CV13 Residues: *Cauchy's Residue Theorem*

A point z_0 is called a singular point of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. A singular point is said to be isolated if there is a deleted neighborhood $0<|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic.

Example

The function $\dfrac{z+1}{z^3(z^2+1)}$ has three isolated singular points $z=0$, $\pm i$.

Example

The origin is a singular point of the principle branch

$$Log\, z = ln\, r + i\,\Theta \quad (r>0, -\pi<\Theta<\pi).$$

It is not an isolated singular point since every deleted ε neighborhood of it contains points on the negative real axis and the branch is not even defined there, i.e., not analytic throughout the deleted ε neighborhood.

Example

The function $\dfrac{1}{sin(\pi/z)}$ has the singular points $z=0, \pm 1, \pm\dfrac{1}{2}, \pm\dfrac{1}{3}, \cdots$, all lying on the segment of the real axis form $z=-1$ to $z=1$. Each singular point except $z=0$ is isolated. The singular point $z=0$ is not isolated because every deleted neighborhood of the origin contains other singular points.

If z_0 is an isolated singular point and $f(z)$ is analytic for $0<|z-z_0|<R_2$, then f is represented by a Laurent series as

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$$

with C being any positively oriented simple closed contour around z_0 and lying in the disk $0<|z-z_0|<R_2$.

When n=1, the complex number

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz$$

is called the ***residue*** of f at ***the isolated singular point*** z_0. We often use the notation

$$\underset{z=z_0}{Res} f(z) = \frac{1}{2\pi i}\int_C f(z)dz$$

or simply B when the point z_0 and the function f are clearly indicated.

Example

Consider the integral $\int_C \frac{dz}{z(z-2)^4}$ where C is the positively oriented circle $|z-2|=1$. Since

$$f(z) = \frac{1}{z(z-2)^4}$$

is analytic everywhere in the finite plane except at the points z=0 and z=2, it has a Laurent series representation valid in the punctured disk $0<|z-2|<2$, i.e., $\left|\frac{z-2}{2}\right|<1$. Hence, the Laurent series is

$$f(z) = \frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4}\cdot\frac{1}{2+(z-2)}$$
$$= \frac{1}{2(z-2)^4}\cdot\frac{1}{1-\left(-\frac{z-2}{2}\right)} = \frac{1}{2(z-2)^4}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-4}$$

The coefficient b_1 of $1/(z-2)$ is the desired residue. Therefore, when n=3, we have $b_1=-1/16$. Consequently,

$$b_1 = \frac{1}{2\pi i}\int_C \frac{dz}{z(z-2)^4} = \frac{-1}{16} \quad \text{or} \quad \int_C \frac{dz}{z(z-2)^4} = -\frac{\pi i}{8}$$

Example

Consider $\int_C e^{1/z^2}dz$, where C is the positively oriented circle around the origin. Since the integrand $f(z)=e^{1/z^2}$ is analytic everywhere except at the points z=0, it has a Laurent series representation valid in the punctured disk $0<|z|<\infty$. Hence, the Laurent series is

$$f(z)=e^{1/z^2}=\sum_{n=0}^{\infty}\frac{(1/z^2)^n}{n!}$$
$$=1+\frac{1}{1!}\cdot\frac{1}{z^2}+\frac{1}{2!}\cdot\frac{1}{z^4}+\frac{1}{3!}\cdot\frac{1}{z^6}+\cdots$$

The coefficient b_1 of $1/z$ is the desired residue. Therefore, $b_1=0$ and $\int_C e^{1/z^2}dz=0$. Although $\int_C e^{1/z^2}dz=0$, the function e^{1/z^2} is not necessary to be analytic throughout the simple closed contour C.

***<u>Cauchy's Residue Theorem</u>*:**

Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k (k=1,2,…,n) inside C, then $\int_C f(z)dz=2\pi i\sum_{k=1}^{n}\underset{z=z_k}{Res}\,f(z)$

<u>Proof</u>

Let the point z_k be centered of positively oriented circle C_k which are interior to C and so small that no two of them have points in common. According to Cauchy-Goursat theorem, we have

$$\int_C f(z)dz-\sum_{k=1}^{n}\int_{C_k}f(z)dz=0$$

Since $\int_{C_k}f(z)dz=2\pi i\,\underset{z=z_k}{Res}\,f(z)$, we have

$$\int_C f(z)dz=2\pi i\sum_{k=1}^{n}\underset{z=z_k}{Res}\,f(z).$$

Example

Consider the integral $\int_C\frac{5z-2}{z(z-1)}dz$, where C is the circle $|z|=2$, described counterclockwise. The integrand has two isolated singularities $z=0$ and $z=1$, both interior to C. Choose C_1 and C_2 to be $|z|=1$ and $|z-1|=1$ for $z=0$ and $z=1$, respectively. Since

$$\frac{5z-2}{z(z-1)}=\left(5-\frac{2}{z}\right)\left(\frac{-1}{1-z}\right)=\left(5-\frac{2}{z}\right)\left(-1-z-z^2-\cdots\right)$$
$$=\frac{2}{z}-3-3z-3z^2-3z^3-\cdots\quad(0<|z|<1)$$

we have B_1=2 and

$$\int_{C_1} \frac{5z-2}{z(z-1)}dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{5z-2}{z(z-1)}\right) = 4\pi i\,.$$

