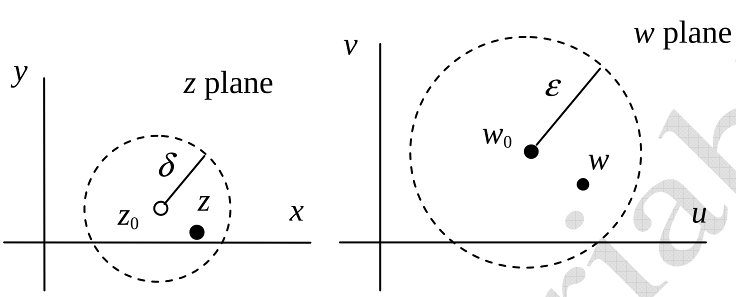
# *CV5 Analytic Functions* : *Limits and Continuity*

Let $f$ be defined at all points $z$ in $0<|z-z_0|<\delta$. If $f(z)$ has a limit $w_0$ as $z$ approaches $z_0$, then

$$\lim_{z\to z_0} f(z)=w_0$$

i.e., for all $\varepsilon>0$, there is a number $\delta>0$ such that $|f(z)-w_0|<\varepsilon$ whenever $0<|z-z_0|<\delta$. Note that $z_0$ can be a boundary point.




It is true that if a limit of $f(z)$ exists at a point $z_0$ then it is unique. To prove this, we suppose that it is not unique, i.e.,

$$\lim_{z\to z_0} f(z)=w_0 \quad \text{and} \quad \lim_{z\to z_0} f(z)=w_1$$

where $w_0\neq w_1$, then for any $\varepsilon>0$, there exist $\delta_0>0$ and $\delta_1>0$ such that

$$|f(z)-w_0|<\varepsilon \quad \text{whenever} \quad 0<|z-z_0|<\delta_0.$$

$$|f(z)-w_1|<\varepsilon \quad \text{whenever} \quad 0<|z-z_0|<\delta_1.$$

Therefore, if $0<|z-z_0|<\delta=min\{\delta_0,\delta_1\}$, then we have $|f(z)-w_0|<\varepsilon$ and $|f(z)-w_1|<\varepsilon$, which leads to

$$\begin{aligned}|w_1-w_0|&<|(f(z)-w_0)+(f(z)-w_1)|\\&\leq|f(z)-w_0|+|f(z)-w_1|<2\varepsilon\end{aligned}$$

Since $\varepsilon$ can be arbitrarily small, it can be concluded that $|w_1-w_0|=0$ or $w_1=w_0$. Clearly, this is contradictory to the assumption $w_0\neq w_1$. Hence, the limit $\lim_{z\to z_0} f(z)=w_0$ is unique.

Example Show that if $f(z)=iz/2$ and $|z|<1$, then $\lim_{z\to 1} f(z)=i/2$.

Clearly, the point $z$=1 is on the boundary of $|z|<1$. When $z$ is in $|z|<1$, we have

$$|f(z)-i/2| = \left|\frac{i}{2}(z-1)\right| = \frac{|z-1|}{2}$$

Hence, for any $\varepsilon$>0, there is a number $\delta$=2$\varepsilon$ such that

$$|f(z)-i/2| < \varepsilon \quad \text{whenever} \quad 0<|z-1|<\delta(=2\varepsilon)$$

According to the definition, we have $\lim_{z\to 1} f(z) = i/2$.

Example If $f(z) = z/\bar{z}$, then what is $\lim_{z\to 0} f(z)$?

Let's consider two cases of $z\to 0$: $z=x+i0\to 0$ and $z=0+iy\to 0$. For the case $z=x+i0\to 0$, it implies $x\to 0$ and then $\lim_{z\to 0} f(z) = \frac{x+i0}{x-i0} = 1$. For the other case $z=0+iy\to 0$, it implies $y\to 0$ and then $\lim_{z\to 0} f(z) = \frac{0+iy}{0-iy} = -1$. Clearly, $\lim_{z\to 0} f(z)$ does not exist since there are two results of $\lim_{z\to 0} f(z)$ while in different approaching ways.

