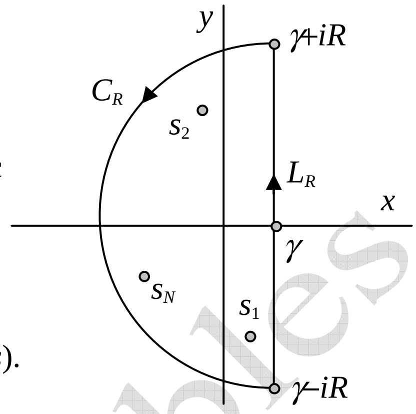
CV17 Residues: Inverse Laplace Transform

One of the important tools in engineering to solve the differential equations is the Laplace transform. For a function *f*(*t*), we define the Laplace transform as below:

$$F(s)=\int_0^{\infty} e^{-st} f(t)dt\ .$$



where the function *F*(*s*) is assumed to be analytic throughout the finite *s* plane except for a finite number of isolated singularities s_i, i=1,2,…,N.

Now, let's discuss how to find *f*(*t*) from *F*(*s*). First, check the following integration

$$g(t)=\frac{1}{2\pi i}\int_C e^{st}F(s)ds \quad (t>0)$$

where the contour *C*, including L_R and C_R as shown in the figure, encloses all the poles of *F*(*s*). Hence,

$$g(t)=\frac{1}{2\pi i}\left(\int_{L_R} e^{st}F(s)ds+\int_{C_R} e^{st}F(s)ds\right) \quad (t>0)$$

For *F*(*s*) on C_R, i.e., $s=\gamma+Re^{i\theta}$ and $\pi/2\le\theta\le3\pi/2$, it satisfies $|F(s)|< M_R$, where M_R tends to zero as *R* tends to infinite. Besides,

$$\int_{C_R} e^{st}F(s)ds=\int_{\pi/2}^{3\pi/2} e^{\gamma t+Rte^{i\theta}}F\left(\gamma+Re^{i\theta}\right)Rie^{i\theta}d\theta$$

Since $\left|e^{\gamma t+Rte^{i\theta}}\right|=e^{\gamma t}e^{Rt\cos\theta}$ and $\left|F\left(\gamma+Re^{i\theta}\right)\right|\le M_R$, we find that

$$\left|\int_{C_R} e^{st}F(s)ds\right|\le e^{\gamma t}M_R R\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta$$

$$=e^{\gamma t}M_R R\int_0^{\pi} e^{-Rt\sin\phi}d\phi< e^{\gamma t}M_R R\left(\frac{\pi}{Rt}\right)$$

$$=\frac{e^{\gamma t}M_R\pi}{t}$$

which leads to $\lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds=0$. As a result, we have

$$g(t)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)ds \quad (t>0)$$

which can be written as

$$g(t)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}\left(\int_0^{\infty} e^{-s\tau}f(\tau)d\tau\right)ds \quad (t>0)$$

On the contour L_R, i.e., $s=\gamma+i\omega$ and $ds=id\omega$ for $-\infty<\omega<\infty$, $g(t)$ is further rearranged as

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\gamma t}e^{i\omega t}\left(\int_0^{\infty} e^{-\gamma\tau}e^{-i\omega\tau} f(\tau)d\tau\right)d\omega \\ &= \int_0^{\infty} f(\tau)e^{\gamma(t-\tau)}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}d\omega\right)d\tau \end{aligned}$$

According to the Fourier transform of $x(t)$ and its inverse, we have the following expression

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$$

It is obvious that letting $x(t)=\delta(t)$ yields

$$X(\omega)=\int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt=\int_{-\infty}^{\infty} \delta(t)dt=1$$

and then

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega$$

It is clear that $g(t)$ can be changed into

$$g(t)=\int_0^{\infty} f(\tau)e^{\gamma(t-\tau)}\delta(t-\tau)d\tau$$

Due to the fact that

$$f(\tau)e^{\gamma(t-\tau)}\delta(t-\tau)=f(t)\delta(t-\tau)$$

we have

$$g(t)=\int_0^{\infty} f(t)\delta(t-\tau)d\tau=f(t)\int_0^{\infty}\delta(t-\tau)d\tau=f(t)$$

which leads to

$$f(t)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)ds \qquad (t>0)$$

To summarize, if the Laplace transform of $f(t)$ is

$$F(s)=\int_0^{\infty} e^{-st}f(t)dt$$

where all the poles of $F(s)$ are enclosed by the contour C, including L_R and C_R as shown in the figure, then $f(t)$ can be determined by the inverse Laplace transform

$$f(t)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)ds \qquad (t>0)$$

Actually, $f(t)$ can be also written as

$$f(t)=\frac{1}{2\pi i}\int_C e^{st}F(s)ds \qquad (t>0)$$

and obtained by the following formula

$$f(t)=\sum_{n=1}^{N}\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] \quad \text{for } t>0$$

where s_n , n=1,2,…,N, are the poles of $F(s)$.

