CV12 Series

An infinite sequence $z_1, z_2, \ldots, z_n, \ldots,$ of complex numbers has a limit z if, for each positive number ε, there exists a positive integer n_0 such that

$$|z_n - z| < \varepsilon \qquad \text{whenever} \quad n > n_0$$

The value of n_0 will depend on the value of ε.

An infinite sequence can have at most one limit, i.e., a limit z is unique if it exists. When the limit exists, the sequence converges to z and expressed as

$$\lim_{n\to\infty} z_n = z$$

If the sequence has no limit, it diverges.

***Theorem*:**

If $z_n = x_n + i\, y_n$ (n=1,2,…) and $z=x+i\,y$, then

$$\lim_{n\to\infty} z_n = z$$

if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Example

$z_n = \frac{1}{n^3} + i$, n=1,2,…, converges to $\lim_{n\to\infty} z_n = 0 + i = i$.

$z_n = -2 + i\frac{(-1)^n}{n^2}$, n=1,2,…, converges to $\lim_{n\to\infty} z_n = -2 + 0\,i = -2$.

Consider an infinite series formed by an infinite sequence $z_1, z_2, \ldots, z_n, \ldots,$ of complex numbers and denoted as

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

The series is said to converge to the sum S if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \qquad (n = 1,2,\ldots)$$

of partial sums converges to $\lim_{n\to\infty} S_n = S$. We write $\sum_{n=1}^{\infty} z_n = S$

and note that a series can have at most one sum. When a series does not converge, we say that it diverges.

***Theorem*:**

Suppose that $z_n = x_n + i\, y_n$ (n=1,2,…) and $S=X+i\,Y$. Then $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

A necessary condition for the convergence of $\sum_{n=1}^{\infty} z_n$ is that $\lim_{n\to\infty} z_n = 0$. A convergent series of complex numbers are bounded, i.e., there exists a positive constant M such that $|z_n| < M$ for each positive integer n.

If $\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$ converges to a positive real number, then $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent. Since $|x_n| \le \sqrt{x_n^2 + y_n^2}$ and $|y_n| \le \sqrt{x_n^2 + y_n^2}$, we know that $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ must converge.

Absolute convergence of a series of complex numbers implies the convergence of that series.

Example

If $|z| < 1$, then $\lim_{N\to\infty} z^N = \lim_{N\to\infty} r^N e^{iN\theta} = 0$, and then

$$\sum_{n=1}^{\infty} z^n = \lim_{N\to\infty} \sum_{n=1}^{N} z^n = \lim_{N\to\infty} \frac{1-z^N}{1-z} = \frac{1}{1-z}.$$

***Theorem*:**

If f is analytic throughout a disk $|z - z_0| < R_0$, then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$ $(n = 0,1,2,......)$. That means $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ will converge to $f(z)$ when $|z - z_0| < R_0$.

For $|z-z_0|<R_0$, a function $f(z)$ can be written into the expansion of Taylor series given as

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots.$$

Clearly, for $|z|<R_0$, i.e., z_0=0, it is called a Maclaurin series and given as

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\cdots.$$

Example

The function $f(z)=e^z$ is entire. Its Maclaurin series is valid for all z. Since $f^{(n)}(z)=e^z$, we have

$$e^z=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=\sum_{n=0}^{\infty}\frac{z^n}{n!}\qquad(|z|<\infty).$$

Hence, the Maclaurin series for z^2e^{3z} is

$$z^2e^{3z}=z^2\sum_{n=0}^{\infty}\frac{(3z)^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n z^{n+2}}{n!}=\sum_{n=2}^{\infty}\frac{3^{n-2}}{(n-2)!}z^n.$$

Example

The function $f(z)=\sin z=\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$ is entire. Its Maclaurin series is valid for all z, expressed as

$$\sin z=\frac{1}{2i}\left(\sum_{n=0}^{\infty}\frac{(iz)^n}{n!}-\sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right)\qquad(|z|<\infty).$$

