CV16 Residues: Special Integral methods

Here we will discuss some special integral methods which can solve the integral of real function.

Fourier Integrals

The Fourier integrals are often required to solve the the following forms

$$\int_{-\infty}^{\infty} f(x) sinax\, dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x) cosax\, dx$$

To obtain their results, we adopt the fact that

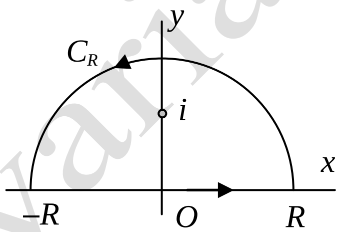
$$\int_{-R}^{R} f(x)e^{iax}dx = \int_{-R}^{R} f(x) cosax\, dx + i\int_{-R}^{R} f(x) sinax\, dx$$

and the fact that $\left|e^{iaz}\right| = \left|e^{ia(x+iy)}\right| = e^{-ay}$ is bounded for y≥0.

Example

Consider the integral $\int_{-\infty}^{\infty} \frac{cos\,3x}{\left(x^2+1\right)^2} dx$

and introduce the contour C. Then



$$\int_{-R}^{R} \frac{e^{i3x}}{\left(x^2+1\right)^2} dx + \int_{C_R} \frac{e^{i3z}}{\left(z^2+1\right)^2} dz = 2\pi i \mathop{Res}_{z=i}\left[\frac{e^{i3z}}{\left(z^2+1\right)^2}\right]$$

where $z=i$ is a pole of order 2.

Let $\frac{e^{i3z}}{\left(z^2+1\right)^2} = \frac{\phi(z)}{(z-i)^2}$ where $\phi(z) = \frac{e^{i3z}}{(z+i)^2}$, and then the residue at $z=i$ is calculated as

$$\mathop{Res}_{z=i}\left[\frac{e^{i3z}}{\left(z^2+1\right)^2}\right] = \phi'(i) = \left.\frac{\left(3i(z+i)-2\right)e^{i3z}}{(z+i)^3}\right|_{z=i} = \frac{1}{ie^3}$$

Besides,

$$\left|\int_{C_R} \frac{e^{i3z}}{\left(z^2+1\right)^2} dz\right| \le \frac{e^{-3y}}{\left(R^2-1\right)^2}\pi R \le \frac{\pi R}{\left(R^2-1\right)^2}$$

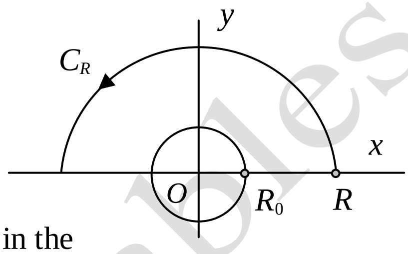
which implies $\int_{C_R} \frac{e^{i3z}}{\left(z^2+1\right)^2} dz = 0$. Therefore,

$$\int_{-R}^{R} \frac{e^{i3x}}{\left(x^2+1\right)^2} dx = 2\pi i \frac{1}{ie^3} = \frac{2\pi}{e^3}$$

Taking the real part leads to $\int_{-R}^{R}\frac{\cos 3x}{\left(x^2+1\right)^2}dx=\frac{2\pi}{e^3}$. Since $\frac{\cos 3x}{\left(x^2+1\right)^2}$ is even, the Cauchy principal value for the integral exists, i.e.,

$$\int_{-\infty}^{\infty}\frac{\cos 3x}{\left(x^2+1\right)^2}dx=2\pi i\frac{1}{ie^3}=\frac{2\pi}{e^3}.$$

Integration Based on Jordan's Lemma



***Jordan's Lemma*:**

Suppose that

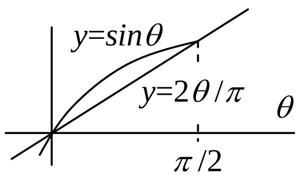
(i) a function $f(z)$ is analytic at all points z in the upper half plane $y\geq 0$ that are exterior to the circle $|z|=R_0$;

(ii) C_R denotes a semicircle $z=Re^{i\theta}$ $(0\leq\theta\leq\pi)$, where $R>R_0$;

(iii) for all points z on C_R, there is a positive constant M_R such that $|f(z)|< M_R$, where $\lim_{R\to\infty}M_R=0$.

