CV14 Residues: Poles and Zeros

If $f(z)$ has an isolated singular point z_0, then it can be represented by a Laurent series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots$$

in a punctured disk $0<|z-z_0|<R_2$. The portion

$$\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots$$

is called the principal part of f at z_0.

If f contains finite terms of the principal part which is given as the following form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\cdots+\frac{b_m}{(z-z_0)^m}$$

for $0<|z-z_0|<R_2$, where $b_m\neq 0$, then the isolated singular point z_0 is called a pole of order m of f. A pole of order $m=1$ is referred to as a simple pole.

Example

The function $f(z)=\dfrac{z^2-2z+3}{z-2}=2+(z-2)+\dfrac{3}{z-2}$, $(0<|z-2|<\infty)$ has a simple pole at $z=2$. Its residue $b_1=3$.

Example

Consider the following function

$$f(z)=\frac{\sinh z}{z^4}=\frac{1}{z^4}\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots\right)$$
$$=\frac{1}{z^3}+\frac{1}{3!\,z}+\frac{z}{5!}+\frac{z^3}{7!}+\cdots \qquad (0<|z|<\infty)$$

Clearly, there is a pole of order 3 at $z=0$ with residue $b_1=1/6$.

If $f(z)$ has an isolated singular point z_0 and contains no principal part shown as $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ where $0<|z-z_0|<R_2$, then z_0 is known as a removable singular point. The residue at a removable singular point is zero. If we define $f(z_0)=a_0$, then the expression of $f(z)$ becomes valid

throughout the entire disk $|z-z_0|<R_2$. It follows that f is analytic at z_0 when it is assigned the value a_0 there. The singular point z_0 is removed.

Example

Consider the following function

$$f(z)=\frac{1-\cos z}{z^2}=\frac{1}{z^2}\left(1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)\right)$$
$$=\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots \qquad (0<|z|<\infty)$$

If $f(0)=1/2$ is assigned, then f becomes entire.

If f contains an infinite number of terms of the principal part as shown below:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots$$

then z_0 is called an essential singular point.

An important result concerning an essential singular point is due to Picard. It states that in each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times.

Example

Consider the following function

$$e^{1/z}=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{1}{z^n}$$
$$=1+\frac{1}{1!}\cdot\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\cdots \quad (0<|z|<\infty)$$

has an essential singular point $z=0$ with residue $b_1=1$. To illustrate Picard's theorem, recall that $e^z=-1$ when $z=(2n+1)\pi i$ $(n\in Z)$.

That means $e^{1/z}=-1$ when

$$z=\frac{1}{(2n+1)\pi i}=-\frac{i}{(2n+1)\pi} \qquad (n=0,\pm1,\pm2,\cdots)$$

and an infinite number of these points lie in any given neighborhood of the origin. Since $e^{1/z} \neq 0$ for any value of z, zero is the exceptional value in Picard's theorem.

***Theorem*:**

An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written as $f(z)=\dfrac{\phi(z)}{(z-z_0)^m}$, where $\phi(z)$ is analytic and $\phi(z_0)\neq 0$. Moreover, $\operatorname*{Res}_{z=z_0} f(z)=\dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ $\qquad m=1,2,\ldots$.

Since $\phi(z)$ is analytic and $\phi(z_0)\neq 0$, its Taylor expansion is

$$\begin{aligned}\phi(z)&=\phi(z_0)+\frac{\phi'(z_0)}{1!}(z-z_0)+\frac{\phi''(z_0)}{2!}(z-z_0)^2+\cdots\\&\quad+\frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1}+\sum_{n=m}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n\end{aligned}$$

in some neighborhood $|z-z_0|<\varepsilon$ of z_0. It follows that

$$\begin{aligned}f(z)&=\frac{\phi(z)}{(z-z_0)^m}\\&=\frac{\phi(z_0)}{(z-z_0)^m}+\frac{\phi'(z_0)}{1!(z-z_0)^{m-1}}+\frac{\phi''(z_0)}{2!(z-z_0)^{m-2}}+\cdots\\&\quad+\frac{\phi^{(m-1)}(z_0)}{(m-1)!(z-z_0)}+\sum_{n=m}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}\end{aligned}$$

which reveals that z_0 is a pole of order m of $f(z)$. The coefficient of $1/(z-z_0)$ tells us that the residue is $\operatorname*{Res}_{z=z_0} f(z)=\dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ where $m=1,2,\ldots$.

