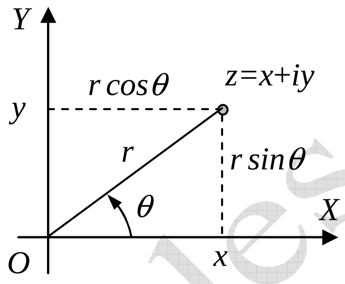
CV2 Complex Numbers: Exponential Form and de Moivre's Formula

A nonzero complex number *z=x+iy* can be written in the polar form as below:

$$z = x + iy = r(\cos\theta + i\sin\theta)$$



where $r = |z|$ is the modulus and θ is the angle or argument. Clearly, we have

$$x = r\cos\theta$$

$$y = r\sin\theta$$

as shown in the figure.

It is true that $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$ for any integer *n*. That implies the argument θ, also represented by *arg z*, is not unique. In general, we define the principal argument as Θ, or *Arg z*, where $-\pi < \Theta \le \pi$ and then the argument can be written as

$$\theta = \Theta + 2n\pi, \qquad n = 0, \pm 1, \pm 2, \cdots$$

For example, the argument of $-1-i$ is $\theta = -\frac{3\pi}{4} + 2n\pi$ and its principal argument is $\Theta = -\frac{3\pi}{4} = -2.3562$. In Matlab, the command angle(z) is used to calculate the principal argument of z in radian.

Matlab:

```
>> z=-1-i;
>> angle(z)    % argument of z in rad
ans =
      -2.3562

>> angle(z)*180/pi    % argument of z in degree
ans =
      -135
```

Actually, the polar form of complex numbers can be also expressed in the exponential form as below:

$$z = r(\cos\theta + i\cdot\sin\theta) = r\cdot e^{i\theta}$$

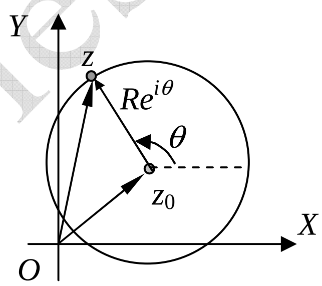
where θ is measured in radian. Such representation is proposed by Euler and thus called the Euler's formula. In fact, the Euler's formula can be derived from the Taylor's expansion of the natural exponential function, which is given as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$e = 2.71828\cdots$

Letting x=1 yields e=2.71828⋯. Similarly, by letting $x=i\theta$, we have

$$e^{i\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)}_{sin\theta}$$
$$= cos\,\theta + i\,sin\,\theta$$



where the real part and the imaginary part are respectively the Taylor's expansion of $cos\theta$ and $sin\theta$. Note that

$$\left|e^{i\theta}\right| = \left|cos\,\theta + i\,sin\,\theta\right| = \sqrt{cos^2\,\theta + sin^2\,\theta} = 1$$

i.e., the modulus of $e^{i\theta}$ is 1 for any argument θ. Based on the Euler's formula, the circle $\left|z - z_0\right| = R$ with radius R and centered at z_0 can be also written as

$$z = z_0 + R \cdot e^{i\theta}$$

which is depicted in the figure.

Most importantly, the Euler's formula makes the multiplication of complex numbers easier since the product of $e^{i\theta_1}e^{i\theta_2}$ can be calculated as below:

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

which can be seen from

$$\left(cos\,\theta_1 + i\cdot sin\,\theta_1\right)\left(cos\,\theta_2 + i\cdot sin\,\theta_2\right) = cos\left(\theta_1 + \theta_2\right) + i\cdot sin\left(\theta_1 + \theta_2\right).$$

Similarly, the product of z_1 and z_2 is also easily obtained by

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 \cdot e^{i(\theta_1+\theta_2)}$$

For the inverse of any nonzero complex number z, we have

$$z^{-1} = \frac{1}{r\left(cos\,\theta + i\cdot sin\,\theta\right)} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$
$$= \frac{1}{r}\left(cos\,\theta - i\cdot sin\,\theta\right)$$

Then, the quotient of z_1 to z_2 becomes

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Most importantly, the famous de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

can be derived by the Euler's formula as below:

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta,$$

for $n \in Z$.