It can be further expressed as

$$\sin z=\frac{1}{2i}\sum_{n=0}^{\infty}\left(1-(-1)^n\right)\frac{i^nz^n}{n!}=\frac{1}{i}\sum_{n=0}^{\infty}\frac{i^{2n+1}z^{2n+1}}{(2n+1)!}$$
$$=\sum_{n=0}^{\infty}\frac{i^{2n}z^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\qquad\left(|z|<\infty\right)$$

Since $\cos z=\frac{d}{dz}\sin z$, we have

$$\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)z^{2n}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2n!}\quad\left(|z|<\infty\right).$$

Example

The function $f(z)=\sinh z=-i\sin(iz)$ is entire. Its Maclaurin series is valid for all z, expressed as

$$sinh\, z = -i \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{i^{2n+2} z^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Similarly, the function $f(z)=cosh\, z= cos(iz)$ is entire. Its Maclaurin series is valid for all z, expressed as

$$cosh\, z = \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n}}{2n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n!} \quad (|z| < \infty).$$

Example

The function $f(z)=1/(1-z)$ is analytic for $|z|<1$ and

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}.$$

Hence, its Maclaurin series is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n \quad (|z|<1).$$

Similarly, by replacing z by $-z$, we have

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \qquad (|z|<1)$$

Further replacing z by $z-1$ leads to

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1|<1)$$

which is the Taylor series for $1/z$ about $z_0=1$ in the region $|z-1|<1$.

Example

Consider $f(z) = \frac{1+2z^2}{z^3+z^5}$, which is further expressed as the following form

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right).$$

We cannot find the Maclaurin series for $f(z)$ since it is not analytic at $z=0$. However,

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (|z|<1).$$

Hence, when $0<|z|<1$, we have

$$f(z)=\frac{1}{z^3}\left(2-\frac{1}{1+z^2}\right)=\frac{1}{z^3}\left(2-\sum_{n=0}^{\infty}(-1)^n z^{2n}\right)$$
$$=\frac{2}{z^3}-\sum_{n=0}^{\infty}(-1)^n z^{2n-3}=\frac{2}{z^3}-\left(\frac{1}{z^3}-\frac{1}{z}+z-z^3+z^5-\cdots\right)$$
$$=\frac{1}{z^3}+\frac{1}{z}-z+z^3-z^5+\cdots$$

which contains terms of negative powers, $1/z$ and $1/z^3$.

Theorem:(Laurent series)

Suppose that a function f is analytic throughout an annular domain $R_1<|z-z_0|<R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ can be expressed as the following Laurent series representation

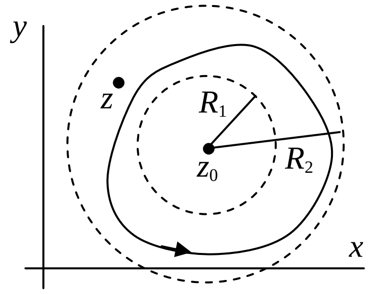
$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} \qquad (R_1<|z-z_0|<R_2)$$

where

$$a_n=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{(z-z_0)^{n+1}} \quad \text{and} \quad b_n=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{(z-z_0)^{-n+1}}.$$

The Laurent series can be also represented as

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n \qquad (R_1<|z-z_0|<R_2)$$



where

$$c_n=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{(z-z_0)^{n+1}} \qquad (n=0,\pm 1,\pm 2,\ldots).$$

Note that if f is analytic throughout the disk $|z-z_0|<R_2$, the term b_n can be written as

$$b_n=\frac{1}{2\pi i}\int_C f(z)(z-z_0)^{n-1}dz, \qquad \text{for } n=1,2,\ldots.$$

Since the integrand $f(z)(z-z_0)^{n-1}$ is also analytic throughout the disk $|z-z_0|<R_2$, we have

$$b_n=\frac{1}{2\pi i}\int_C f(z)(z-z_0)^{n-1}dz=0, \text{ for } n=1,2,\ldots.$$

Thus, the Laurent series is reduced to the Taylor series as

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0|<R_2)$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)dz}{(z-z_0)^{n+1}} = \dfrac{f^{(n)}(z_0)}{n!}$.

Example

It is known that $e^z = \displaystyle\sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1+\dfrac{z}{1!}+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\cdots \qquad (|z|<\infty)$.

