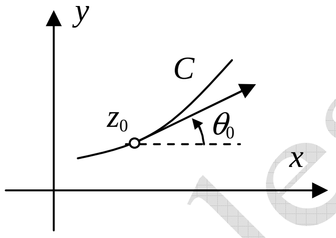
CV20 Mapping: Conformal Transformation

First, let's discuss the conformal transformation which is related to the preservation of angle. Consider a smooth arc C represented by $z(t)$ and a function $f(z(t))$ defined at all points on C, where $a \le t \le b$. Suppose that C passes through a point $z_0 = z(t_0)$, $a<t<b$, at which f is analytic and that $f'(z_0) \neq 0$. If $w(t) = f(z(t))$, then

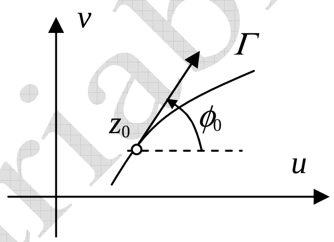


$$w'(t_0) = f'(z(t_0))z'(t_0)$$

which implies

$$arg\,w'(t_0) = arg\ f'(z(t_0)) + arg\ z'(t_0)$$

From the Figures of C and Γ, we know that $\theta_0 = arg\ z'(t_0)$ is the angle of inclination of a directed line tangent to C at z_0 and $\phi_0 = arg\,w'(t_0)$ is the angle of inclination of a directed line tangent to the image Γ at the point $w_0 = f(z_0)$. Define $\psi_0 = arg\ f'(z_0)$,



then $\psi_0 = \phi_0 - \theta_0$ which is an angle of rotation from θ_0 to ϕ_0.

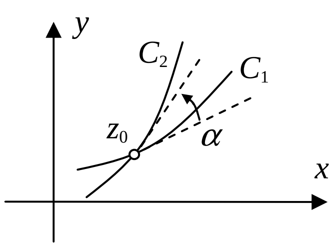
Now let C_1 and C_2 be two smooth arcs passing thorugh z_0, let θ_1 and θ_2 be angles of inclination of directed lines tangent to C_1 and C_2 at z_0 and let ϕ_1 and ϕ_2 be angles of inclination of directed lines tangent to the images Γ_1 and Γ_2 at the point $w_0 = f(z_0)$. Thus,

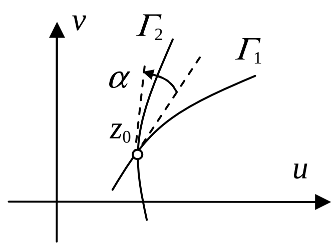
$$\psi_0 = arg\ f'(z_0) = \phi_1 - \theta_1 = \phi_2 - \theta_2$$

As a result,

$$\alpha = \phi_2 - \phi_1 = \theta_2 - \theta_1$$

which shows that the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 is the same as the angle $\theta_2 - \theta_1$ from C_1 to C_2.





Because of this angle-preserving property, a transformation $w = f(z)$ is said to be conformal at a point z_0 if f is analytic theree and $f'(z_0) \neq 0$. Such a transformation is actually conformal at each point in aneighborhood of z_0 for *f* must be analytic in a neighborhood of z_0 and *f'* is continuous at z_0, which implies that $f'(z) \neq 0$ throughout the neighborhood of z_0.

A transformation $w = f(z)$, defined on a domain *D*, is referred to as a conformal transformation, or conformal mapping, when it is conformal at each point in *D*. Hence, the mapping is conformal in *D* if *f* is analytic in *D* and its *f'* has no zero there. From the definition, the exponential function, logarithmic function, trigonometic function and hyperbolic function can be used to define a transformation that is conformal in some domain.

Example

The mapping $w=e^z$ is conformal throughout the entire *z* plane since $(e^z)' = e^z \neq 0$.

A mapping that preserves the magnitude of the angle between two smooth arcs but not the orientation is called an isogonal mapping.

Example

The transformation $w = \bar{z}$, which is a reflection in the real axis, is isogonal but not conformal.

Suppose that *f* is not a constant function and is analytic at a point z_0. If $f'(z_0) = 0$, then z_0 is called a critical point of the transformation $w = f(z)$.

Example

The point $z=0$ is a critical point of the transformation $w=1+z^2$. Choose C_1:$z=y+iy$ and C_2:$z=iy$, and then the angle between two arcs at $z=0$ is $\pi/4$.

They are mapped to Γ_1:w=1+i(2y^2) and Γ_2:z=1–y^2 and the angle between Γ_1 and Γ_2 is π/2, doubled by the transformation, not conformal.

Another property of a transformation $w = f(z)$ that is conformal at a point z_0 is related to the scale factor $|f'(z_0)|$, the modulus of $f'(z_0)$.

