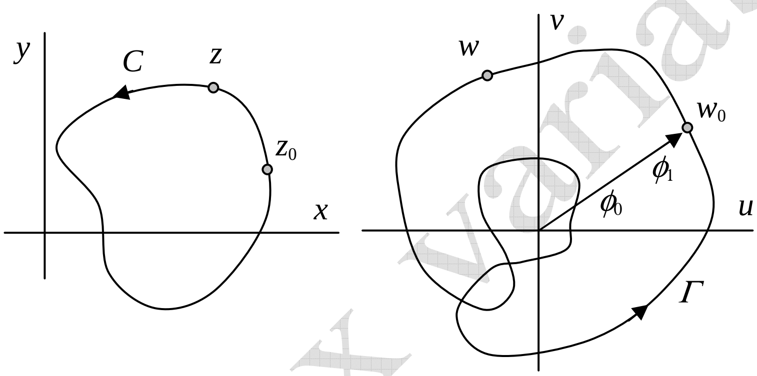
CV18 Argument Principle

A function *f* is said to be *meromorphic* in a domain *D* if it is analytic throughout *D* except for poles. The terminology comes from the ancient Greek "meros" meaning "part", as opposed to "holos", meaning "whole".

Suppose *f* is meromorphic in the domain interior to a positively oriented simple closed contour *C* and that is analytic and nonzero on *C*. The image Γ of *C* under the transformation $w=f(z)$ is a closed contour, not necessary simple, in the *w* plane. Since *f* has no zeros on *C*, the contour does not pass through the origin in the *w* plane.



Let *w* and w_0 be points on Γ, where w_0 is fixed and ϕ_0 is a value of $\arg w_0$. Then let $\arg w$ vary continuously, starting with ϕ_0, as the point *w* begins at the point w_0 and traverses Γ once in the direction of the orientation assigned to it by the mapping $w=f(z)$. When *w* returns to the point w_0, where it started, $\arg w$ assumes a particular value of $\arg w_0$, denoted by ϕ_1. Thus the change in $\arg w$ as *w* describes Γ once is $\phi_1-\phi_0$. We write

$$\Delta_C \arg f(z) = \phi_1 - \phi_0$$

which is an integral multiple of 2π and the integer, called the winding number of Γ,

$$\frac{1}{2\pi}\Delta_C \arg f(z) = \frac{\phi_1 - \phi_0}{2\pi}$$

represents the number of times the point *w* winds around the origin in the *w* plane.

The winding number is positive if Γ winds around the origin in the counterclockwise direction and negative if it winds clockwise. The winding number is zero when Γ does not enclose the origin.

***Theorem*: Argument principle**

Suppose that

(i) a function f is *meromorphic* in the domain interior to a positively oriented simple closed contour C;

(ii) $f(z)$ is analytic and nonzero on C;

(iii) counting multiplicities, Z is the number of zeros and P is the number of poles of $f(z)$ inside C.

Then $\frac{1}{2\pi}\Delta_C \, arg \; f(z) = Z - P$.

***Proof*:**

First, let's evaluate $\int_C \frac{f'(z)}{f(z)} dz$, where $z=z(t)$ ($a \le t \le b$) around C.

Hence,

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'(z(t))z'(t)}{f(z(t))} dt .$$

Under the transformation

$$w=f(z)=\rho(t)e^{i\phi(t)},$$

the image Γ of C never passes through the origin in the w plane.

Thus,

$$f'(z(t))z'(t) = \frac{d}{dt} f(z(t)) = \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t)$$

which leads to

$$\begin{aligned}\int_C \frac{f'(z(t))}{f(z(t))} dz &= \int_a^b \frac{\rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t)}{\rho(t)e^{i\phi(t)}} dt \\ &= \int_a^b \frac{\rho'(t)}{\rho(t)} dt + \int_a^b i\phi'(t)dt \\ &= ln\,\rho(t)\Big|_a^b + i\phi(t)\Big|_a^b \\ &= ln\,\rho(b) - ln\,\rho(a) + i(\phi(b) - \phi(a))\end{aligned}$$

Since $ln\,\rho(b) = ln\,\rho(a)$ and $\phi(b) - \phi(a) = \Delta_C \, arg \; f(z)$, we have

$$\int_C \frac{f'(z(t))}{f(z(t))}dz = i\Delta_C \, arg \; f(z)$$

Next let's evaluate $\int_C \frac{f'(z)}{f(z)}dz$ by Cauchy's residue theorem. If $f(z)$ has q zeros z_k with multiplicities m_k, k=1,2,…,q and p poles z_l with multiplicities n_l, l=1,2,…,p, that is,

$$f(z) = \frac{\prod_{k=1}^{q}(z-z_k)^{m_k}}{\prod_{l=1}^{q}(z-z_l)^{n_l}} g(z)$$

where $g(z)$ is analytic and nonzero in C. Clearly, the total number of zeros and poles are

$$Z=\sum_{k=1}^{q} m_k \quad \text{and} \quad P=\sum_{l=1}^{p} n_l \, .$$

Since

$$\begin{aligned} f'(z) = & \frac{\prod_{k=1}^{q}(z-z_k)^{m_k}}{\prod_{l=1}^{q}(z-z_l)^{n_l}} g'(z) \\ & + \sum_{s=1}^{q} \frac{m_s(z-z_s)^{m_s-1}\prod_{k=1,k\neq s}^{q}(z-z_k)^{m_k}}{\prod_{l=1}^{q}(z-z_l)^{n_l}} g(z) \\ & + \sum_{t=1}^{p} \frac{(-n_t)\prod_{k=1}^{q}(z-z_k)^{m_k}}{(z-z_t)^{n_t+1}\prod_{l=1,l\neq t}^{q}(z-z_l)^{n_l}} g(z) \end{aligned}$$

we have

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{s=1}^{q}\frac{m_s}{(z-z_s)} + \sum_{t=1}^{p}\frac{(-n_t)}{(z-z_t)}$$

