CV19 Mapping: *Elementary Transformations*

First, let's consider the simpliest mapping, or transformation, which is given as

$$w = Az$$

where $A = ae^{i\alpha}$ and $z = re^{i\theta}$ ($ar \neq 0$). As a result, we have

$$w = (ar)e^{i(\alpha+\theta)}$$

where the module of z is expanded or contracted with the factor $a=|A|$ and rotates an angle $\alpha = arg\ A$. Next, let's consider the mapping $w = z + B$, where $B=b_1+ib_2$ is any complex constant, then $z=x+iy$ translates to $w=u+iv$ by means of vector addition in w plane as below:

$$(u,v) = (x,y) + (b_1,b_2) = (x+b_1, y+b_2).$$

Based on these two transformations, the general linear transformation can be expressed as

$$w = Az + B \qquad (A \neq 0)$$

which is a composition of the transformations

$Z=Az$ and $w=Z+B$

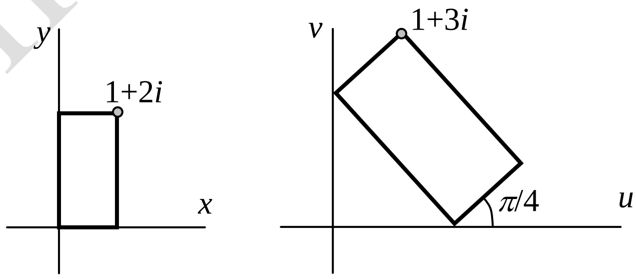
i.e., an expansion or contraction and a rotation, followed by a translation.

$\boxed{\text{Example}}$

Let z be a rectangular in the z plane and then the mapping

$$w = (1+i)z + 2 = \sqrt{2}\, e^{i\pi/4} z + 2$$

transforms the rectangular as below:

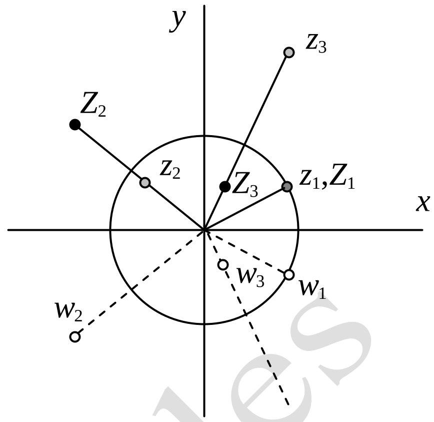


Following the linear transformation, let's discuss a nonlinear transformation, called the inverse transformation and given as

$$w = \frac{1}{z}$$

What is the relation between z and w in the complex plane? It is easy to check that $z\bar{z}=|z|^2$ and then we define

$$Z=\frac{1}{\bar{z}}=\frac{z}{|z|^2}$$



which implies Z and z are in the same direction. Besides, taking the module of Z yields

$$|Z|=\frac{|z|}{|z|^2}=\frac{1}{|z|}$$

Clearly, for Z and z, if one is on the unit circle, then the other is on the unit circle, shown by Z_1 and z_1 in the Figure. Similarly, if one is in the unit circle, then the other is outside the unit circle, shown by Z_2 and z_2 or by Z_3 and z_3 in the Figure.

Hence, from $Z=\frac{1}{\bar{z}}$ and $w=\frac{1}{z}$, it is obvious that $w=\bar{Z}$ as shown in the Figure similarly. Now, let's discuss the mapping of $w=\frac{1}{z}$ more detailedly.

When a point $w=u+iv$ is the image of a nonzero point $z=x+iy$ under the transformation $w=1/z$, we have

$$u=\frac{x}{x^2+y^2},\quad v=\frac{-y}{x^2+y^2}$$

or $$x=\frac{u}{u^2+v^2},\quad y=\frac{-v}{u^2+v^2}$$

Such mapping $w=1/z$ transforms circles and lines in the z plane as

$$A\left(x^2+y^2\right)+Bx+Cy+D=0$$

where $A=0$ for a line and $A\neq 0$ with $B^2+C^2>4AD$ for a circle into the w plane as

$$D\left(u^2+v^2\right)+Bu-Cv+A=0$$

where $D=0$ for a line and $D\neq 0$ with $B^2+C^2>4AD$ for a circle.

A circle ($A\neq 0$) which is not passing through the origin ($D\neq 0$) in the z plane, described as

$$A\left(x^2+y^2\right)+Bx+Cy+D=0$$

i.e., $$\left(x+\frac{B}{2A}\right)^2+\left(y+\frac{C}{2A}\right)^2=\left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$$

is transformed into a circle not passing through the origin in the w plane

$$D\left(u^2+v^2\right)+Bu-Cv+A=0$$

i.e., $$\left(u+\frac{B}{2D}\right)^2+\left(v-\frac{C}{2D}\right)^2=\left(\frac{\sqrt{B^2+C^2-4AD}}{2D}\right)^2.$$

A circle ($A\neq 0$) through the origin ($D=0$) in the z plane, given as

$$A\left(x^2+y^2\right)+Bx+Cy=0$$

i.e., $$\left(x+\frac{B}{2A}\right)^2+\left(y+\frac{C}{2A}\right)^2=\left(\frac{\sqrt{B^2+C^2}}{2A}\right)^2$$

is transformed into a line

$$Bu-Cv+A=0.$$

that does not pass through the origin in the w plane. A line ($A=0$) not passing through the origin ($D\neq 0$) in the z plane

$$Bx+Cy+D=0$$

is transformed into a circle

$$D\left(u^2+v^2\right)+Bu-Cv=0$$

through the origin in the w plane. A line ($A=0$) through the origin ($D=0$) in the z plane

$$Bx+Cy=0$$

is transformed into a line

$$Bu-Cv=0.$$

through the origin in the w plane.

