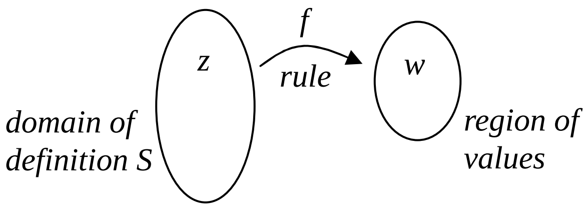
CV4 Analytic Functions : *Functions of Complex Variables*

Let S be a set of complex numbers. A function f of $z \in S$ is defined as

$$w=f(z)$$



where w is a complex number and called the value of f at z. The set of S is the domain of definition of the function. If the domain of definition is not mentioned, we often take the largest possible set.

Let $z=x+iy$ or $z=r\, e^{i\theta}$, then the value of the function f at z can be expressed as

$$f(z)=u(x,y)+i\cdot v(x,y)$$

or

$$f(z)=u(r,\theta)+i\cdot v(r,\theta)$$

where u and v are the real part and imaginary part of $f(z)$.

Example

Let $f(z)=z^2$, then

$$f(z)=x^2-y^2+i\,2xy=u(x,y)+i\cdot v(x,y)$$

where $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$.

If the polar coordinates are used, then

$$\begin{aligned} f(z)=r^2e^{i2\theta} &= r^2 cos\,2\theta+i\cdot r^2 sin\,2\theta \\ &= u(r,\theta)+i\cdot v(r,\theta) \end{aligned}$$

where $u(r,\theta)=r^2 cos\,2\theta$ and $v(r,\theta)=r^2 sin\,2\theta$.

If $f(z)$ is real, then $f(z)$ is a real-valued function. For example, $f(z)=|z|$ is a real-valued function, which can be represented as. $f(z)=|z|=\sqrt{x^2+y^2}$ or $f(z)=|z|=r$.

Let the complex polynomial of degree n with complex coefficients a_i, i=0,1,2,…,n, be given as

$$P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$$

then the complex rational function can be defined as

$$R(z)=\frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are complex polynomials and $Q(z)\neq 0$.

A function may assign more than one value to a point z in the domain of definition, which is called a multiple-valued function. For example, the function $z^{1/2}$ takes two values $\sqrt{r}\,e^{i\theta/2}$ and $-\sqrt{r}\,e^{i\theta/2}$ for $z=re^{i\theta}$ and thus $z^{1/2}$ is a multiple-valued function. However, when a multiple-valued function is considered, we only take one possible value.

Example

Let z be nonzero complex number, then $f(z)=z^{1/2}$ may be assigned by two possible values:

$$z^{1/2}=\pm\sqrt{r}\,e^{i\Theta/2}$$

where $r=|z|$ and $-\pi<\Theta\le\pi$.

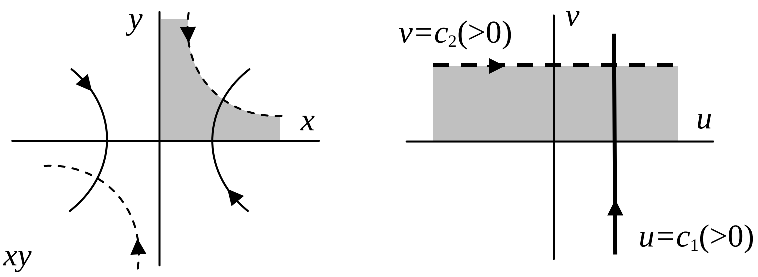
If we choose $f(z)=\sqrt{r}\,e^{i\Theta/2}$ for $r\ge 0$ and $-\pi<\Theta\le\pi$, then f is well defined on the entire z plane.

The information of $w=f(z)=u+iv$ can be displayed by pairs of $z=(x,y)$ in z plane and $w=(u,v)$ in w plane. This often refers to as a mapping or transformation.

Example

Let $f(z)=z^2$ and $z=x+iy$,

then



$$w=f(z)=x^2-y^2+i\,2xy$$
$$=u(x,y)+i\cdot v(x,y)$$

Hence,

$$u(x,y)=x^2-y^2,\qquad v(x,y)=2xy$$

The domain $x>0$, $y>0$, $xy<c_2/2$ is mapped on to $0<v<c_2$.