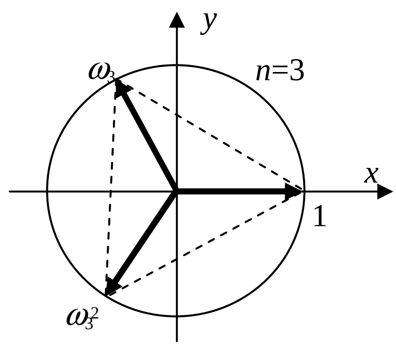
The de Moivre's formula has been widely used in engineering analysis. For example, it can be used to determine the roots of the equation $z^n = r e^{i\Theta}$. First, let's define

$$c_k = \sqrt[n]{r}\, e^{i\left(\frac{\Theta}{n}+\frac{2k\pi}{n}\right)}, \quad k = 0, 1, 2, \cdots, n-1$$

and it is easy to check that

$$c_k^n = \left[\sqrt[n]{r}\, e^{i\left(\frac{\Theta}{n}+\frac{2k\pi}{n}\right)}\right]^n = r\, e^{i(\Theta+2k\pi)} = re^{i\Theta}$$

which means all the c_k's are the n roots of $z^n = r e^{i\Theta}$. Note that all these roots lie on the circle $|z| = \sqrt[n]{r}$ and are equally spaced every $2\pi/n$ rad, starting with the principal root $c_0 = \sqrt[n]{r}\, e^{i\frac{\Theta}{n}}$.



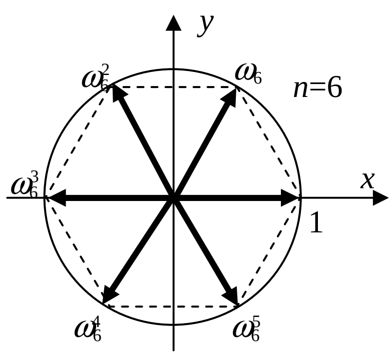
Now, let's consider the special case of Θ=0 and r=1. The n roots of $z^n = 1$ are given as

$$c_k = e^{i\left(\frac{2k\pi}{n}\right)} = \omega_n^k, \quad k = 0, 1, 2, \cdots, n-1$$

where

$$\omega_n = e^{i\frac{2\pi}{n}}.$$

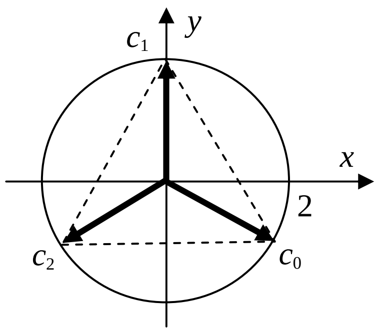
The roots for n=3 and n=6 are shown on the right where all the roots are located along the unit circle $|z| = 1$.



We can adopt the term $\omega_n = e^{i\frac{2\pi}{n}}$, a rotator with rotating angle $2\pi/n$, to represent the roots of $z^n = re^{i\Theta}$, which are given as

$$c_k = \sqrt[n]{r}\, e^{i\frac{\Theta}{n}} e^{i\frac{2k\pi}{n}} = \sqrt[n]{r}\, e^{i\frac{\Theta}{n}} \omega_n^k = c_0\, \omega_n^k$$

for $k = 0,1,\pm 2,\cdots,n-1$, where $c_0 = \sqrt[n]{r}\, e^{i\frac{\Theta}{n}}$. The figure on the right shows the roots of $z^n = re^{i\Theta}$, where n=3, r=8, Θ=$-\pi$/2.



P2-1 Find the principal argument *Argz* for

(a) $z = \dfrac{i}{-2-2i}$; (b) $z = \left(\sqrt{3}-i\right)^6$.

Ans: (a) $-3\pi/4$ (b) π

P2-2 Use de Moivre's formula to derive the trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta$;

(b) $\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$.

P2-3 Establish the identity $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$, $z \neq 1$,

and then use it to derive Lagarnge's trigonometric identity:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}$$

where $0<\theta<2\pi$.

P2-4 Find the principal root of c_0 $z^6 = -8$ and all the roots $c_0\omega^k$, k=0,1,…,5. Then verify $(c_0\omega)^6 = -8$ by Matlab.

Ans: $c_0 = \sqrt{2}e^{i\frac{\pi}{6}}$, $\omega = e^{i\frac{\pi}{3}}$

Matlab:

```
>> c0=2^0.5*exp(i*pi/6)
c0 =
     1.2247 + 0.7071i
>> w=exp(i*pi/3)
w =
     0.5000 + 0.8660i
>> (c0*w)^6
ans =
   -8.0000 + 0.0000i
```

$\boxed{\textbf{\textit{P2-5}}}$ Consider the quadratic equation $az^2+bz+c=0$, where a, b, and c are complex numbers and $a\neq 0$. Derive the quadratic formula

$z=\dfrac{-b+\left(b^2-4ac\right)^{1/2}}{2a}$ and then solve $z^2+2z+(1-i)=0$.

Ans: $\left(-1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}},\ \left(-1-\dfrac{1}{\sqrt{2}}\right)-\dfrac{i}{\sqrt{2}}$