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***Theorem:*** Suppose that

$$f(z)=u(x,y)+iv(x,y),\quad z_0 = x_0+iy_0 \quad \text{and} \quad w_0 = u_0+iv_0,$$

then $\lim_{z\to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

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***Theorem:*** Suppose that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$, then

$$\lim_{z\to z_0}\big(f(z)+F(z)\big) = w_0+W_0$$

$$\lim_{z\to z_0}\big(f(z)F(z)\big) = w_0W_0$$

$$\lim_{z\to z_0}\big(f(z)/F(z)\big) = w_0/W_0, \quad \text{for } W_0 \neq 0.$$

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Since $\lim_{z\to z_0} c = c$ and $\lim_{z\to z_0} z = z_0$, according to the above theorem we have

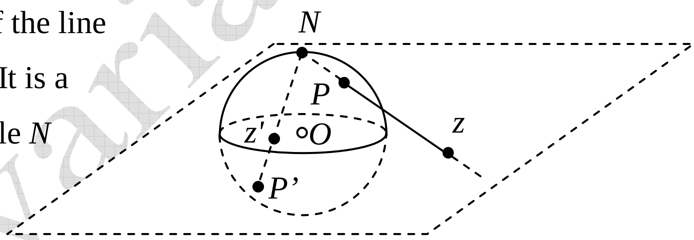
$$\lim_{z\to z_0} z^n = z_0^n \qquad \text{for} \quad n=1,2,\ldots$$

and

$$\lim_{z\to z_0} P(z) = P(z_0)$$

where $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$.

Sometimes, the limit of $f(z)$ includes the point at infinite $z=\infty$. The $z$ plane together with the point at infinity $\infty$, is called the extended complex plane. Think of the $z$ plane as passing through the equator of a unit sphere, the *Riemann sphere*, centered at $z=0$. To each point $z$ corresponds just one point $P$, determined by the intersection of the line through the point $z$ and the north pole $N$. It is a stereographic projection and the north pole $N$ is projected to the point at infinity.

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***Theorem:*** If $z_0$ and $w_0$ are points in $z$ and $w$ planes, respectively, then

$$\lim_{z\to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z\to \infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z\to \infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to 0} \frac{1}{f(1/z)} = 0$$

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Example

$$\lim_{z\to -1} \frac{iz+3}{z+1} = \infty \quad \text{if and only if} \quad \lim_{z\to -1} \frac{z+1}{iz+3} = 0,$$

$$\lim_{z\to \infty} \frac{2z+i}{z+1} = 2 \quad \text{if and only if} \quad \lim_{z\to 0} \frac{(2/z)+i}{(1/z)+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2,$$

$$\lim_{z\to \infty} \frac{2z^3-1}{z^2+1} = \infty \quad \text{if and only if} \quad \lim_{z\to 0} \frac{(1/z)^2+1}{2(1/z)^3-1} = \lim_{z\to 0} \frac{z+z^3}{2-z^3} = 0.$$

A function $f(z)$ is *continuous* at $z=z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$. If $f_1(z)$ and $f_2(z)$ are continuous at $z_0$, then $f_1(z)+f_2(z)$ and $f_1(z)f_2(z)$ are continuous at $z_0$, and $f_1(z)/f_2(z)$ is continuous at $z_0$ when $f_2(z_0) \neq 0$. As for the polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, it is continuous in the entire $z$ plane.

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***Theorem*** **:** A *composition* of continuous functions is itself continuous.

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***Theorem*** **:** If a function $f(z)$ is continuous and nonzero at a point $z_0$, then $f(z)\neq 0$ throughout some neighborhood of that point.

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The continuity of $f(z)=u(x,y)+iv(x,y)$ is closely related to the continuity of its components $u(x,y)$ and $v(x,y)$.

***P5-1*** Prove that

(a) $\lim_{z\to z_0} Re\, z = Re\, z_0$, (b) $\lim_{z\to z_0} Re\, \bar{z} = Re\, \bar{z}_0$, (c) $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$.

***P5-2*** Let $n$ be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0)\neq 0$. Find

(a) $\lim_{z\to z_0} \frac{1}{z^n} \, (z_0 \neq 0)$, (b) $\lim_{z\to i} \frac{iz^3 - 1}{z + i}$, (c) $\lim_{z\to z_0} \frac{P(z)}{Q(z)}$.

***Ans*****:** (a) $\frac{1}{z_0^n}$, (b) 0, (c) $\frac{P(z_0)}{Q(z_0)}$

***P5-3*** Use the theorem in this section to show that

(a) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$, (b) $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$, (c) $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$.