The linear fractional transformation, or Möbius transformation, is described as the following form

$$w=\frac{az+b}{cz+d}\quad\left(ad-bc\neq 0\right)$$

where a, b, c, and d are complex constants. It can be further written into

$$Azw + Bz + Cw + D = 0 \quad (AD - BC \neq 0)$$

which is linear in z and linear in w, or bilinear in z and w, and thus is also called bilinear transformation.

It is easy to check that when c=0 the fractional transformation becomes a linear transformation

$$w = \frac{a}{d}z + \frac{b}{d} \quad (ad \neq 0).$$

As for $c \neq 0$, we let

$$Z = cz + d \quad \text{and} \quad W = \frac{1}{Z}$$

then

$$w = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz + d} = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{Z} \quad (ad - bc \neq 0)$$

i.e.,

$$w = \frac{a}{c} + \frac{bc - ad}{c}W \qquad (ad - bc \neq 0).$$

Clearly, a linear fractional transformation is a composition of linear transformation and a inverse transformation. Regardless of whether c is zero or nonzero, any linear fractional transformation transforms circles and lines into circles and lines.

It is known that the linear fractional transformation is a one to one mapping of the extended z plane onto the extended w plane. Let

$$w = T(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)$$

then $T(z_1) \neq T(z_2)$ for $z_1 \neq z_2$. Hence, there is an inverse transformation T^{-1} defined on the extended w plane. The inverse transformation of T is expressed as

$$T^{-1}(w) = z \quad \text{if and only if} \quad T(z) = w.$$

It can be also expressed as

$$T^{-1}(w) = \frac{-dw + b}{cw - a} \quad (ad - bc \neq 0)$$

There is always a linear fractional transformation that maps three given distinct points z_1, z_2 and z_3 onto three specified distinct points w_1, w_2 and w_3, respectively.

Example

Find $w=\dfrac{az+b}{cz+d}$ such that $z_1=-1$, $z_2=0$ and $z_3=1$ are mapped onto three specified distinct points $w_1=-i$, $w_2=1$ and $w_3=i$.

The mapping of $z_2=0$ onto $w_2=1$ results in $d=b$ and then $w=\dfrac{az+b}{cz+b}$. Moreover, The mapping of $z_1=-1$ and $z_3=1$ onto $w_1=-i$ and $w_3=i$ leads to $ci-bi=-a+b$ and $ci+bi=a+b$. Hence, $a=ib$ and $c=-ib$. Therefore, $w=\dfrac{iz+1}{-iz+1}=\dfrac{-z+i}{z+i}$.

Example

Find $w=\dfrac{az+b}{cz+d}$ such that $z_1=1$, $z_2=0$ and $z_3=-1$ are mapped onto three specified distinct points $w_1=i$, $w_2=\infty$ and $w_3=1$.

The mapping of $z_2=0$ onto $w_2=\infty$ results in $d=0$, $b\neq 0$ and then $w=\dfrac{az+b}{cz}$. Moreover, The mapping of $z_1=1$ and $z_3=-1$ onto $w_1=i$ and $w_3=1$ leads to $ci=a+b$ and $-c=-a+b$. Hence, $a=(1+i)c/2$ and $b=(-1+i)c/2$. Therefore, $w=\dfrac{(1+i)z+(-1+i)}{2z}$.

To fulfill the linear fractional transformation, we can apply the following eqation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

since it maps three distinct points z_1, z_2 and z_3 onto three specified distinct points w_1, w_2 and w_3, respectively.

Example

Find $w=\dfrac{az+b}{cz+d}$ such that $z_1=-1$, $z_2=0$ and $z_3=1$ are mapped onto three specified distinct points $w_1=-i$, $w_2=1$ and $w_3=i$. From the formula, we have

$$\frac{(w+i)(1-i)}{(w-i)(1+i)}=\frac{(z+1)(0-1)}{(z-1)(0+1)}$$

i.e., $$w=\frac{-z+i}{z+i}.$$

A fixed point of a transformation $w=f(z)$ is a point z_0 such that $f(z_0)=z_0$. Every linear fractional transformation except $w=z+k$, where k is a constant complex number, has at most two fixed points in the extended plane.

The linear fractional transformation $w=\dfrac{az+b}{cz+d}$ $(ad-bc\neq 0)$ at the fixed point z_0 is

$$z_0=\frac{az_0+b}{cz_0+d}$$

For $c=0$, we have

$$z_0=\frac{az_0+b}{d}\ \ (ad\neq 0)$$

Clearly, if $a=d$, i.e., $w=z+k$, where $k=b/d$, then z_0 can be any complex number for $k=0$ and z_0 does not exist for $k\neq 0$, which has been ignored in the statement. For $c\neq 0$, we have

$$cz_0^2+(d-a)z_0-b=0$$

Clearly, we can find one or two values for z_0 for the second order equation of z_0. This completes the statement.

P19-1

Find the region onto which the half plane $y>0$ is mapped by the transformation $w=(1+i)z$ by using (a) polar coordinate and (b) rectangular coordinates. Sketch the region.

P19-2

Find the image of $x>1$, $y>0$ under the transformation $w=1/z$.

P19-3

Find the linear fractionaltransformation that maps the points $z_1=2$, $z_2=i$ and $z_3=-2$ onto the points $w_1=1$, $w_2=i$ and $w_3=-1$.

P19-4

Find the fixed points of the transformation

(a) $w=\dfrac{z-1}{z+1}$; (b) $w=\dfrac{6z-9}{z}$.

P19-5

Let $T(z)=\dfrac{az+b}{cz+d}$, where $ad-bc\neq 0$.

Show that $T^{-1}=T$ if and only if $d=-a$.