Example

Suppose $F(s)$ has a pole of order m at a real point s_0 and its Laurent series representation in a punctured disk $0<|s-s_0|<R_2$ has principal part

$$F_{P_{s_0}}(s)=\frac{b_1}{s-s_0}+\frac{b_2}{(s-s_0)^2}+\cdots+\frac{b_m}{(s-s_0)^m} \quad (b_m\neq 0)$$

Then, $\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]=\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[e^{st}F_{P_{s0}}(s)(s-s_0)^m\right]_{s=s_0}$

For m=1,

$$\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]=\left[e^{st}b_1\right]_{s=s_0}=b_1e^{s_0t}$$

For m=2,

$$\begin{aligned}\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]&=\frac{d}{ds}\left[e^{st}\left(b_1(s-s_0)+b_2\right)\right]_{s=s_0}\\&=\left[e^{st}\left(t\left(b_1(s-s_0)+b_2\right)+b_1\right)\right]_{s=s_0}\\&=e^{s_0t}\left(b_2t+b_1\right)\end{aligned}$$

For m=3,

$$\begin{aligned}\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]&=\frac{1}{2}\frac{d^2}{ds^2}\left[e^{st}\left(b_1(s-s_0)^2+b_2(s-s_0)+b_3\right)\right]_{s=s_0}\\&=e^{s_0t}\left(\frac{1}{2}b_3t^2+b_2t+b_1\right)\end{aligned}$$

Therefore, we have

$$\begin{aligned}\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]&=\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[e^{st}F_{P_{s0}}(s)(s-s_0)^m\right]_{s=s_0}\\&=e^{s_0t}\left[b_1+\frac{b_2}{1!}t+\cdots+\frac{b_m}{(m-1)!}t^{m-1}\right]\end{aligned}$$

If $F(s)$ has a pole $s_0=\alpha+i\beta$ ($\beta\neq 0$) of order m and $\overline{F(s)}=F(\bar{s})$ at points of analyticity of $F(s)$, then $\bar{s}_0=\alpha-i\beta$ is also a pole of order m. We have

$$\underset{s=s_0}{Res}\left[e^{st}F(s)\right]+\underset{s=\bar{s}_0}{Res}\left[e^{st}F(s)\right]$$
$$=2e^{\alpha t}Re\left[e^{i\beta t}\left(b_1+\frac{b_2}{1!}t+\cdots+\frac{b_m}{(m-1)!}t^{m-1}\right)\right]$$

Example

Consider $F(s)=\dfrac{s}{\left(s^2+a^2\right)^2}$. If $g(t)=sin(at)$, then its Laplace transform $G(s)=\dfrac{a}{s^2+a^2}$. According to the property $\mathcal{L}\{t\,g(t)\}=-\dfrac{d}{ds}G(s)$, we have $\mathcal{L}\{t\,g(t)\}=\dfrac{2as}{\left(s^2+a^2\right)^2}=2aF(s)$ and then the inverse Laplace transform of $F(s)$ is

$$f(t)=\frac{t\,g(t)}{2a} \quad \text{or} \quad f(t)=\frac{1}{2a}t\,sin(at).$$

The above method has been widely used in courses required to solve inverse Laplace transform. We will apply the formula introduced in this section. First, there is a pole $s_0=ia$ of order 2, we have

$$\underset{s=ia}{Res}\left[e^{st}F(s)\right]=\frac{d}{ds}\left[e^{st}F(s)(s-ia)^2\right]_{s=ia}=-it\frac{e^{iat}}{4a}$$

Therefore,

$$f(t)=\underset{s=ia}{Res}\left[e^{st}F(s)\right]+\underset{s=-ia}{Res}\left[e^{st}F(s)\right]$$
$$=-it\frac{e^{iat}}{4a}+it\frac{e^{-iat}}{4a}=\frac{1}{2a}t\,sin(at)$$