Since

$$\begin{aligned}\frac{5z-2}{z(z-1)} &= \frac{5(z-1)+3}{(1+(z-1))(z-1)}\\ &= \left(5+\frac{3}{z-1}\right)\left(\frac{1}{1+(z-1)}\right)\\ &= \left(5+\frac{3}{z-1}\right)\left(1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right)\\ &= \frac{3}{z-1}+2-2(z-1)+2(z-1)^2-2(z-1)^3+\cdots\end{aligned}$$

where $0<|z-1|<1$, we have B_2=3 and

$$\int_{C_2} \frac{5z-2}{z(z-1)}dz = 2\pi i \operatorname*{Res}_{z=1}\left(\frac{5z-2}{z(z-1)}\right) = 6\pi i\,.$$

Thus, $$\int_{C} \frac{5z-2}{z(z-1)}dz = 2\pi i(B_1+B_2) = 10\pi i$$

Note that in this example, it is easy to write

$$\frac{5z-2}{z(z-1)} = \frac{2}{z}+\frac{3}{z-1} = \frac{B_1}{z}+\frac{B_2}{z-1}$$

and obtain the result

$$\int_{C} \frac{5z-2}{z(z-1)}dz = 2\pi i(B_1+B_2) = 10\pi i\,.$$

***<u>Theorem</u>*:**

If a function f is analytic everywhere in the finite plane except for a finite number singular points interior to a positively oriented simple closed contour C, then

$$\int_{C} f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

***<u>Proof</u>*:**

Construct a circle $|z|=R_1$ which is large enough so that the contour C is interior to it. If C_0 denotes a positively oriented circle $|z|= R_0$, where $R_0> R_1$, then from Laurent's theorem we have

$$f(z)=\sum_{n=-\infty}^{\infty} c_n z^n \qquad (R_1<|z|<\infty)$$

where $c_n=\dfrac{1}{2\pi i}\displaystyle\int_{C_0}\frac{f(z)dz}{z^{n+1}} \qquad (n=0,\pm 1,\pm 2,\cdots).$

By writing $n=-1$, we find that

$$\int_{C_0} f(z)dz = 2\pi i\, c_{-1}$$

Note that c_{-1} is not the residue of f at $z=0$ since it is valid for $R_1<|z|<\infty$, not the type of $0<|z|<R_2$, and $z=0$ may not even be a singular point of f. However, if we replace z by $1/z$, we see that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right)=\sum_{n=-\infty}^{\infty} c_n\left(\frac{1}{z^{n+2}}\right) \qquad \left(0<|z|<\frac{1}{R_1}\right)$$

Clearly,

$$c_{-1}=\underset{z=0}{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Since f is analytic throughout the closed region bounded by C and C_0, we have

$$\int_C f(z)dz=\int_{C_0} f(z)dz = 2\pi i\,\underset{z=0}{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

This completes the proof.

Example

Consider the integral $\displaystyle\int_C \frac{5z-2}{z(z-1)}dz$, where C is the circle $|z|=2$, described counterclockwise. The integrand $f(z)=\dfrac{5z-2}{z(z-1)}$ has two isolated singularities $z=0$ and $z=1$, both interior to C.

Hence,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right)=\frac{5-2z}{z(1-z)}=\frac{5}{z}+\frac{3}{1-z}$$

The residue $\underset{z=0}{Res}\left[\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right)\right]=5$. Then $\displaystyle\int_C \frac{5z-2}{z(z-1)}dz=10\pi i$.

P13-1

Find the residue at z=0 of the function

(a) $\frac{1}{z+z^2}$; (b) $z\cos\left(\frac{1}{z}\right)$; (c) $\frac{z-\sin z}{z}$; (d) $\frac{\cot z}{z^4}$; (e) $\frac{\sinh z}{z^4\left(1-z^2\right)}$

P13-2

Evaluate the integral around |z|=3 in the positive sense:

(a) $\frac{\exp(-z)}{z^2}$; (b) $\frac{\exp(-z)}{(z-1)^2}$; (c) $z^2\exp\left(\frac{1}{z}\right)$; (d) $\frac{z+1}{z^2-2z}$

P13-3

Use the theorem involving a single residue to evaluate the integral around |z|=2 in the positive sense:

(a) $\frac{z^5}{1-z^3}$; (b) $\frac{1}{1+z^2}$; (c) $\frac{1}{z}$.

P13-4

Let

$$P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n \quad (a_n\neq 0)$$
$$Q(z)=b_0+b_1z+b_2z^2+\cdots+b_mz^m \quad (b_m\neq 0)$$

and $m\geq n+2$. Use the theorem involving a single residue to show that if all the zeros of $Q(z)$ are interior to a simple closed contour C, then $\int_C \frac{P(z)}{Q(z)}dz$.