Then, for every positive constant a,

$$\lim_{R\to\infty}\int_{C_R}f(z)e^{iaz}dz=0.$$



***Proof*:**

From the sine function, it is known that $\sin\theta>2\theta/\pi$ for $0\leq\theta\leq\pi/2$.

If $R>0$, then

$$e^{-R\sin\theta}\leq e^{-2R\theta/\pi},\qquad \text{when } 0\leq\theta\leq\pi/2.$$

This leads to

$$\int_0^{\pi/2}e^{-R\sin\theta}d\theta\leq\int_0^{\pi/2}e^{-2R\theta/\pi}d\theta=\frac{\pi}{2R}\left(1-e^{-R}\right)<\frac{\pi}{2R}$$

Since $\sin\theta$ is symmetric with respect to $\theta=\pi/2$, we have

$$\int_0^{\pi}e^{-R\sin\theta}d\theta<\frac{\pi}{R}$$

which is known as Jordan's inequality. According to the statements (i)-(iii), it can be attained that

$$\int_{C_R}f(z)e^{iaz}dz=\int_0^{\pi}f\left(Re^{i\theta}\right)e^{iaRe^{i\theta}}iRe^{i\theta}d\theta$$

Since

$$\left|f\left(Re^{i\theta}\right)\right| \le M_R$$

$$\left|e^{iaRe^{i\theta}}\right| = \left|e^{iaR(cos\theta+isin\theta)}\right| \le e^{-aRsin\theta},$$

we have

$$\left|\int_{C_R} f(z)e^{iaz}dz\right| \le \int_0^{\pi}\left|f\left(Re^{i\theta}\right)e^{iaRe^{i\theta}}iRe^{i\theta}\right|d\theta$$

$$\le M_R R\int_0^{\pi}\left|e^{iaRe^{i\theta}}\right|d\theta \le M_R R\int_0^{\pi} e^{-aRsin\theta}d\theta$$

$$= M_R R\frac{\pi}{aR} = \frac{M_R\pi}{a}$$

From $\lim\limits_{R\to\infty} M_R = 0$, we have $\lim\limits_{R\to\infty}\int_{C_R} f(z)e^{iaz}dz = 0$.

Example

Find the Cauchy principal value $\int_{-\infty}^{\infty}\frac{x\,sin\,x\,dx}{x^2+2x+2}$.

Let's consider

$$f(z) = \frac{z}{z^2+2z+2} = \frac{z}{(z-z_1)(z-\overline{z_1})}$$

where $z_1 = -1+i$ is a simple pole of $f(z)e^{iz}$ and lies above the real axis.

The residue of $f(z)e^{iz}$ at $z_1 = -1+i$ is

$$B_1 = \frac{z_1e^{iz_1}}{z_1-\overline{z_1}} = \frac{(cos1+sin1)}{2e} + i\frac{(cos1-sin1)}{2e}$$

Hence, when $R > \sqrt{2}$ and C_R denotes the upper half of the positively oriented circle $|z|=R$,

$$\int_{-R}^{R}\frac{x\,e^{ix}\,dx}{x^2+2x+2} + \int_{C_R} f(z)e^{iz}dz = 2\pi iB_1$$

which means

$$\int_{-R}^{R}\frac{x\,sinx\,dx}{x^2+2x+2} + Im\left(\int_{C_R} f(z)e^{iz}dz\right)$$

$$= Im(2\pi iB_1) = \frac{\pi(cos1+sin1)}{e}$$

Now,

$$\left|Im\left(\int_{C_R} f(z)e^{iz}dz\right)\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right|$$

where $|f(z)| \le M_R$, $M_R = \dfrac{R}{\left(R-\sqrt{2}\right)^2}$, and $\lim\limits_{R\to\infty} M_R = 0$.

