CV11 Integrals **:** *Cauchy Integral Formula*

Based on the contour integrals, let's introduce the concept of antiderivative of a continuous function $f(z)$ on a domain D. First, consider the following three statements and show that if any one of the following statements is true, then so are the others:

(1) $f(z)$ has an antiderivative $F(z)$, i.e., $F'(z)=f(z)$, in D;

(2) the integrals of $f(z)$ along contours lying entirely in D and extending from one fixed point to the other fixed point all have the same value;

(3) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

Let's assume (1) is true. If a contour C: $z=z(t)$, for $a \le t \le b$, lying in D, is a smooth arc, then

$$\frac{d}{dt}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t) \quad (a \le t \le b)$$

and thus,

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = F(z(t))\Big|_a^b = F(z_2) - F(z_1)$$

where z_1=z(a) and z_2=z(b). Evidently, $\int_C f(z)\,dz$ has the same value for all the contour extending from z_1 to z_2. If a contour C consists of finite number of smooth arcs C_k, k=1,2,…,n, then

$$\begin{aligned}\int_C f(z)\,dz &= \sum_{k=1}^{n}\int_{C_k} f(z)\,dz \\ &= \sum_{k=1}^{n}\left(F(z_{k+1}) - F(z_k)\right) = F(z_{n+1}) - F(z_1)\end{aligned}$$

Again, $\int_C f(z)\,dz$ has the same value for all the contour extending from z_1 to z_{n+1}. Clearly, statement (2) follows from statement (1). Besides, once the contour C is closed, i.e., $z_1 = z_{n+1}$, we have

$$\int_C f(z)\,dz = F(z_{n+1}) - F(z_1) = 0\,.$$

which leads to statement (3).

Example

The continuous function $f(z)=z^2$ has an antiderivative $F(z)=z^3/3$ on the complex plane. Hence,

$$\int_0^{1+i} z^2\, dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$$

Example

The function $f(z)=1/z^2$, continuous everywhere except at the origin, has an antiderivative $F(z)=-1/z$ in the domain $|z|>0$, consisting of the entire plane with the origin deleted.

Consequently,

$$\int_C \frac{dz}{z^2} = 0$$

when C is any contour around the origin.

Example

The integral of function $f(z)=1/z$ for C being any contour around the origin can not be evaluated in a similar way since its antiderivative $F(z)=log\ z$ fails to exist at the branch cut. Hence,

$$\int_C \frac{dz}{z} \neq 0$$

when C is any contour around the origin. To determine its value, Let the contour start from $z_1=re^{-i\pi}$ to $z_2=re^{i\pi}$ in positively oriented direction. Actually, z_1 and z_2 represent the same point in z-plane. Let C' be the same contour as C except that z_1 (or z_2) is deleted. Hence,

$$\int_C \frac{dz}{z} = \int_{C'} \frac{dz}{z}$$

Since $F(z)$ is differential in the branch for $-\pi<\theta<\pi$, we have

$$\int_{C'} \frac{dz}{z} = F(z_2^-) - F(z_1^+)$$

where $z_1^+ = re^{-i\pi^+}$ and $z_2^- = re^{i\pi^-}$. Therefore,

$$\int_C \frac{dz}{z} = F(z_2^-) - F(z_1^+) = log\left(re^{i\pi^-}\right) - log\left(re^{-i\pi^+}\right)$$

$$= log\left(re^{i\pi}\right) - log\left(re^{-i\pi}\right) = 2\pi i$$

Clearly, it is true that $\int_C \frac{dz}{z} \neq 0$.

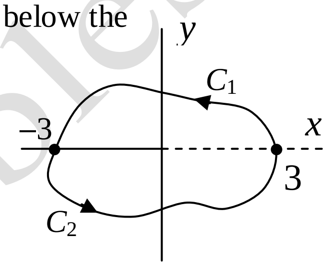
Example

The integral of $f(z)=z^{1/2}$ for the closed contour C is $\int_C z^{1/2}dz$ where the integrand is the branch

$$z^{1/2}=\sqrt{r}\,e^{i\theta/2}\quad (r>0, 0<\theta<2\pi)$$

of the square root function. The contour C includes a point z=3 on ray θ=0, not in the above branch. Hence, $\int_C z^{1/2}dz$ is not necessary zero.