According to the residue theorem,

$$\begin{aligned} \int_C \frac{f'(z)}{f(z)}dz &= 2\pi i\left(\sum_{s=1}^{q} m_s + \sum_{t=1}^{p}(-n_t)\right) \\ &= 2\pi i(Z-P) \end{aligned}$$

Hence, $i\Delta_C \, arg \; f(z) = 2\pi i(Z-P)$

that is,

$$\frac{1}{2\pi}\Delta_C \, arg \; f(z) = Z-P \, .$$

This completes the proof.

Example

The only singularity of $1/z^2$ is a pole of order 2 at the origin, and there are no zeros in the finite plane. Let C denote the positively oriented circle around the origin, then

$$\frac{1}{2\pi}\Delta_C\, arg\left(1/z^2\right)=-2\,.$$

That is, the image Γ of C winds around the origin w=0 twice in the clockwise direction. It can be verified directly by $w=e^{-i2\theta}$ $(0\le\theta\le 2\pi)$.

***Theorem*: (Rouche's Theorem)**

Suppose that

(i) two functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C;

(ii) $|f(z)|>|g(z)|$ at each point on C.

Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, inside C.

***Proof*:**

The orientation of C is immaterial, so we assume the orientation is positive. Since $|f(z)|>|g(z)|\ge 0$ and $|f(z)+g(z)|\ge||f(z)|-|g(z)||>0$ when z is on C, neither $f(z)$ nor $f(z)+g(z)$ has a zero on C. Let Z_f and Z_{f+g} denote the number of zeros of f and $f+g$ inside C, we know that

$$Z_f=\frac{1}{2\pi}\Delta_C\, arg\ f(z)$$

$$Z_{f+g}=\frac{1}{2\pi}\Delta_C\, arg\left[f(z)+g(z)\right]$$

Since

$$\begin{aligned}&\Delta_C\, arg\left[f(z)+g(z)\right]\\&\quad=\Delta_C\, arg\left[f(z)\left(1+\frac{g(z)}{f(z)}\right)\right]\\&\quad=\Delta_C\, arg\ f(z)+\Delta_C\, arg\left(1+\frac{g(z)}{f(z)}\right)\end{aligned}$$

it is clear that

$$Z_{f+g}=Z_f+\frac{1}{2\pi}\Delta_C\, arg\ F(z)$$

where $F(z)=1+\dfrac{g(z)}{f(z)}$. But, $|F(z)-1|=\dfrac{|g(z)|}{|f(z)|}<1$, which means under the transformation $w=F(z)$, the image of C lies in the open disk $|w-1|<1$. That image does not enclose the origin $w=0$ and then $\Delta_C \, arg\, F(z)=0$.

This completes the proof of $Z_{f+g}=Z_f$.

Example

To determine the number of roots of the equation

$$z^7-4z^3+z-1=0$$

inside the circle $|z|=1$, we write

$$f(z)=-4z^3 \quad \text{and} \quad g(z)=z^7+z-1$$

Since $|f(z)|=4$ and $|g(z)|\le|z|^7+|z|+1=3$ when $|z|=1$, the condition $|f(z)|>|g(z)|$ of the Rouché's Theorem is satisfied and then $f(z)$ and $f(z)+g(z)$ have the same number of zeros or roots. Consequently, there are three roots of $f(z)+g(z)=z^7-4z^3+z-1$ because $f(z)=-4z^3$ has three roots.

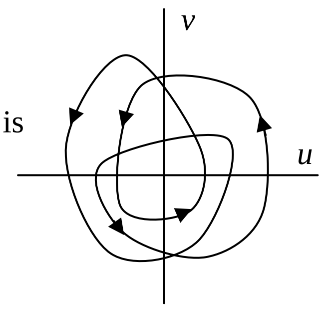
P18-1

Let C denote the unit circle $|z|=1$ in the positive sense. Determine $\Delta_C \, arg\, f(z)$ when

(a) $f(z)=z^2$; (b) $f(z)=(z^3+2)/z$; (c) $f(z)=(2z-1)^7/z^3$.

P18-2

Let f be analytic inside and on a simple closed contour C, and f is nonzero on C. The image of C is shown in the figure. Determine the value of $\Delta_C \, arg\, f(z)$ and the number of zeros, counting multiplicities, of f interior to C.



P18-3

Let f be analytic inside and on a simple closed contour C, and f has no zeros on C. Show that if f has n zeros z_k (k=1,2,…,n) inside C, each z_k

with multiplicity m_k, then

$$\int_C \frac{z\, f'(z)}{f(z)}\,dz = 2\pi i \sum_{k=1}^{n} m_k\, z_k\ .$$

P18-4

Determine the number of zeros, counting multiplicity, of

(a) z^4+3z^3+6, (b) $z^4-2z^3+9z^2+z-1$, and (c) $z^5+3z^3+z^2+1$

inside the circle $|z|=2$.

P18-5

Determine the number of roots, counting multiplicity, of the equation

$2z^5-6z^2+z+1=0$ in $1\le|z|<2$.