Example

The function $f(z)=\dfrac{z+1}{z^2+9}$ has an isolated singular point $z=3i$. Then, it can be rewritten as $f(z)=\dfrac{\phi(z)}{z-3i}$ where $\phi(z)=\dfrac{z+1}{z+3i}$. Since $\phi(z)$ is analytic at $z=3i$ and $\phi(3i)=\dfrac{1}{6}(3-i)\neq 0$. Hence,

$$\operatorname*{Res}_{z=3i} f(z)=\left.\frac{\phi^{(m-1)}(3i)}{(m-1)!}\right|_{m=1}=\phi(3i)=\frac{3-i}{6}.$$

Example

Since $f(z)=\dfrac{z^3+2z}{(z-i)^3}$ has a pole of order 3 at $z=i$, it can be written as $f(z)=\dfrac{\phi(z)}{(z-i)^3}$, where $\phi(z)=z^3+2z$ is entire and $\phi(i)=i\neq 0$. Hence, the residue at $z=i$ of $f(z)$ is

$$\operatorname*{Res}_{z=i} f(z)=\left.\frac{\phi^{(m-1)}(i)}{(m-1)!}\right|_{m=3}=\frac{\phi''(i)}{2}=3i\,.$$

Example

Consider $f(z)=\dfrac{(log\ z)^3}{z^2+1}$ with the branch $log\ z=lnr+i\theta$ for $0<\theta<2\pi$ is to be used. Since $f(z)$ has a pole at $z=i$, it can be written as $f(z)=\dfrac{\phi(z)}{z-i}$, where $\phi(z)=\dfrac{(log\ z)^3}{z+i}$ is analytic at $z=i$ and $\phi(i)=\dfrac{(log\ i)^3}{2i}=-\dfrac{\pi^3}{16}\neq 0$.

Hence, the residue at $z=i$ of $f(z)$ is

$$\operatorname*{Res}_{z=i} f(z)=\left.\frac{\phi^{(m-1)}(i)}{(m-1)!}\right|_{m=1}=\phi(i)=-\frac{\pi^3}{16}\,.$$

Example

$f(z)=\dfrac{sinh\ z}{z^4}$ has a pole of order 3 at $z=0$ since

$$f(z)=\frac{sinh\ z}{z^4}=\frac{1}{z^4}\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots\right)$$
$$=\frac{1}{z^3}+\frac{1}{3!\,z}+\frac{z}{5!}+\frac{z^3}{7!}+\cdots$$

Therefore, f(z) has a pole of order 3 at $z=0$ and its residue $b_1=1/6$. You can not choose $\phi(z)=sinh\ z$, which is wrong in this case.

Example

Consider $f(z)=\dfrac{1}{z(e^z-1)}=\dfrac{1}{z\left(\sum_{n=1}^{\infty}\dfrac{z^n}{n!}\right)}=\dfrac{1}{z^2\left(1+\dfrac{z}{2!}+\dfrac{z^2}{3!}+\cdots\right)}$ which has a pole at z=0 of order 2. Let $f(z)=\dfrac{\phi(z)}{z^2}$, then $\phi(z)=\dfrac{z}{e^z-1}$ is analytic at z=0 and $\phi(0)=1\neq 0$. Hence, the residue at z=0 of $f(z)$ is

$$\underset{z=0}{Res}\ f(z)=\left.\frac{\phi^{(m-1)}(0)}{(m-1)!}\right|_{m=2}=\phi'(0)=\left.\frac{(e^z-1)-z(e^z)}{(e^z-1)^2}\right|_{z=0}.$$

$$=\left.\frac{-z}{2(e^z-1)}\right|_{z=0}=\left.\frac{-1}{2(e^z)}\right|_{z=0}=-\frac{1}{2}$$

If $f(z_0)$=0 and if there is a positive integer m such that $f^{(m)}(z_0)\neq 0$ and each derivative of lower order vanishes at z_0, then f has a zero of order m at z_0.