To evaluate the integral of the closed contour, separate itinto two paths, C_1 from z=3 to z=–3 above the x-axis and C_2 from z=–3 to z=3 below the x-axis. The contour C_1 is in the branch of $z^{1/2}$ for r>0 and $-\pi/2<\theta<3\pi/2$. Therefore, the integral of $z^{1/2}$ along C_1, can be obtained from its antiderivative $\frac{2}{3}z^{3/2}$ as



$$\int_{C_1} z^{1/2}dz=\frac{2}{3}z^{3/2}\Big|_{z=3}^{z=-3}=\frac{2}{3}(3)^{3/2}\left(e^{i3\pi/2}-e^{i0}\right)=2\sqrt{3}(-i-1)$$

Similarly, the contour C_2 is in the branch of $z^{1/2}$ for r>0 and $\pi/2<\theta<5\pi/2$. The integral of $z^{1/2}$ along C_2, can be also obtainedas

$$\int_{C_2} z^{1/2}dz=\frac{2}{3}z^{3/2}\Big|_{z=-3}^{z=3}=\frac{2}{3}(3)^{3/2}\left(e^{i3\pi}-e^{i3\pi/2}\right)=2\sqrt{3}(-1+i)$$

Hence,

$$\int_C z^{1/2}dz=\int_{C_1} z^{1/2}dz+\int_{C_2} z^{1/2}dz$$
$$=2\sqrt{3}(-i-1)+2\sqrt{3}(-1+i)=-4\sqrt{3}$$

Example

Calculate $\int_0^{\pi+i} z\cos(2z)\,dz$. Since $z\cos(2z)$ is entire, its antiderivative is $\frac{1}{2}z\sin(2z)+\frac{1}{4}\cos(2z)$. We have

$$\int_0^{\pi+i} z\cos(2z)\,dz=\frac{1}{2}z\sin(2z)+\frac{1}{4}\cos(2z)\Big|_0^{\pi+i}$$

$$=\frac{1}{2}(\pi+i)\sin(2\pi+2i)+\frac{1}{4}\cos(2\pi+2i)-\frac{1}{4}$$
$$=\frac{1}{2}(\pi i-1)\sin 2i+\frac{1}{4}\cos 2i-\frac{1}{4}$$

When a continuous function $f(z)$ has an antiderivative in a domain D, the integral of $f(z)$ around any given closed contour C (not necessary to be simple closed contour) lying entirely in D has value zero.

***<u>Green's theorem</u>*:**

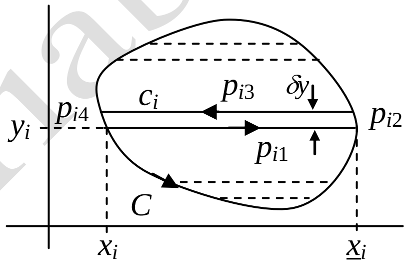
It is true for two real-valued functions $P(x,y)$ and $Q(x,y)$, together with their first-order partial derivatives Q_x and P_y, that

$$\int_C P\,dx + Q\,dy = \int\int_R \left(Q_x - P_y\right) dA$$

where P, Q, Q_x and P_y are all continuous throughout a closed region R consisting of all points interior to and on the simple closed contour C.

