CV15 Residues: *Evaluation of Improper Integrals*

In calculus, the improper integral of a continuous function $f(x)$ over the semi-infinite interval $x\geq 0$ is defined by means of the equation

$$\int_0^{\infty} f(x)dx = \lim_{R\to\infty}\int_0^{R} f(x)dx$$

which converges if the limit on the right exists.

If $f(x)$ is continuous for all x, its improper integral over $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx$$

which converges if both of the limits exist.

Cauchy principal value (P.V.) is defined as

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

which converges if the limit on the right exists.

If improper integral converges, then Cauchy principal value of the integral converges, i.e.,

$$\int_{-\infty}^{\infty} f(x)dx \text{ converges} \Rightarrow P.V.\int_{-\infty}^{\infty} f(x)dx \text{ converges}$$

However, it is not true that if Cauchy principal value of an integral converges, then its improper integral converges, i.e.,

$$P.V.\int_{-\infty}^{\infty} f(x)dx \text{ converges} \not\Rightarrow \int_{-\infty}^{\infty} f(x)dx \text{ converges}$$

Example

$$P.V.\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{-R}^{R} = 0$$

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty}\int_0^{R_2} x\,dx$$

$$= \lim_{R_1\to\infty}\left[\frac{x^2}{2}\right]_{R_1}^{0} + \lim_{R_2\to\infty}\left[\frac{x^2}{2}\right]_0^{R_2} = -\lim_{R_1\to\infty}\frac{R_1^2}{2} + \lim_{R_2\to\infty}\frac{R_2^2}{2}$$

Clearly,

$$P.V.\int_{-\infty}^{\infty} f(x)dx \text{ converges} \not\Rightarrow \int_{-\infty}^{\infty} f(x)dx \text{ converges}.$$

Suppose that $f(x)$ is an even function for all x, that is,

$$f(-x) = f(x) \qquad \text{for } -\infty < x < \infty .$$

Then, it is true that

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx .$$

Consider an even rational function

$$f(x)=f(-x)=p(x)/q(x),$$

where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. Besides, $q(z)$ has no real zeros but has zeros z_1, z_2, …, z_n, above the real axis. That means $f(z)$ has no real poles but has poles z_1, z_2, …, z_n, above the real axis. To evaluate $\int_{-\infty}^{\infty} f(x)dx$, choose a contour consisting of the segment form $z=-R$ to $z=R$ and the top half of the circle $|z|=R$, denoted as C_R, where R is large enough that all the zeros z_1, z_2, …, z_n, lie inside the closed path. According to Cauchy Residue Theorem,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z).$$

If $\qquad \lim_{R\to\infty}\int_{C_R} f(z)dz = 0 ,$

then

$$P.V.\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z).$$

Since $f(x)$ is even, we have

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z)$$

and

$$\int_{0}^{\infty} f(x)dx = \pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z).$$

Example

To evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx$, where $f(x)=\frac{x^2}{x^6+1}$ is even and $f(z)=\frac{z^2}{z^6+1}$ has three poles $z_1=e^{i\pi/6}$, $z_2=e^{i\pi/2}$, $z_3=e^{i5\pi/6}$, lying in the upper half plane. Hence,

$$\int_{-\infty}^{\infty} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{3} \underset{z=z_k}{Res}\, f(z) = 2\pi i (B_1 + B_2 + B_3).$$

Since the three poles are simple, we have

$$B_k = \underset{z=z_k}{Res} \frac{z^2}{z^6+1} = \left.\frac{z^2}{6z^5}\right|_{z=z_k} = \frac{1}{6} z_k^{-3},$$

i.e., $B_1 = \frac{1}{6} e^{-i\pi/2} = -\frac{1}{6} i$, $B_2 = \frac{1}{6} e^{-i3\pi/2} = \frac{1}{6} i$, $B_3 = \frac{1}{6} e^{-i5\pi/2} = -\frac{1}{6} i$.