***Theorem*:**

A function f that is analytic at a point z_0 has a zero of order m there if and only if there is a function g, which is analytic at z_0 and $g(z_0)\neq 0$, such that $f(z)=(z-z_0)^m\, g(z)$.

***Theorem*:**

Suppose that

(i) two functions p and q are analytic at z_0,

(ii) $p(z_0)\neq 0$ and q has a zero of order m at z_0.

Then $p(z)/q(z)$ has a pole of order m at z_0.

Example

Let $p(z)$=1 and $q(z)=z(e^z-1)$, where $q(z)$ has a zero of order 2 at z=0. Hence, the quotient $p(z)/q(z)$ has a pole of order 2 at z=0.

***Theorem*:**

Let p and q be analytic at z_0. If $p(z_0)\neq 0$, $q(z_0)=0$, and $q'(z_0)\neq 0$, then z_0 is a simple pole of $p(z)/q(z)$ and $\underset{z=z_0}{Res}\dfrac{p(z)}{q(z)}=\dfrac{p(z_0)}{q'(z_0)}$.

Proof:

From the conditions $q(z_0)=0$, and $q'(z_0)\neq 0$, the point z_0 is a simple zero. That means $q(z)=(z-z_0)g(z)$, where $g(z)$ is analytic at z_0 and $g(0)\neq 0$. Therefore, z_0 is a simple pole of $p(z)/q(z)$ and $\dfrac{p(z)}{q(z)}=\dfrac{p(z)/g(z)}{z-z_0}=\dfrac{\phi(z)}{z-z_0}$

where $\phi(z)=\dfrac{p(z)}{g(z)}$. Hence, $\operatorname*{Res}_{z=z_0}\dfrac{p(z)}{q(z)}=\phi(z_0)=\dfrac{p(z_0)}{g(z_0)}$.

Note that $q'(z)=g(z)+(z-z_0)g'(z)$ and $\operatorname*{Res}_{z=z_0}\dfrac{p(z)}{q(z)}=\dfrac{p(z_0)}{q'(z_0)}=\dfrac{p(z_0)}{g(z_0)}$.

Example

Consider $f(z)=\cot z=\cos z/\sin z$, which is a quotient $p(z)/q(z)$ of $p(z)=\cos z$ and $q(z)=\sin z$. The singularities occur at the zeros $z=n\pi$ (n=0, 1,±2,…). Since $p(n\pi)=(-1)^n\neq 0$ and $q(n\pi)=0$, and $q'(n\pi)=(-1)^n\neq 0$, each singular point $z=n\pi$ of f is a simple pole, with residue $\operatorname*{Res}_{z=n\pi}\dfrac{p(z)}{q(z)}=\dfrac{p(n\pi)}{q'(n\pi)}=1$.

Example

Find the residue of $f(z)=\tanh z/(z^2)=\sinh z/(z^2\cosh z)$ at the zero $z=\pi i/2$ of $\cosh z$. It is a quotient $p(z)/q(z)$ with $p(z)=\sinh z$ and $q(z)=z^2\cosh z$. Since

$p(\pi i/2)=\sinh(\pi i/2)=i\sin(\pi/2)=i\neq 0$

$q(\pi i/2)=0$

$q'(\pi i/2)=(\pi i/2)^2\sinh(\pi i/2)=-\pi^2 i/4\neq 0$,

we know that $z=\pi i/2$ is a simple pole of f and the residue can be determined as $\operatorname*{Res}_{z=\pi i/2}\dfrac{p(z)}{q(z)}=\dfrac{p(\pi i/2)}{q'(\pi i/2)}=-\dfrac{4}{\pi^2}$.