Example

When $f(z) = z^2$, the transformation

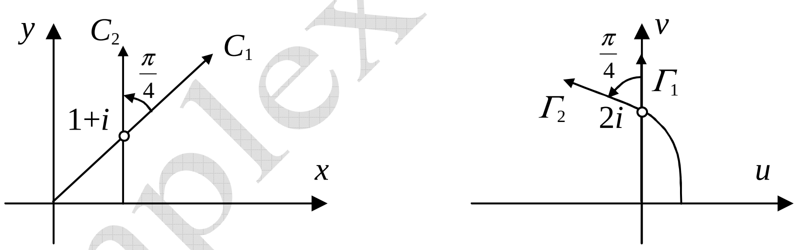
$$w = f(z) = x^2 - y^2 + i2xy$$

is conformal at z=1+i, where the half lines C_1: y=x(x≥0) and C_2: x=1(x≥0) intersect. Their images are

Γ_1: u=0 and v=2x^2

Γ_2: u=1–y^2 and v=2y.

At z=1+i, the angle from C_1 to C_2 is π/4 and at w=(1+i)^2=2i, the angle from Γ_1 to Γ_2 is also π/4. The scalar factor is $|f'(1+i)| = |2(1+i)| = 2\sqrt{2}$.



A transformation $w = f(z)$ that is conformal at a point z_0 has a local inverse there. That is, if $w_0 = f(z_0)$, then there exists a unique transformation $z = g(w)$, which is defined and analytic in a neighborhood N of w_0, such that $g(w_0) = z_0$ and $f(g(w)) = z$ for all points w in N. The derivative of $g(w)$ is

$$g'(w) = \frac{dz}{dw} = \frac{1}{f'(z)}$$

Hence, the transformation $z = g(w)$ is itself conformal at w_0.

It is known that a harmonic function $u(x,y)$ defined on a simply connected domain D satisfies

$$u_{xx} + u_{yy} = 0$$

and always has a harmonic conjugate $v(x,y)$ in D, which is expressed as

$$v(x,y) = \int_C [-u_t(s,t)ds + u_s(s,t)dt]$$

with an arbitrary contour C from (x_0,y_0) to (x,y).

Example

Consider $u(x,y) = xy$, which is harmonic throughout the xy plane. Then

$$v(x,y) = \int_C [-u_t(s,t)ds + u_s(s,t)dt]$$
$$= \int_{(0,0)}^{(x,y)} [-sds + tdt] = -\frac{1}{2}x^2 + \frac{1}{2}y^2$$

where C is from (0,0) to (x,y). Hence, the corresponding analytic function is

$$f(z) = xy - \frac{i}{2}(x^2 - y^2) = -\frac{i}{2}z^2$$

The problem of finding a function that is harmonic in a specified domain and satisfies prescribed conditions on the boundary of the domain is prominent in applied mathematics.

If the values of the function are prescribed along the boundary, the problem is known as a boundary value problem of the first kind, or a Dirichlet problem.

If the values of the normal derivative of the function are prescribed along the boundary, the problem is known as a boundary value problem of the second kind, or a Neumann problem.

Example

The function $T(x,y) = e^{-y}\sin x$ satisfies a certain Dirichlet problem for the strip $0<x<\pi$, $y>0$ and noted that it represents a solution of a temperature problem. The function $T(x,y) = e^{-y}\sin x$ is harmonic throughout the *xy* plane and is the real part of the entire function

$$-ie^{iz} = e^{-y} \sin x - ie^{-y} \cos x$$

or the imagainary part of the entire function e^{iz}.

Theorem

Suppose that an analytic function

$$w = f(z) = u(x,y) + iv(x,y)$$

maps a domain D_z in the *z* plane on to a domain D_w in the *w* plane. If $h(u,v)$ is a harmonic function defined on D_w, then the function

$$H(x,y) = h(u(x,y),v(x,y))$$

is harmonic in D_z.

Example

The function $h(u,v) = e^{-v} \sin u$ is harmonic in the domain D_w consisting of all the points in the upper half plane $v>0$. If the transformation is $w=z^2$, then $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$; moreover the domain D_z in the *z* plane consisting of the points in the first quadrant $x>0$, $y>0$ is mapped on to the domain D_w. Hence, the function

$$H(x,y) = h(u(x,y),v(x,y)) = e^{-2xy} \sin(x^2 - y^2)$$

is harmanic in D_z.

Example

The function $h(u,v) = v$ is harmonic in the horizontal strip $-\pi/2<v<\pi/2$. The transformation is $w=Logz=ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$ maps $x>0$ onto the strip. Hence,

$$H(x,y) = \tan^{-1}\frac{y}{x}$$

is harmanic in the half plane $x>0$.