Hence, replacing z by $1/z$, we have

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = 1+\frac{1}{1!\,z}+\frac{1}{2!\,z^2}+\cdots \qquad (0<|z|<\infty)$$

From Laurent series,

$$e^{1/z} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \qquad (0<|z|<\infty)$$

where, with C being any positively oriented simple closed contour around origin,

$$a_n = \frac{1}{2\pi i}\int_C \frac{e^{1/z}dz}{z^{n+1}}, \quad (n=0,1,2,\ldots)$$

$$b_n = \frac{1}{2\pi i}\int_C e^{1/z} z^{n-1} dz, \quad (n=1,2,\ldots).$$

Clearly,

$$a_0 = \frac{1}{2\pi i}\int_C \frac{e^{1/z}dz}{z} = 1$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{e^{1/z}dz}{z^{n+1}} = 0 \qquad \text{for } n=1,2,\ldots$$

$$b_n = \frac{1}{2\pi i}\int_C e^{1/z} z^{n-1} dz = \frac{1}{n!} \quad \text{for } n=1,2,\ldots.$$

This is a kind of method to evaluate certain integrals around simple closed contours.

Example

The function $f(z)=1/(z-i)^2$ is already in the form of a Laurent series, where $z_0=i$. That is

$$f(z)=\frac{1}{(z-i)^2}=\sum_{n=-\infty}^{\infty}c_n(z-i)^n \qquad (0<|z-i|<\infty)$$

where

$$c_n=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{(z-i)^{n+1}}=\frac{1}{2\pi i}\int_C\frac{dz}{(z-i)^{n+3}}=\begin{cases}0 & n\neq-2\\ 1 & n=-2\end{cases}$$

Example

The function $f(z)=\frac{-1}{(z-1)(z-2)}=\frac{1}{z-1}-\frac{1}{z-2}$, which has two singular points $z=1$ and $z=2$, is analytic in the domains $|z|<1$, $1<|z|<2$, and $2<|z|<\infty$. In each of these domains, $f(z)$ has series representations in powers of z.

For $|z|<1$, we have $|z/2|<1$ and then

$$f(z)=-\frac{1}{1-z}+\frac{1}{2}\cdot\frac{1}{1-(z/2)}=-\sum_{n=0}^{\infty}z^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
$$=\sum_{n=0}^{\infty}\left(2^{-n-1}-1\right)z^n$$

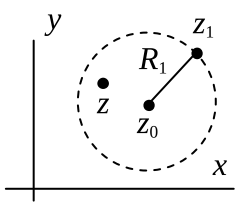
For $1<|z|<2$, we have $|1/z|<1$ and $|z/2|<1$, then

$$f(z)=\frac{1}{z}\cdot\frac{1}{1-(1/z)}+\frac{1}{2}\cdot\frac{1}{1-(z/2)}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
$$=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{z^n}$$

For $2<|z|<\infty$, we have $|1/z|<1$ and $|2/z|<1$, then

$$f(z)=\frac{1}{z}\cdot\frac{1}{1-(1/z)}-\frac{1}{z}\cdot\frac{1}{1-(2/z)}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$
$$=\sum_{n=1}^{\infty}\left(1-2^{n-1}\right)\frac{1}{z^n}$$

Theorem



If a power series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ converges when $z=z_1$ ($z_1\neq z_0$), then it is absolutely convergent at each point z in the open disk $|z-z_0|<R_1$, where $R_1=|z_1-z_0|$.

Theorem

If z_1 is a point inside the circle of convergence $|z-z_0|=R$ of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, then that series must be uniformly convergent in the closed disk $|z-z_0|\leq R_1$, where $R_1=|z_1-z_0|$.

Theorem

A power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z-z_0|=R$.

Theorem

Let C denote any contour interior to the circle of convergence of the power series $S(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$, and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n \, dz .$$

Corollary

The sum $S(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic at each point z interior to the circle of convergence of that series.

Theorem

The power series $S(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$ can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series, $S'(z)=\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$.

Theorem

If a series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z-z_0|=R$, then it is the Taylor series expansion for f in powers of $z-z_0$.

