f'(z)$

(a) $f(z)=\sqrt{r}e^{i\theta/2}$ ($r>0$, $\alpha<\theta<\alpha+2\pi$)

(b) $f(z)=e^{-\theta}\cos(\ln r)+ie^{-\theta}\sin(\ln r)$ ($r>0$, $0<\theta<2\pi$)