From Jordan's Lemma,

we have $\lim\limits_{R\to\infty}\int_{C_R} f(z)e^{iz}dz = 0$ and thus

$$P.V.\int_{-\infty}^{\infty}\frac{x\,sinx\,dx}{x^2+2x+2} = \frac{\pi(cos1+sin1)}{e}.$$

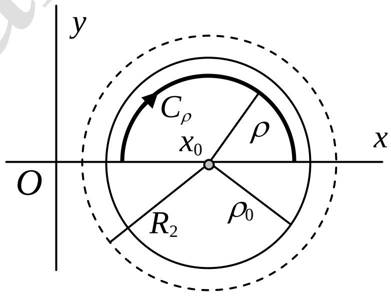
Integration of Indented Paths

Theorem

Suppose that

(i) $f(z)$ has a simple pole at $z=x_0$ on the real axis, with a Laurent series representation in a punctured disk $0<|z-x_0|<R_2$ and with residue B_0;

(ii) C_ρ denotes the upper half of a circle $|z-x_0|=\rho$, where $\rho<R_2$ and the the clockwise direction is taken.



Then $\lim\limits_{\rho\to 0}\int_{C_\rho} f(z)dz = -B_0\,\pi i$.

***Proof*:**

Assuming (i) and (ii) are satisfied, then the Laurent series is written as

$$f(z) = g(z) + \frac{B_0}{z-x_0}$$

where $g(z) = \sum_{n=0}^{\infty} a_n (z-x_0)^n$ for $|z-x_0|<R_2$. Thus,

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + B_0\int_{C_\rho}\frac{dz}{z-x_0}$$

If choose a number ρ_0 such that $\rho<\rho_0<R_2$, $g(z)$ must be bounded on the closed disk $|z-x_0|<\rho_0$, i.e., there is a nonnegative constant M such that

$$|g(z)| \le M \quad \text{whenever} \quad |z-x_0| \le \rho_0.$$

It follows that $\left|\int_{C_\rho} g(z)dz\right| \le M\,\pi\rho$ and consequently, $\lim\limits_{\rho\to 0}\int_{C_\rho} g(z)dz = 0$.

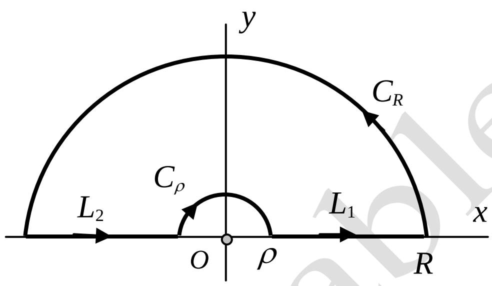
Besides,

$$\int_{C_\rho}\frac{dz}{z-x_0}=-\int_{-C_\rho}\frac{dz}{z-x_0}=-\int_0^{\pi}\frac{1}{\rho e^{i\theta}}\rho ie^{i\theta}d\theta=-i\pi$$

Thus, $\lim_{\rho\to 0}\int_{C_\rho} f(z)dz=-B_0\pi i$. This completes the proof.

Example

Determine $\int_0^{\infty}\frac{sinx}{x}dx$.



We consider

$$f(z)=\frac{e^{iz}}{z}$$

and the contour including C_R-L_2-C_ρ- L_1 as shown. The semicircle C_ρ is introduced to avoid integrating through the singularity z=0.