Example

Consider $F(s)=\dfrac{tanhs}{s^2}=\dfrac{sinhs}{s^2coshs}$. Note that $F(s)$ has isolated singularities at $s=0$ and the zeros of $coshs$, i.e., $s_n=\dfrac{(2n-1)\pi}{2}i$ and $\bar{s}_n=-\dfrac{(2n-1)\pi}{2}i$, where $n=1, 2,\ldots$. Actually, the singularity $s=0$ is only a simple pole since the Laurent series of $F(s)$ is

$$F(s)=\frac{sinhs}{s^2coshs}=\frac{1}{s}-\frac{1}{3}s+\cdots \qquad \text{for } 0<|s|<\pi/2.$$

which also tells us that

$$\underset{s=0}{Res}\left[e^{st}F(s)\right]=1$$

Hence,

$$f(t)=\underset{s=0}{Res}\left[e^{st}F(s)\right]+\sum_{n=1}^{\infty}\left\{\underset{s=s_n}{Res}\left[e^{st}F(s)\right]+\underset{s=\bar{s}_n}{Res}\left[e^{st}F(s)\right]\right\}$$

To be specific, we write

$$e^{st}F(s)=\frac{p(s)}{q(s)}=\frac{e^{st}sinhs}{s^2coshs}$$

and observe that

$$p(s_n)=e^{s_nt}sinhs_n\neq 0,\ q(s_n)=0,$$
$$q'(s_n)=s_n^2sinhs_n\neq 0$$

which results in

$$\underset{s=s_n}{Res}\left[e^{st}F(s)\right]=\frac{p(s_n)}{q'(s_n)}=\frac{e^{s_nt}}{s_n^2}=-\frac{4e^{s_nt}}{\pi^2(2n-1)^2}$$

and then

$$\underset{s=s_n}{Res}\left[e^{st}F(s)\right]+\underset{s=\bar{s}_n}{Res}\left[e^{st}F(s)\right]=-\frac{4\left(e^{s_nt}+e^{\bar{s}_nt}\right)}{\pi^2(2n-1)^2}$$
$$=-\frac{8}{\pi^2(2n-1)^2}cos\frac{(2n-1)\pi t}{2}$$

Therefore,

$$f(t)=1-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}cos\frac{(2n-1)\pi t}{2} \qquad (t>0)$$

Example

Consider $F(s)=\dfrac{sinh\left(xs^{1/2}\right)}{s\ sinh\left(s^{1/2}\right)}$, $0<x<1$, where $s^{1/2}$ denotes any branch of this double-valued function. We agree to use the same branch in the numerator and denominator, so that $F(s)$ has singularities at s=0 and $s^{1/2}=\pm n\pi i$, n=1,2,…, which are located at

$$s=0 \quad \text{and} \quad s_n=-n^2\pi^2 \quad (n=1,2,...)$$

Hence,

$$f(t) = \operatorname*{Res}_{s=0}\left[e^{st}F(s)\right] + \sum_{n=1}^{\infty} \operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right]$$

Since

$$F(s) = \frac{xs^{1/2} + \left(xs^{1/2}\right)^3/3! + \cdots}{s\left[s^{1/2} + \left(s^{1/2}\right)^3/3! + \cdots\right]} = \frac{x + x^3 s/6 + \cdots}{s + s^2/6 + \cdots}$$

it is easy to check that $\operatorname*{Res}_{s=0}\left[e^{st}F(s)\right] = x$. As for the residue of $e^{st}F(s)$, we write

$$e^{st}F(s) = \frac{p(s)}{q(s)} = \frac{e^{st}\sinh\left(xs^{1/2}\right)}{s\sinh\left(s^{1/2}\right)}$$

and observe that

$$p(s_n) = e^{s_n t}\sinh(xs_n) \neq 0,\ \ q(s_n) = 0,$$

$$q'(s_n) = \frac{1}{2}s_n^{1/2}\cosh\left(s_n^{1/2}\right) \neq 0,$$

which results in

$$\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] = \frac{2}{\pi}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin(n\pi x)$$

Therefore,

$$f(t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin(n\pi x) \quad (t>0)$$

P17-1

Determine $f(t)$ when its Laplace transform is

(a) $F(s) = \dfrac{2s^3}{s^4 - 4}$; (b) $F(s) = \dfrac{2s - 2}{(s+1)(s^2 + 2s + 5)}$;

(c) $F(s) = \dfrac{12}{s^3 + 8}$; (d) $F(s) = \dfrac{s^2 - a^2}{\left(s^2 + a^2\right)^2}$.

P17-2

Determine $f(t)$ when its Laplace transform is

(a) $F(s) = \dfrac{\coth(\pi s/2)}{s^2 + 1}$; (b) $F(s) = \dfrac{1}{s^2} - \dfrac{1}{s\sinh s}$.