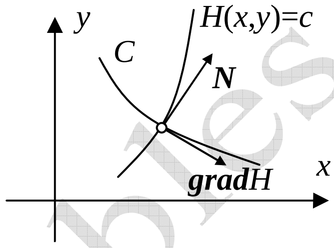
Theorem

Suppose that a transformation

$$w = f(z) = u(x,y) + iv(x,y)$$

is conformal on a smooth arc C, and let Γ be the image of C under that transformation. If, along Γ, $h(u,v)$ satisfies either of the conditions

$$h = h_0 \quad \text{or} \quad \frac{dh}{dn} = (\textbf{\textit{grad}}\ h)\cdot \boldsymbol{n} = 0$$

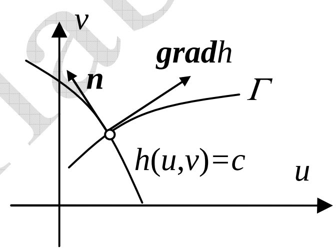


where h_0 is a real constant and dh/dn denotes derivatives normal to Γ, then, along C, the functio

$$H(x,y) = h(u(x,y), v(x,y))$$

satisfies the corresponding condition

$$H = h_0 \quad \text{or} \quad \frac{dH}{dN} = (\textbf{\textit{grad}}\ H)\cdot \boldsymbol{N} = 0$$



where dH/dN denotes derivatives normal to C.

Example

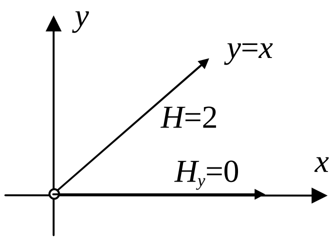
Consider the function $h(u,v) = v + 2$. The transformation

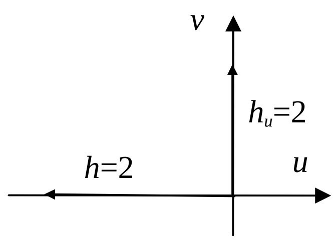
$$w = iz^2 = -2xy + i(x^2 - y^2)$$

is conformal when $z\neq 0$. It maps the half line $y=x$ ($x>0$) onto the negative u axis, where $h=2$, and the positive x axis onto the positive v axis, where the normal derivative h_u is 0. According to the theorem, the function

$$H(x,y) = x^2 - y^2 + 2$$

must satisfy the condition $H=2$ along the half line $y=x$ ($x>0$) and $H_y=0$ along the positive x axis, as one can verify directly.





Next we will focus on the concept of complex potential and apply it to electrostatic potential which is introduced in Electrical Engineering.

In an electrostatic force field, the field intensity at a point is a vector representing the force exerted on a unit positive charge placed at that point. The electrostatic potential *V* is a scalar function of the space coordinates such that, at each point in regions free of charges, the potential *V* is a harmoniic function. For simplicity, we will only discuss the cases in two dimensions *x* and *y*, then the potential *V* must satisfy the following harmonic condition

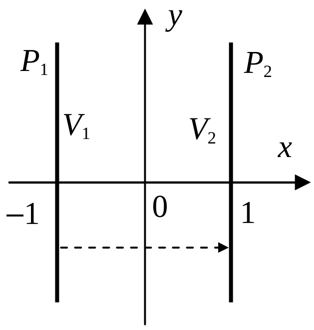
$$V_{xx}(x,y)+V_{yy}(x,y)=0$$

known as Laplace's equation.

Since the potential $V(x,y)$ is harmonic in a domain *D*, then there exists a harmonic conjugate $U(x,y)$ in the same domain *D*. Define a mapping in *z* plane as

$$w=F(z)=V(x,y)+iU(x,y)$$

then according to the conformal property, both the curves $V(x,y)=c_1$ and $U(x,y)=c_2$ are orthogonal except at the point $F'(z)=0$. Hence, the direction of $U(x,y)=c_2$ is the direction of electric force, i.e., all the positive charges in this electrostatic field will move along this direction..