The Cauchy-Goursat theorem tells us that

$$\int_{L_1}\frac{e^{iz}}{z}dz+\int_{C_R}\frac{e^{iz}}{z}dz+\int_{L_2}\frac{e^{iz}}{z}dz+\int_{C_\rho}\frac{e^{iz}}{z}dz=0$$

Since $\frac{e^{iz}}{z}=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{i^n z^{n-1}}{n!}$ for 0<|z|<∞, its residue at z=0 is 1. Hence,

$$\lim_{\rho\to 0}\int_{C_\rho}\frac{e^{iz}}{z}dz=-\pi i\,.$$

Besides, $\left|\frac{1}{z}\right|=\frac{1}{R}$ on C_R. According to Jordan's lemma we have

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z}dz=0$$

Now,

$$\int_{L_1}\frac{e^{iz}}{z}dz+\int_{L_2}\frac{e^{iz}}{z}dz=\pi i$$

which can be further expressed as

$$\int_\rho^R\frac{e^{ir}}{r}dr+\int_{-R}^{-\rho}\frac{e^{ir}}{r}dr=\int_\rho^R\frac{e^{ir}}{r}dr-\int_\rho^R\frac{e^{-ir}}{r}dr=\pi i$$

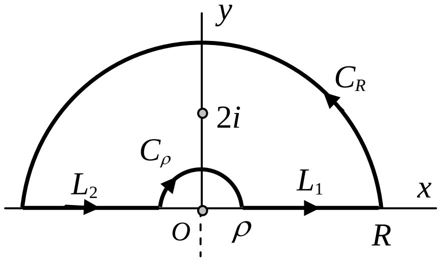
Thus,

$$\int_0^{\infty}\frac{sinr}{r}dr=\frac{\pi}{2}\;\Rightarrow\;\int_0^{\infty}\frac{sinx}{x}dx=\frac{\pi}{2}$$

Integration around Branch Point

Example

Determine $\int_0^\infty \frac{\ln x}{\left(x^2+4\right)^2}dx$.



We consider the branch

$$f(z)=\frac{\log z}{\left(z^2+4\right)^2}$$

for |z|>0, $-\pi/2<\arg z<3\pi/2$ and the contour including C_R-L_2-C_ρ-L_1 as shown. The semicircle C_ρ is introduced to avoid integrating through the singularity z=0 and the pole z=2i of order 2 is within the contour. According to Cauchy residue theorem,

$$\int_{L_1} f(z)dz+\int_{C_R} f(z)dz+\int_{L_2} f(z)dz+\int_{C_\rho} f(z)dz$$
$$=2\pi i \operatorname*{Res}_{z=2i} f(z)$$

where the residue is evaluated from the first derivative of

$$\phi(z)=(z-2i)^2\ f(z)=\frac{\log z}{(z+2i)^2}$$

at z=2i, that is,

$$\operatorname*{Res}_{z=2i} f(z)=\phi'(2i)=\frac{\pi}{64}+i\frac{1-\ln 2}{32}$$

Moreover,

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=2\int_\rho^R\frac{\ln r}{\left(r^2+4\right)^2}dr+i\pi\int_\rho^R\frac{dr}{\left(r^2+4\right)^2}$$

Taking the real parts yields

$$2\int_\rho^R\frac{\ln r}{\left(r^2+4\right)^2}dr+Re\int_{C_R} f(z)dz+Re\int_{C_\rho} f(z)dz=\frac{\pi}{16}(\ln 2-1)$$

Since

$$\left|Re\int_{C_R} f(z)dz\right|\le\left|\int_{C_R} f(z)dz\right|\le\frac{\ln R+\pi}{\left(R^2-4\right)^2}\pi R$$

and

$$\left|Re\int_{C_\rho} f(z)dz\right|\le\left|\int_{C_\rho} f(z)dz\right|\le\frac{-\ln\rho+\pi}{\left(4-\rho^2\right)^2}\pi\rho$$

where $\lim_{R\to\infty}\frac{\ln R+\pi}{\left(R^2-4\right)^2}\pi R=0$ and $\lim_{\rho\to 0}\frac{-\ln\rho+\pi}{\left(4-\rho^2\right)^2}\pi\rho=0$, we have