<u>Explanation:</u>

The equation is further partitioned into



$$\int_C P\,dx = \int\int_R \left(-P_y\right) dA = -\int\int_R \frac{\partial P}{\partial y}\,dxdy$$

$$\int_C Q\,dy = \int\int_R \left(Q_x\right) dA = \int\int_R \frac{\partial Q}{\partial x}\,dxdy$$

Let's consider the first term

$$\int_C P\,dx = -\int\int_R \frac{\partial P}{\partial y}\,dxdy$$

Decompose the contour into $C = c_1 + c_2 + \cdots + c_i + \cdots$, where all the closed contours c_i are horizontal to the real axis and each has four paths

p_{i1}: $(x_i, y_i) \rightarrow (\underline{x}_i, y_i)$, a horizontal line, $y=y_i$

p_{i2}: $(\underline{x}_i, y_i) \rightarrow (\underline{x}_{i+1}, y_{i+1})$, a sub-contour of C with $\delta y= y_{i+1}- y_i \rightarrow 0$.

p_{i3}: $(\underline{x}_{i+1}, y_{i+1}) \rightarrow (x_{i+1}, y_{i+1})$, a horizontal line, $y=y_{i+1}$

p_{i4}: $(x_{i+1}, y_{i+1}) \rightarrow (x_i, y_i)$, a sub-contour of C with $\delta y= y_{i+1}- y_i \rightarrow 0$.

Hence,

$$\begin{aligned}\int_{c_i} P\,dx &= \int_{p_{i1}} P(x,y_i)\,dx + \int_{p_{i2}} P(x,y)\,dx \\ &\quad + \int_{p_{i3}} P(x,y_{i+1})\,dx + \int_{p_{i4}} P(x,y)\,dx \\ &= \int_{p_i} P(x,y_i)\,dx + \int_{p_{i3}} P(x,y_{i+1})\,dx \\ &= \int_{x_i}^{\underline{x}_i} P(x,y_i)\,dx + \int_{\underline{x}_{i+1}}^{x_{i+1}} P(x,y_i+\delta y)\,dx\end{aligned}$$

where $\int_{p_{i2}} P(x,y)\,dx = \int_{p_{i4}} P(x,y)\,dx = 0$ since the length of p_{i2} and p_{i4} are infinitesimal. Furthermore, based on the fact of $\delta y \to 0$, we have

$$\int_{\underline{x}_{i+1}}^{x_{i+1}} P(x, y_i + \delta y)\, dx = -\int_{x_i}^{\underline{x}_i} P(x, y_i + \delta y)\, dx$$
$$= -\int_{x_i}^{\underline{x}_i} \left[P(x, y_i) + P_y(x, y_i)\delta y \right] dx$$

Therefore,

$$\int_{c_i} P\, dx = \int_{x_i}^{\underline{x}_i} P(x, y_i)\, dx + \int_{\underline{x}_{i+1}}^{x_{i+1}} P(x, y_i + \delta y)\, dx$$
$$= \int_{x_i}^{\underline{x}_i} P(x, y_i)\, dx - \int_{x_i}^{\underline{x}_i} \left[P(x, y_i) + P_y(x, y_i)\delta y \right] dx$$
$$= -\int_{x_i}^{\underline{x}_i} P_y(x, y_i)\delta y\, dx$$

which is the area bounded between the curve $P(x,y_i)$ and $P(x,y_{i+1})$ but takes the negative numeric sign. Consequently, we have

$$\int_C P\, dx = \sum_{i=1}^{\infty} \int_{c_i} P\, dx = \sum_{i=1}^{\infty} -\int_{x_i}^{\underline{x}_i} P_y(x, y_i)\delta y\, dx$$
$$= -\int\int_R \frac{\partial P}{\partial y} dxdy$$

In a similar process, we can derive that

$$\int_C Q\, dy = \int\int_R \frac{\partial Q}{\partial x} dxdy$$

and then obtain the ***Green's theorem***.