Theorem

If a series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z-z_0$ for that domain.

Theorem

Suppose $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z)=\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge within some circle $|z-z_0|=R$. They are analytic functions in the disk $|z-z_0|<R$ and their product has a Taylor series expansion, called the Cauchy product and expressed as $f(z)g(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^n$ with $c_n=\sum_{k=0}^{\infty} a_k b_{n-k}$ for $|z-z_0|<R$.

Suppose that $g(z)\neq 0$ when $|z-z_0|<R$. The quotient $f(z)/g(z)$ is analytic throughout the disk $|z-z_0|<R$ and has a Taylor series representation

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n(z-z_0)^n \quad \text{for } |z-z_0|<R$$

where the coefficients d_n can be found by formally carrying out the division. It is usually only the first few terms that are needed in practice.

_P12-1_

Show that if $0<r<1$ (also valid for $r=0$), then

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2}$$

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta + r^2}.$$

_P12-2_

Obtain the Taylor series $e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$ $(|z-1|<\infty)$ for the function $f(z)=e^z$ by (a) using $f^{(n)}(1)$ (n=0,1,2,…); (b) writing $e^z=e^{z-1}e$.

P12-3

Find the Maclaurin series expansion of the function

$$f(z)=\frac{z}{z^4+9}=\frac{z}{9}\cdot\frac{1}{1+\left(z^4/9\right)}.$$

P12-4

Derive the Taylor series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad \left(|z-i|<\sqrt{2}\right).$$

P12-5

Expand $cos\, z$ into a Taylor series about $z_0=\pi/2$.

Hint: $cos\, z=-sin\left(z-\frac{\pi}{2}\right)$

P12-6

Show that when $0<|z|<4$, $\frac{1}{4z-z^2}=\frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$.

P12-7

Find the Laurent series to represent the function $f(z)=z^2\, sin\left(\frac{1}{z^2}\right)$ in the domain $0<|z|<\infty$.

P12-8

Give two Laurent series in powers of z for the function $f(z)=\frac{1}{z^2(1-z)}$ and specify the regions in which those expansions are valid.

P12-9

Show that when $0<|z|<2$, $\frac{z}{(z-1)(z-3)}=-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}-\frac{1}{2(z-1)}$.

P12-10

Let a denote a real number, where $-1<a<1$, and derive the Laurent series

$$\frac{a}{z-a}=\sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad \left(|a|<|z|<\infty\right).$$

P12-11

The z-transform of $x[n]$ is $X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$ in the annulus domain $R_1<|z|<R_2$. If $|z|=1$ is in the annulus domain, then the inverse z-transform is $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{i\theta}\right)e^{in\theta}d\theta \quad (n=0,\pm1,\pm2,\cdots)$.

Hint:

$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n$ converges in $R_1<|z-z_0|<R_2$

where $c_n=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{(z-z_0)^{n+1}}$ (n=0,± 1,±2,…)

P12-12

Find the Taylor series for $\frac{1}{z}=\frac{1}{2+(z-2)}=\frac{1}{2}\cdot\frac{1}{1+(z-2)/2}$ about z_0=2.

Then, differentiate it and show that

$$\frac{1}{z^2}=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \quad (|z-2|<2).$$

P12-13

Prove that if

$$f(z)=\begin{cases}\dfrac{\cos z}{z^2-(\pi/2)^2} & \text{when } z\neq\pm\pi/2\\ -\dfrac{1}{\pi} & \text{when } z=\pm\pi/2\end{cases}$$

then f is an entire function.

P12-14

Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)}=\frac{1}{z}+1-\frac{1}{2}z-\frac{5}{6}z^2+\cdots \quad (0<|z|<1).$$

P12-15

Use division to obtain the Laurent series

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\cdots \quad (0<|z|<2\pi).$$

P12-16

The Euler numbers are E_n in the Maclaurin series

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \quad \left(|z| < \pi/2\right).$$

Point out why this representation is valid in the indicated disk and why

$E_{2n+1}=0$ (n=0,1,2,…) Then show that $E_0=1$, $E_2=-1$, $E_4=5$, and $E_6=-61$.