Therefore,

$$\int_{-\infty}^{\infty} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \left(-\frac{1}{6} i\right) = \frac{\pi}{3}.$$

Moreover,

$$\left| \int_{C_R} f(z)dz \right| \le M_R L .$$

where $|f(z)| = \left| \frac{z^2}{z^6+1} \right| \le \frac{|z|^2}{|z|^6 - 1} \le \frac{R^2}{R^6-1} = M_R$ and $L=\pi R$. Thus,

$$\left| \int_{C_R} f(z)dz \right| \le \frac{\pi R^3}{R^6-1} \quad \text{for } R \to \infty .$$

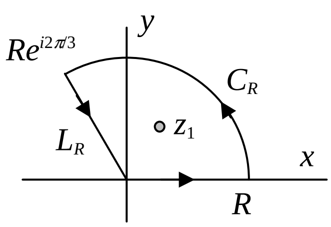
which implies $\int_{C_R} f(z)dz = 0$. This leads to

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{3} \quad \text{or} \quad \int_{0}^{\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{6}$$

Example

To evaluate $\int_0^{\infty} \frac{dx}{x^3+1}$, we choose the contour below.

Since $f(z) = \frac{1}{z^3+1}$ has one simple pole $z_1=e^{i\pi/3}$ lying in the contour, we have



$$\int_0^{\infty} f(x)dx + \int_{C_R} f(z)dz + \int_{L_R} f(z)dz$$
$$= 2\pi i \underset{z=z_1}{Res} \frac{1}{z^3+1} = 2\pi i \frac{1}{3z_1^2} = \frac{2\pi i}{3} e^{-i2\pi/3}.$$

where

$$\left| \int_{C_R} f(z)dz \right| \le \frac{1}{R^3-1} \cdot \frac{2\pi R}{3} = \frac{2\pi R}{3(R^3-1)}$$

$$\begin{aligned}\int_{L_R} f(z)dz &= \int_R^0 \frac{e^{i2\pi/3}}{\left(re^{i2\pi/3}\right)^3+1}dr \\ &= \int_R^0 \frac{dr}{r^3 e^{i4\pi/3}+e^{-i2\pi/3}} \\ &= \int_R^0 \frac{dr}{r^3 e^{-i2\pi/3}+e^{-i2\pi/3}} \\ &= \int_R^0 \frac{e^{i2\pi/3}dr}{r^3+1} = -e^{i2\pi/3}\int_0^R \frac{dx}{x^3+1}\end{aligned}.$$

Clearly, when $R\to\infty$, we have

$$\int_{C_R} f(z)dz = 0 \quad \text{and} \quad \int_{L_R} f(z)dz = -e^{i2\pi/3}\int_0^R \frac{dx}{x^3+1}$$

which result in

$$\int_0^\infty \frac{1}{x^3+1}dx - e^{i2\pi/3}\int_0^\infty \frac{1}{x^3+1}dx = \frac{2\pi i}{3}e^{-i2\pi/3}$$

Consequently,

$$\int_0^\infty \frac{1}{x^3+1}dx = \frac{2\pi i}{3\left(1-e^{i2\pi/3}\right)}e^{-i2\pi/3} = \frac{2\pi}{3\sqrt{3}}.$$

P15-1

Evaluate the improper integrals

(a) $\int_0^\infty \frac{1}{\left(x^2+1\right)^2}dx$; (b) $\int_0^\infty \frac{1}{x^4+1}dx$; (c) $\int_0^\infty \frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)}dx$.

P15-2

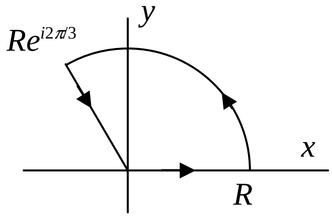
Evaluate the improper integrals

(a) $\int_{-\infty}^\infty \frac{1}{x^2+2x+2}dx$; (b) $\int_{-\infty}^\infty \frac{x}{\left(x^2+1\right)\left(x^2+2x+2\right)}dx$.

P15-3

Use residues and the contour with $R>1$ to show

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$



P15-4

Show that $\int_0^\infty \frac{x^{2m}}{x^{2n}+1}dx = \frac{\pi}{2n}csc\left(\frac{2m+1}{2n}\pi\right).$