Let C denote a simple closed contour $z=z(t)$ ($a \le t \le b$), described in the positive sense (counterclockwise), and assume that f is analytic at each point interior to and on C. If $f(z) = u(x,y) + i\, v(x,y)$, then

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt = \int_a^b (u + i\, v)(x' + i\, y') dt$$
$$= \int_a^b (u\, x' - v\, y') dt + i \int_a^b (v\, x' + u\, y') dt$$
$$= \int_C (u\, dx - v\, dy) + i \int_C (v\, dx + u\, dy)$$

According to Green's theorem, we have

$$\int_C f(z) dz = -\int\int_R (v_x + u_y) dA + i \int\int_R (u_x - v_y) dA$$

where R is the region enclosed by C. Since f is analytic, that is $u_x = v_y$ and $u_y = -v_x$, we obtain $\int_C f(z) dz = 0$.

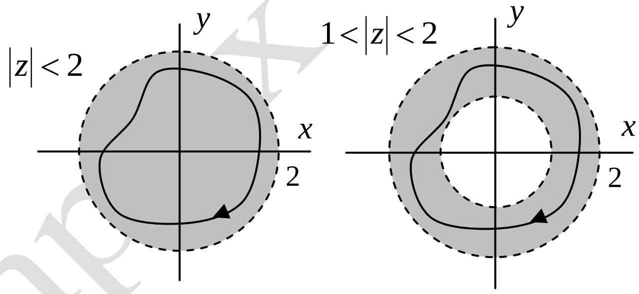
Example

It is true that $\int_C e^{z^3} dz = 0$ where C is any simple closed contour since $f(z) = e^{z^3}$ is analytic everywhere and its derivative $f'(z) = 3z^2 e^{z^3}$ is continuous everywhere.

***Cauchy-Goursat Theorem*:**

If a function f is analytic at all points interior to and on a simple closed contour, then $\int_C f(z)dz = 0$.

A ***simply connected domain*** D is a domain such that every simple closed contour within it enclosed only points of D, such as $|z| < 2$. A domain that is not simply connected is said to be ***multiply connected***, such as $1 < |z| < 2$.



***Theorem*:**

If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour C lying in D.

***Corollary*:**

A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Example Entire functions always possess antiderivatives.

Theorem

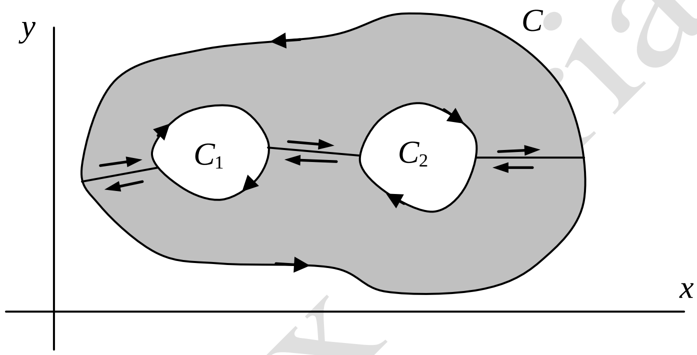
Suppose that

(1) C is a simple closed contour in the counterclockwise direction;

(2) C_k (k=1,2,…,n) are simple closed contours interior to C, all in the clockwise direction, that are disjoint and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k, then

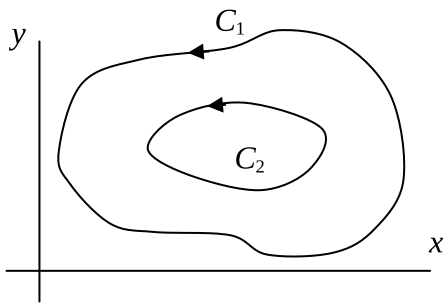
$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$



Corollary

Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1. If a function is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



Example

When C is any positively oriented simple closed contour surrounding the origin, the integral $\int_C \frac{dz}{z}$ can be determined as

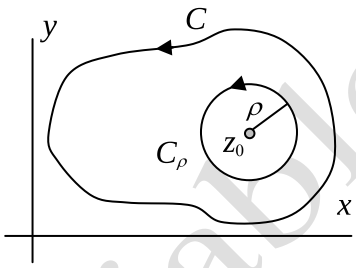
$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z} = 2\pi i$$

where C_0 is a positively oriented circle with center at origin and lying entirely inside C. Note that $1/z$ is analytic everywhere except at z=0.