First, let's consider two plate conductors P_1 and P_1 placed in parallel at $x=-1$ and $x=1$, where the potentials are constant V_1 and V_2 as shown in the Figure.
What is the potential distribution of the region between these two conductors? According to the structure, the potential $V(x,y)$ is only related to *x*. Besides, because there are no charges within the region, the potential must satisfy the Laplace equation $V_{xx}(x,y)=0$, which leads to

$$V(x,y)=ax+b$$

From the boundary conditions $V(-1,y)=V_1$ and $V(1,y)=V_2$ we obtain

$$a=\frac{V_2-V_1}{2} \quad \text{and} \quad b=\frac{V_2+V_1}{2}$$

Clearly, the surface $x=x_0$ is an equipotential surface with potential

$$V(x_0, y) = \frac{V_2 - V_1}{2} x_0 + \frac{V_2 + V_1}{2}.$$

Since the harmonic conjugate of $V(x,y)$ is

$$U(x,y) = \int \left[-V_y(x,y)dx + V_x(x,y)dy\right] = ay$$

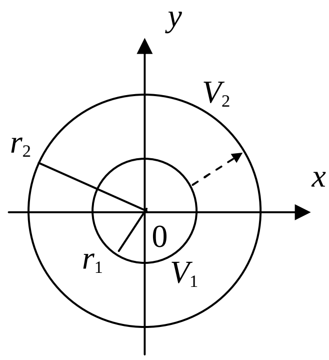
The complex potential is

$$F(z) = ax + b + iay = az + b$$

and the electrostatic force is along the direction of the line $y=y_0$ where y_0 is a constant, i.e., the force is in the direction parallel to x axis, shown as the dashed line in the Figure.

Next, let's consider two coaxial cylinders with radiuses r_1 and r_2 and constant potentials V_1 and V_2 as shown in the Figure. Due to the symmetrical structure, the potential in the region $r_1<r<r_2$ is only related to r. Besides, because there are no charges within the region, the potential must satisfy the Laplace equation $r^2V_{rr}(r,\theta)+rV_r(r,\theta)+V_{\theta\theta}(r,\theta)=0$. Since $V_{\theta\theta}(r,\theta)=0$, we have $r^2V_{rr}(r,\theta)+rV_r(r,\theta)=0$ or

$$\frac{\partial V_r(r,\theta)}{V_r(r,\theta)} = -\frac{1}{r}\partial r$$



which results in

$$lnV_r(r,\theta) = -lnr + c = ln\frac{a}{r}$$

with $c=lna$. Hence,

$$V_r(r,\theta) = \frac{\partial V(r,\theta)}{\partial r} = \frac{a}{r}$$

i.e.,

$$V(r,\theta) = alnr + b$$

From the boundary conditions $V(r_1,\theta)=V_1$ and $V(r_2,\theta)=V_2$ we obtain

$$a = \frac{V_1 - V_2}{lnr_1 - lnr_2} \quad \text{and} \quad b = \frac{V_2 lnr_1 - V_1 lnr_2}{lnr_1 - lnr_2}$$

Clearly, the surface $r=r_0$ is an equipotential surface with potential

$$V(r_0,\theta) = \frac{V_1 - V_2}{lnr_1 - lnr_2} r_0 + \frac{V_2 lnr_1 - V_1 lnr_2}{lnr_1 - lnr_2}.$$

Since the harmonic conjugate of $V(r,\theta) = a\ln r + b$ is

$$U(r,\theta) = a\theta$$

The complex potential is

$$F(z) = a\,\ln r + b + ia\theta = a\mathrm{Ln}z + b$$

where θ is chosen as principal argument. The electrostatic force is along the direction of the line $\theta=\theta_0$ where θ_0 is a constant, i.e., the force is in radial direction, shown as the dashed line in the Figure.

P20-1

What angle of rotation is produced by the transformation $w=1/z$ at the point (a) $z=1$, (b) $z=i$.

P20-2

Find the local inverse of the transformation $w=z^2$ at the point
(a) $z_0=2$, (b) $z_0=-2$, (c) $z_0=-i$.

P20-3

Use $v(x,y) = \int_{(0,0)}^{(x,y)} [-sds + tdt]$ to find a harmonic conjugate of the harmonic function $u(x,y) = x^3 - 3xy^2$.

P20-4

Under the transformation $w=e^z$, the image of the segment $0\le y\le\pi$ of the y axis is the semicircle $u^2+v^2=1$, $v\ge 0$. Also the function

$$h(u,v) = Re\left(2 - w + \frac{1}{w}\right) = 2 - u + \frac{u}{u^2+v^2}$$

is harmonic everywhere in the w plane except for the origin; and it assumes the value $h=2$ on the semicircle. Write the function $H(x,y)$, then illustrate that $H=2$ along the segment $0\le y\le\pi$ of the y axis.

P20-5

The transformation $w=z^2$ maps the positive x and y axes and the origin in the z plane onto the u axis in the w plane. Consider the harmonic function

$$h(u,v) = Re\left(e^{-w}\right) = e^{-u} cosv$$

and obseve that its normal derivative h_v along the u axis is zero.Then illustrate that the normal derivative of $H(x,y)$ is zero along both positive axes in the z plane. (Note that the transformation $w=z^2$ is not conformal at the origin.)