$$Re\int_{C_R} f(z)dz=0 \quad\text{and}\quad Re\int_{C_\rho} f(z)dz=0.$$

Therefore,

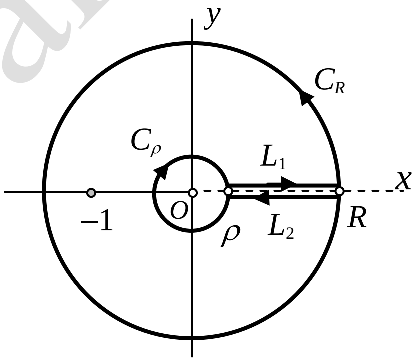
$$\int_0^\infty \frac{\ln r}{\left(r^2+4\right)^2}dr=\frac{\pi}{32}(\ln 2-1)$$

that is, $\int_0^\infty \frac{\ln x}{\left(x^2+4\right)^2}dx=\frac{\pi}{32}(\ln 2-1)$. Note that from the imaginary part of the above example, we can obtain

$$\int_0^\infty \frac{dx}{\left(x^2+4\right)^2}=\frac{\pi}{32}.$$

Example

Determine $\int_0^\infty \frac{x^{-a}}{x+1}dx$ where $0<a<1$.



Consider the branch $f(z)=\frac{z^{-a}}{z+1}$

for $|z|>0$ and $0<\arg z<2\pi$,

where z^{-a} is defined as $e^{-a\log z}$.

Consider the contour including C_R-L_2-C_ρ-L_1 as shown,

where $L_1(\theta=0)$ and $L_2(\theta=2\pi)$ are the upper and lower edges of the branch cut, respectively. The simple pole $z=-1$ is within the contour.

According to Cauchy residue theorem,

$$\int_{L_1} f(z)dz+\int_{C_R} f(z)dz+\int_{L_2} f(z)dz+\int_{C_\rho} f(z)dz=2\pi i\,\underset{z=-1}{Res}\, f(z)$$

where the residue at $z=-1$ is evaluated from

$$\phi(z)=(z+1)f(z)=z^{-a}=e^{-a\log z}=e^{-a(\ln r+i\theta)}$$

as $\underset{z=-1}{Res}\, f(z)=\phi(-1)=e^{-a(\ln 1+i\pi)}=e^{-ia\pi}\neq 0$. Moreover,

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=\int_\rho^R\frac{r^{-a}}{r+1}dr-\int_\rho^R\frac{r^{-a}e^{-i2a\pi}}{r+1}dr$$
$$=\left(1-e^{-i2a\pi}\right)\int_\rho^R\frac{r^{-a}}{r+1}dr$$

Hence,

$$\left(1-e^{-i2a\pi}\right)\int_{\rho}^{R}\frac{r^{-a}}{r+1}dr+\int_{C_R}f(z)dz+\int_{C_\rho}f(z)dz=2\pi ie^{-ia\pi}$$

Since

$$\left|\int_{C_R}f(z)dz\right|\le\frac{R^{-a}}{R-1}2\pi R=\frac{2\pi R}{R-1}\cdot\frac{1}{R^a}$$

$$\left|\int_{C_\rho}f(z)dz\right|\le\frac{\rho^{-a}}{1-\rho}2\pi\rho=\frac{2\pi}{1-\rho}\rho^{1-a}$$

where $\lim\limits_{R\to\infty}\frac{2\pi R}{R-1}\cdot\frac{1}{R^a}=0$ and $\lim\limits_{\rho\to 0}\frac{2\pi}{1-\rho}\rho^{1-a}=0$, we have

$\int_{C_R}f(z)dz=0$ and $\int_{C_\rho}f(z)dz=0$

Therefore,

$$\int_0^\infty\frac{r^{-a}}{r+1}dr=\frac{2\pi ie^{-ia\pi}}{1-e^{-i2a\pi}}=\frac{2\pi i}{e^{ia\pi}-e^{-ia\pi}}=\frac{\pi}{sin a\pi}$$

that is, $\int_0^\infty\frac{x^{-a}}{x+1}dr=\frac{\pi}{sin a\pi}\qquad(0<a<1)$.

Definite Integration involving Sines and Cosines

The method of residues is also useful in evaluating $\int_0^{2\pi}F(sin\theta,cos\theta)d\theta$.