Example

Find the residue of $f(z)=z/(z^4+4)$ at the isolated singular point $z=1+i$. It is a quotient $p(z)/q(z)$ with $p(z)=z$ and $q(z)=z^4+4$. Since $p(1+i)=1+i\neq 0$ and $q(1+i)=0$, and $q'(1+i)=4(1+i)^3\neq 0$, we know that $z=1+i$ is a simple pole of f with residue $\operatorname*{Res}_{z=1+i}\dfrac{p(z)}{q(z)}=\dfrac{p(1+i)}{q'(1+i)}=\dfrac{1}{4(1+i)^2}=\dfrac{1}{8i}=-\dfrac{i}{8}$.

***Theorem*:**

If z_0 is a pole of a function f, then $\lim_{z \to z_0} f(z) = \infty$.

***Theorem*:**

If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0<|z-z_0|<\varepsilon$ of z_0.

***Lemma*:**

Suppose that a function f is analytic and bounded in some deleted neighborhood $0<|z-z_0|<\varepsilon$ of z_0. If f is not analytic at z_0, then it has a removable singularity there.

***Casorati-Weierstrass Theorem*:**

Suppose that z_0 is an essential singularity of a function f, and let ω_0 be any complex number. Then, for any positive number ε, the inequality $|f(z)-\omega_0|<\varepsilon$ is satisfied at some point z in each deleted neighborhood $0<|z-z_0|<\delta$ of z_0.

It states that, in each deleted neighborhood of an essential singular point, f assumes values arbitrarily close to any given number.

P14-1

In each case, write the principal part at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z\,exp\left(\frac{1}{z}\right)$; (b) $\frac{z^2}{1+z}$; (c) $\frac{sin\,z}{z}$; (d) $\frac{cos\,z}{z}$ (e) $\frac{1}{(2-z)^3}$

P14-2

Show that the singular poiont of each of the following function is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\frac{1-cosh\,z}{z^3}$; (b) $\frac{1-exp(2z)}{z^4}$; (c) $\frac{exp(2z)}{(z-1)^2}$.

P14-3

Find the value of $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$ taken counterclockwise around (a) $|z-2|=2$; (b) $|z|=4$.

P14-4

Show that

(a) $\underset{z=-1}{Res}\frac{z^{1/4}}{z+1}=\frac{1+i}{\sqrt{2}} \quad (|z|>0, 0<arg\, z<2\pi)$;

(b) $\underset{z=i}{Res}\frac{Log\, z}{(z^2+1)^2}=\frac{\pi+2i}{8}$;

(c) $\underset{z=i}{Res}\frac{z^{1/2}}{(z^2+1)^2}=\frac{1-i}{8\sqrt{2}} \quad (|z|>0, 0<arg\, z<2\pi)$.

P14-5

Show that

(a) $\underset{z=\pi i}{Res}\frac{z-sinh\, z}{z^2\, sinh\, z}=\frac{i}{\pi}$;

(b) $\underset{z=\pi i}{Res}\frac{exp(zt)}{sinh\, z}+\underset{z=-\pi i}{Res}\frac{exp(zt)}{sinh\, z}=-2\cos \pi t$.

P14-6

Let C denote the positively oriented ciecle $|z|=2$ and evaluate the integral

(a) $\int_C tan\, z\, dz$; (b) $\int_C \frac{dz}{sinh\, 2z}$.

P14-7

Consider $f(z)=\frac{1}{[q(z)]^2}$ where q is analytic at z_0, $q(z_0)=0$, and $q'(z_0)\neq 0$.

Show that z_0 is a pole of order $m=2$, with residue

$$B_0=-\frac{q''(z_0)}{[q'(z_0)]^3}$$

Use the above result to find the residue at $z=0$ of the function

(a) $f(z)=csc^2 z$; (b) $f(z)=\frac{1}{(z+z^2)^2}$.