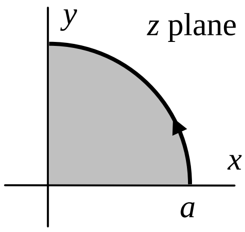
Example

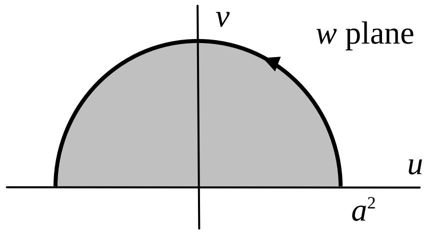
Let $f(z) = z^2$ and $z = r\,e^{i\theta}$,

then

$$w = f(z) = r^2 e^{i2\theta}$$

If $0 \le \theta \le \pi/2$,





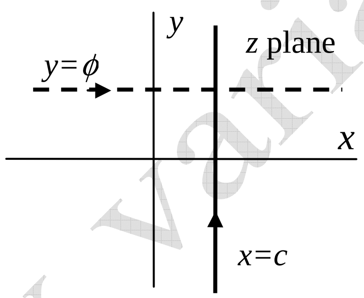
$w = z^2$ is one-to-one transformation onto the half upper *w* plane.
However, if $0 \le \theta \le \pi$, it maps onto the entire *w* plane, but not one-to-one, since both the positive and negative real axes in the *z* plane, i.e., $\theta=0$ and $\theta=\pi$, are mapped onto the same positive real axis in the *w* plane, i.e., $v=0$, $u>0$.

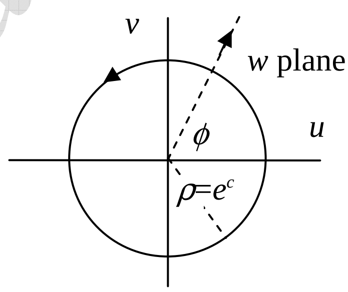
Example Let $w = e^z$.

If $z=x+iy$, then

$$w = e^x e^{iy} = \rho e^{i\phi}$$

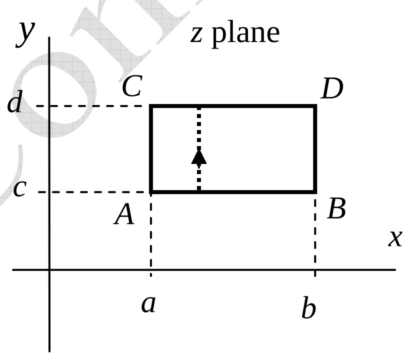
where $\rho = e^x$ and $\phi=y$.

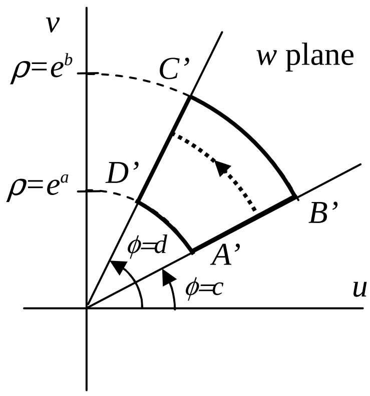




Example Let $w = e^z = e^x e^{iy} = \rho e^{i\phi}$.

If $a \le x \le b$ and $c \le y \le d < 2\pi + c$, then it is mapped onto *w* plane as below:





P4-1

Write the function $f(z) = z^3 + z + 1$ in the form of $f(z) = u(x,y) + iv(x,y)$.

***Ans*:** $\left(x^3 - 3xy^2 + x + 1\right) + i\left(3x^2y - y^3 + y\right)$

P4-2

Write the function $f(z) = z + \frac{1}{z}$ ($z \neq 0$) in the form of $f(z) = u(r,\theta) + iv(r,\theta)$.

***Ans*:** $\left(r + \frac{1}{r}\right)cos\,\theta + i\left(r - \frac{1}{r}\right)sin\,\theta$

P4-3

Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by the transformation

(a) $w=z^2$, (b) $w=z^3$, (c) $w=z^4$.

Supplementary:

Sketch the curve onto which the curve of r=2 and $0 \leq \theta \leq 2\pi$ is mapped by the transformation $w = z + \frac{1}{z}$.

Sol:

Since $z = re^{i\theta}$ is a circle with radius r=2, it is mapped onto a curve with transformation $w = z + \frac{1}{z}$. Then, we obtain

$$w = re^{i\theta} + \frac{1}{r}e^{-i\theta} = \left(r + \frac{1}{r}\right)cos\,\theta + i\left(r - \frac{1}{r}\right)sin\,\theta,$$

whose real part is $Re(w) = \left(r + \frac{1}{r}\right)cos\,\theta$ and imaginary part is $Im(w) = \left(r - \frac{1}{r}\right)sin\,\theta$. Hence, $\frac{Re^2(w)}{\left(r + \frac{1}{r}\right)^2} + \frac{Im^2(w)}{\left(r - \frac{1}{r}\right)^2} = 1$.

With r=2, we have $\left(\frac{Re(w)}{2.5}\right)^2+\left(\frac{Im(w)}{1.5}\right)^2=1$, which is an ellipse with long radius 2.5 and short radius 1.5.

Plot in Matlab=======

```
>> r=2; k=400; dq=exp(i*2*pi/k);
>> for n=1:k+1
    z(n)=r*dq^(n-1);
    w(n)=z(n)+1/z(n);
   end
>> plot(w)
>> clear z w
```