By letting $z=e^{i\theta}\quad(0\le\theta\le 2\pi)$, we have

$$dz=ie^{i\theta}d\theta=iz\,d\theta$$

and $sin\theta=\frac{z-z^{-1}}{2i}$, $cos\theta=\frac{z+z^{-1}}{2}$, $d\theta=\frac{dz}{iz}$. Thus,

$$\int_C F\left(\frac{z-z^{-1}}{2i},\frac{z+z^{-1}}{2}\right)\frac{dz}{iz}$$

where C is the unit circle around the origin in the positive direction.

Example

To evaluate the integral

$$\int_0^{2\pi}\frac{d\theta}{1+a\,sin\theta}\qquad(-1<a<1)$$

we have

$$\int_C \frac{2/a}{z^2+(2i/a)z-1}dz = \int_C \frac{2/a}{(z-z_1)(z-z_2)}dz$$

where C is the positively oriented circle $|z|=1$ and the two poles are

$$z_1 = \left(\frac{-1+\sqrt{1-a^2}}{a}\right)i \quad \text{and} \quad z_2 = \left(\frac{-1-\sqrt{1-a^2}}{a}\right)i\,.$$

Note that because $|a|<1$ and $|z_1 z_2| = 1$, we have $|z_2| = \dfrac{1+\sqrt{1-a^2}}{|a|} > 1$ and

$|z_1| < 1$. That means no singular point is on C and the pole z_1 is in it.

The corresponding residue is obtained by

$$\mathop{Res}_{z=z_1} \frac{2/a}{(z-z_1)(z-z_2)} = \phi(z_1) = \left.\frac{2/a}{z-z_2}\right|_{z=z_1} = \frac{1}{i\sqrt{1-a^2}}$$

Consequently,

$$\int_C \frac{2/a}{z^2+(2i/a)z-1}dz = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$

that is,

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$$

P16-1

Evaluate the improper integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)}$ $(a>b>0)$; (b) $\displaystyle\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2}dx$ $(a>0, b>0)$;

(c) $\displaystyle\int_0^{\infty} \frac{x\sin 2x}{x^2+3}dx$; (d) $\displaystyle\int_0^{\infty} \frac{x^3 \sin x}{(x^2+1)(x^2+9)}dx$; (e) $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx$.

P16-2

Show that

$$\int_0^{\infty} \frac{\cos(ax)-\cos(bx)}{x^2}dx = \frac{\pi}{2}(b-a) \quad (a \ge 0, b \ge 0).$$

Point out how it follows that $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.

P16-3

Use $f(z)=\dfrac{z^{1/3}\log z}{z^2+1}=\dfrac{e^{(1/3)\log z}\log z}{z^2+1}$ for |z|>0, $-\pi/2<\arg z<3\pi/2$, to show that $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}dx=\frac{\pi^2}{6}$ and $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}dx=\frac{\pi}{\sqrt{3}}$

P16-4

Use $f(z)=\dfrac{(\log z)^2}{z^2+1}$ for |z|>0, $-\pi/2<\arg z<3\pi/2$, to show that

$\displaystyle\int_0^\infty \frac{(\ln x)^2}{x^2+1}dx=\frac{\pi^3}{8}$ and $\displaystyle\int_0^\infty \frac{\ln x}{x^2+1}dx=0$.

P16-5

Use $f(z)=\dfrac{z^{1/3}}{(z+a)(z+b)}=\dfrac{e^{(1/3)\log z}}{(z+a)(z+b)}$ for |z|>0, $0<\arg z<2\pi$ and the contour in this section to show that

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx=\frac{2\pi}{\sqrt{3}}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}\quad (a>b>0).$$

P16-6

Evaluate the definite integrals

(a) $\displaystyle\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}$; (b) $\displaystyle\int_0^{2\pi}\frac{d\theta}{1+a\cos\theta}$ $(-1<a<1)$;

(c) $\displaystyle\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}$ $(a>1)$; (d) $\displaystyle\int_0^{\pi}\sin^{2n}\theta\, d\theta$ $(n=1,2,\ldots)$.