Theorem (Cauchy Integral Formula)

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0}$$



Proof:

Let C_ρ denote a positively oriented circle $|z-z_0|=\rho$,

where ρ is small enough that C_ρ is interior to C. Since $f(z)$ is analytic everywhere, $\dfrac{f(z)}{z-z_0}$ is then analytic between and on the contours C and C_ρ. It follows that

$$\int_C \frac{f(z)dz}{z-z_0}=\int_{C_\rho}\frac{f(z)dz}{z-z_0}$$

then

$$\int_C \frac{f(z)dz}{z-z_0}-f(z_0)\int_{C_\rho}\frac{dz}{z-z_0}=\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz$$

Since $\displaystyle\int_{C_\rho}\frac{dz}{z-z_0}=2\pi i$, we have

$$\int_C \frac{f(z)dz}{z-z_0}=2\pi i\ f(z_0)+\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz$$

Due to the fact that $f(z)$ is analytic, and therefore continuous, at z_0 ensures that for each positive number ε, there is a positive number δ such that

$$|f(z)-f(z_0)|<\varepsilon \qquad \text{whenever} \quad |z-z_0|<\delta$$

Hence,

$$\left|\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz\right|<ML$$

where $M = \left|\dfrac{f(z)-f(z_0)}{z-z_0}\right| < \dfrac{\varepsilon}{\rho}$ on the contour C_ρ and $L=2\pi\rho$ is the length of the contour C_ρ. This results in

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon$$

Since ε can be arbitrarily small, $\int_{C_\rho} \dfrac{f(z)-f(z_0)}{z-z_0}dz$ must be zero. This completes the proof of

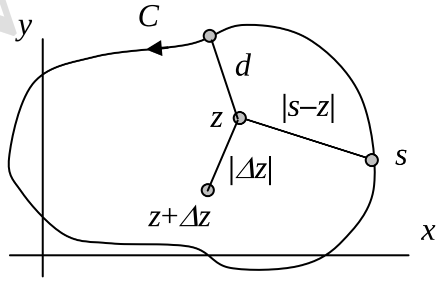
$$\int_C \frac{f(z)dz}{z-z_0} = 2\pi i\, f(z_0) \quad \text{or} \quad f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0} \;.$$

Lemma

Suppose that a function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z is any point interior to C, then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)ds}{(s-z)^3}$$



***Proof*:**

Since $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(s)ds}{s-z}$, we have

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds$$
$$= \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z-\Delta z)(s-z)}$$

Let d denote the smallest distance from z to C, then $|s-z| \ge d$ and $|\Delta z| < d$. It also leads to

$$|s-z-\Delta z| \ge \big||s-z| - |\Delta z|\big| \ge d - |\Delta z| > 0$$

Then,

$$\left|\int_C \frac{\Delta z\; f(s)ds}{(s-z-\Delta z)(s-z)^2}\right| \le \frac{|\Delta z| ML}{(d-|\Delta z|)d^2}$$

where $M \geq |f(z)|$ and L is the length of C. Clearly,

$$\lim_{\Delta z\to 0}\int_C \frac{\Delta z\ f(s)ds}{(s-z-\Delta z)(s-z)^2}=0$$

Now, further express

$$\begin{aligned}&\frac{f(z+\Delta z)-f(z)}{\Delta z}\\ &=\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}+\frac{1}{2\pi i}\int_C \frac{\Delta z\ f(s)ds}{(s-z-\Delta z)(s-z)^2}\end{aligned}$$

then

$$\begin{aligned}f'(z)&=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}\\ &=\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}+\lim_{\Delta z\to 0}\frac{1}{2\pi i}\int_C \frac{\Delta z\ f(s)ds}{(s-z-\Delta z)(s-z)^2}\end{aligned}$$

Since $\lim_{\Delta z\to 0}\int_C \frac{\Delta z\ f(s)ds}{(s-z-\Delta z)(s-z)^2}=0$, we complete the proof of

$$f'(z)=\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$

The second derivative $f''(z)$ can be proved in the same way.

Theorem

If a function is analytic at a point, then its derivatives of all orders exist at that point. Those derivatives are, moreover, all analytic there.

Corollary

If a function $f(z)=u(x,y)+i\ v(x,y)$ is defined and analytic at a point $z=(x,y)$ then the component functions u and v have continuous partial derivatives of all orders at z.

Example

$f'(z)=u_x+i\ v_x=v_y-i\ u_y$ and $f''(z)=u_{xx}+i\ v_{xx}=v_{yx}-i\ u_{yx}$.

The general form of n^{th} derivative

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}\qquad (n=0,1,2,......)$$

which can be used to evaluate the following integral

$$\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0,1,2,......)$$

Example

If C is the positively oriented unit circle $|z|=1$ and $f(z)=e^{2z}$, then

$$\int_C \frac{e^{2z}dz}{z^4} = \int_C \frac{e^{2z}dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{8\pi i}{3!}$$

Example

Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z)=1$, we have

$$\int_C \frac{dz}{z-z_0} = 2\pi i \quad \text{and} \quad \int_C \frac{dz}{(z-z_0)^{n+1}} = 0 \quad (n = 1,2,......)$$

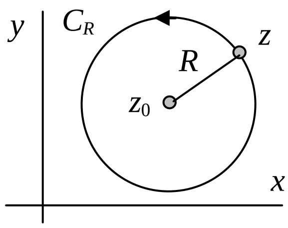
Theorem

Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour C lying in D, then f is analytic throughout D.

Lemma

Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then

$$\left|f^{(n)}(z_0)\right| \le \frac{n!\,M_R}{R^n} \qquad (n = 1,2,......)$$



which is called the Cauchy's inequality.

***Proof*:**

The inequality can be directly derived as below

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{C_R} \frac{f(z)dz}{(z-z_0)^{n+1}}\right| \le \frac{n!}{2\pi}\frac{M_R}{R^{n+1}} 2\pi R = \frac{n!\,M_R}{R^n}\,.$$

Liouville's Theorem

If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

***Proof*:**

Since f is entire and bounded in the complex plane, the Cauchy's inequality with n=1 is hold for any choice of z_0 and R, i.e.,

$$|f'(z_0)| \le \frac{M}{R}$$

where $|f(z)| \le M$ for all z and M is independent to R. Hence, the inequality is true for arbitrarily large value of R only if $f'(z_0)=0$. Because z_0 is arbitrarily selected, that means $f'(z)=0$ everywhere in the complex plane. Clearly, f is a constant function.

***The Fundamental Theorem Of Algebra*:**

Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \ne 0)$$

of degree n ($n \ge 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_0)=0$.

***Proof*:**

By contradiction, suppose that $P(z)$ is not zero for any value of z. Then $f(z)=1/P(z)$ is clearly entire and bounded in the complex plane. Following from the Liouville's theorem, $f(z)$, and consequently $P(z)$, must be constant. But $P(z)$ is not constant, which reaches a contradiction. That means there exists at least one point z_0 such that $P(z_0)=0$.

The fundament theorem tells us that any polynomial $P(z)$ of degree n ($n \ge 1$) can be expressed as a product of linear factors:

$$P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n)$$

where c and z_k (k=1,2,…,n) are complex constants. That also implies $P(z)$ can have no more than n distinct zeros.

<u>Lemma</u>**:**

Suppose that $|f(z)| \le |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

<u>Maximum Modulus Principle</u>**:**

If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it.

<u>Explanation:</u>

Let $f(z) = u(x,y) + iv(x,y)$, whose modulus is

$$r(x,y) = |f(z)| = \sqrt{u^2(x,y) + v^2(x,y)}$$

Assume $r(x,y)$ is a maximal value, then

$$r_x(x,y) = r_y(x,y) = 0, \quad r_{xx}(x,y) < 0, \quad r_{yy}(x,y) < 0$$

Therefore,

$$r_x(x,y) = \frac{\partial r(x,y)}{\partial x} = \frac{uu_x + vv_x}{r(x,y)} = 0$$

$$r_y(x,y) = \frac{\partial r(x,y)}{\partial y} = \frac{uu_y + vv_y}{r(x,y)} = 0$$

$$r_{xx}(x,y) = \frac{\left(uu_{xx} + u_x^2 + vv_{xx} + v_x^2\right)}{r(x,y)} + \frac{\left(uu_x + vv_x\right)^2}{r^3(x,y)} < 0$$

$$r_{yy}(x,y) = \frac{\left(uu_{yy} + u_y^2 + vv_y + v_y^2\right)}{r(x,y)} + \frac{\left(uu_y + vv_y\right)^2}{r^3(x,y)} < 0$$

which results in $r_{xx}(x,y) + r_{yy}(x,y) < 0$, i.e.,

$$\frac{\left(uu_{xx} + u_x^2 + vv_{xx} + v_x^2\right)}{r(x,y)} + \frac{\left(uu_x + vv_x\right)^2}{r^3(x,y)} + \frac{\left(uu_{yy} + u_y^2 + vv_{yy} + v_y^2\right)}{r(x,y)} + \frac{\left(uu_y + vv_y\right)^2}{r^3(x,y)} < 0$$

It is rewritten as

$$\frac{u\left(u_{xx} + u_{yy}\right) + v\left(v_{xx} + v_{yy}\right)}{r(x,y)} + \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{r(x,y)} + \frac{\left(uu_x + vv_x\right)^2 + \left(uu_y + vv_y\right)^2}{r^3(x,y)} < 0$$

Since $f(z)$ is analytic, it is true that

$$u_{xx}+u_{yy}=0 \quad \text{and} \quad v_{xx}+v_{yy}=0$$

Substituting them into the above equation leads to

$$\frac{u_x^2+u_y^2+v_x^2+v_y^2}{r(x,y)}+\frac{(uu_x+vv_x)^2+(uu_y+vv_y)^2}{r^3(x,y)}<0$$

which is obviously contradictory. Therefore, there is no maximal value of $r(x,y)=|f(z)|$.

***Corollary*:**

Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

When $f(z)=u(x,y)+i\,v(x,y)$, the component $u(x,y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior, where it is harmonic.

***Explanation*:**

It can be seen from the composite function $g(z)=exp(f(z))$ is continuous in R and analytic and not constant in the interior. Consequently, its modulus $|g(z)|=exp[u(x,y)]$, which is continuous in R, must assume its maximum value in R on the boundary. Because of the increasing nature of the exponential function, it follows that the maximum value of $u(x,y)$ also occurs on the boundary.

***Corollary (Minimum Modulus Principle)*:**

Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Assuming that $f(z)\neq 0$ anywhere in R, then the minimum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior. [This can be explained by $g(z)=1/f(z)$]

P11-1

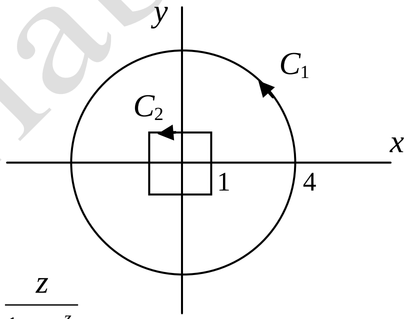
By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) $\int_i^{i/2} e^{\pi z} dz$; (b) $\int_0^{\pi+2i} cos\left(\frac{z}{2}\right) dz$; (c) $\int_1^3 (z-2)^3 dz$.

P11-2

Show that $\int_{-1}^{1} z^i dz = \frac{1+e^{-\pi}}{2}(1-i)$ where z^i denotes the principal branch $z^i = exp(iLog\ z)$, ($|z|>0$, $-\pi<Arg\ z<\pi$) and where the path of integration is any contour from $z=-1$ to $z=1$ that, except for its end points, lies above the real axis.

P11-3



Given the circle contour C_1 and square contour C_2, point out why $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ when

(a) $f(z) = \frac{1}{3z^2+1}$; (b) $f(z) = \frac{z+2}{sin(z/2)}$; (c) $f(z) = \frac{z}{1-e^z}$

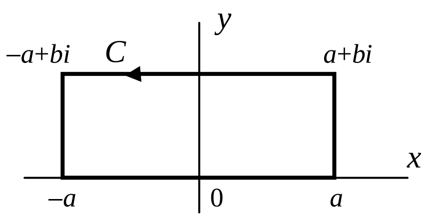
P11-4

Show that

$$\int_C (z-2-i)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \cdots \\ 2\pi i & \text{when } n = 0 \end{cases}$$

where the positively oriented contour C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$.

P11-5



Consider the following rectangular contour C Show that the sum of the integrals of $exp(-z^2)$ along the lower and upper horizontal legs of C can be written as

$$2\int_0^a e^{-x^2}\, dx - 2e^{b^2}\int_0^a e^{-x^2} \cos 2bx\, dx$$

and the sum of the integrals along the vertical legs on the right and left can be written as

$$ie^{-a^2}\int_0^b e^{y^2} e^{-i2ay}\, dy - ie^{-a^2}\int_0^b e^{y^2} e^{i2ay}\, dy$$

Thus, from Cauchy-Goursat theorem, show that

$$\int_0^a e^{-x^2} \cos 2bx\, dx = e^{-b^2}\int_0^a e^{-x^2}\, dx + e^{-\left(a^2+b^2\right)}\int_0^b e^{y^2} \sin 2ay\, dy$$

and then for $a\to\infty$, show that

$$\int_0^\infty e^{-x^2} \cos 2bx\, dx = e^{-b^2}\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} .$$

P11-6

Let C denote the positively oriented boundary of the half disk $0\le r\le 1$, $0\le\theta\le\pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0)=0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \quad (r>0,\ -\pi/2<\theta<3\pi/2)$$

of the multiple-valued function $z^{1/2}$. Show that $\int_C f(z)dz = 0$ by evaluating the integrals of $f(z)$ over the semmicircle and the two radii. Why does the Cauchy-Goursat theorem not apply?

P11-7

Let C denote the positively oriented boundary of the square whose sides lie along the lines $x=\pm 2$ and $y=\pm 2$. Evaluate each of these integrals:

(a) $\int_C \frac{e^{-z}dz}{z-(\pi i/2)}$; (b) $\int_C \frac{\cos z}{z\left(z^2+8\right)}dz$; (c) $\int_C \frac{z\, dz}{2z+1}$;

(d) $\int_C \frac{\cosh z}{z^4}dz$; (e) $\int_C \frac{\tan(z/2)}{\left(z-x_0\right)^2}dz \quad \left(-2<x_0<2\right)$

P11-8

Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ in the positive sense when

(a) $g(z) = \frac{1}{z^2+4}$; (b) $g(z) = \frac{1}{\left(z^2+4\right)^2}$.

P11-9

Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(\omega) = \int_C \frac{z^3 + 2z}{(z-\omega)^3}\,dz$$

Show that $g(\omega)=6\pi i\omega$ when ω is inside C and that $g(\omega)=0$ when ω is outside C.

P11-10

Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)dz}{z - z_0} = \int_C \frac{f(z)dz}{(z - z_0)^2}.$$

P11-11

Consider $f(z) = (z+1)^2$ and the closed triangular region R with vertices at the points at $z=0$, $z=2$ and $z=i$. Find points in R where $|f(z)|$ has its maximum and minimum values.

P11-12

Let $f(z) = e^z$ and R the rectangular region $0 \le x \le 1$, $0 \le y \le \pi$. Find points in R where the component $u(x,y)=Re[f(z)]$ reaches its maximum and minimum values.