CV10 Integrals : *Contour Integrals*

Befroe discuss the contour integrals, let's consider a complex-valued function $w(t)$ of a real variable t, expressed as

$$w(t)= u(t) + i\, v(t)$$

where $u(t)$ and $v(t)$ are real-valued functions of t. Then, its derivative is written as

$$w'(t)=\frac{dw(t)}{dt}=u'(t)+i\,v'(t)$$

provided u' and v' exist at t, and various rules learned in calculus are also applicable, such as

$$\frac{d}{dt}[w(-t)]=-w'(-t),\quad \frac{d}{dt}\left[w^2(t)\right]=2\,w(t)w'(t)$$

$$\frac{d}{dt}[z_0 w(t)]=z_0 w'(t),\quad \frac{d}{dt}\left[e^{z_0 t}\right]=z_0 e^{z_0 t}$$

where z_0 is a complex value.

Even if $w'(t)$ exists when $a<t<b$, the ***mean value theorem*** is ***not*** necessary true that $w'(c)=\dfrac{w(b)-w(a)}{b-a}$ for c on the interval $a<t<b$. For example, the function $w(t)=e^{it}$ on the interval $0\le t\le 2\pi$ and $w'(t)=i\, e^{it}$. Clearly, $w'(c)$ is never zero for c on the interval $0<t<2\pi$, but $w(2\pi)-w(0)=0$.

Now, let's further define the integral of a complex-valued function as below:

$$\int_a^b w(t)\,dt=\int_a^b u(t)\,dt+i\int_a^b v(t)\,dt$$

where $u(t)$ and $v(t)$ are piecewise continuous on the interval $a\le t\le b$. For example,

$$\int_0^1 (1+it)^2\,dt=\int_0^1\left(1-t^2\right)dt+i\int_0^1 2t\,dt=\frac{2}{3}+i$$

Besides, it is also true that

$$\int_a^b w(t)\,dt=\int_a^c w(t)\,dt+\int_c^b w(t)\,dt$$

where c is on the interval $a\le t\le b$.

Suppose $w(t) = u(t) + i\, v(t)$ and $W(t) = U(t) + i\, V(t)$ are continuous on the interval $a \le t \le b$. If $W'(t)=w(t)$, then $U'(t)=u(t)$ and $V'(t)=v(t)$. Hence,

$$\begin{aligned}\int_a^b w(t)\,dt &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt = U(t)\Big|_a^b + i\,V(t)\Big|_a^b \\ &= [U(b)+i\,V(b)] - [U(a)+i\,V(a)] \\ &= W(b) - W(a) = W(t)\Big|_a^b\end{aligned}$$

Based on the above operation, let's find the integral $\int_0^{\pi/4} e^{it}\,dt$. *Since* $(e^{it})' = i\,e^{it}$, we have

$$\begin{aligned}\int_0^{\pi/4} e^{it}\,dt &= -i\int_0^{\pi/4} \left(ie^{it}\right)dt = -i\,e^{it}\Big|_0^{4\pi} \\ &= -i\left(\frac{1}{\sqrt{2}}(1+i) - 1\right) = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)i\end{aligned}$$

Moreover, in polar form we let $\int_a^b w(t)\,dt = re^{i\theta}$, where r and θ are real.

Note that $r = \left|\int_a^b w(t)\,dt\right| > 0$ and $r = \int_a^b e^{-i\theta} w(t)\,dt \in R$. Hence,

$$\left|\int_a^b w(t)\,dt\right| = Re\int_a^b e^{-i\theta} w(t)\,dt = \int_a^b Re\left(e^{-i\theta} w(t)\right)dt$$

Due to the fact that $Re\left(e^{-i\theta} w(t)\right) < \left|e^{-i\theta} w(t)\right| = |w(t)|$, we have

$$\left|\int_a^b w(t)\,dt\right| < \int_a^b |w(t)|\,dt\,.$$

which also implies $\left|\int_a^\infty w(t)\,dt\right| < \int_a^\infty |w(t)|\,dt\,.$

Example

Consider the Legendre polynomial

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \left(x + i\sqrt{1-x^2}\,\cos\theta\right)^n d\theta$$

for $n=0,1,2,\ldots$ and $-1 \le x \le 1$. It is easy to check that

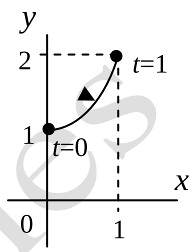
$$\begin{aligned}|P_n(x)| &< \frac{1}{\pi}\int_0^\pi \left|x + i\sqrt{1-x^2}\,\cos\theta\right|^n d\theta \\ &= \frac{1}{\pi}\int_0^\pi \left|x^2 + \left(1-x^2\right)\cos^2\theta\right|^n d\theta \\ &< \frac{1}{\pi}\int_0^\pi \left|x^2 + \left(1-x^2\right)\right|^n d\theta = \frac{1}{\pi}\int_0^\pi (1)^n\,d\theta = 1\end{aligned}$$

Next, we will focus on the contour integral, which is an integral along a contour, not within an interval.

Let's first introduce the concept of an *arc* in the z-plane, which is denoted

$$z(t) = x(t) + i\, y(t) \qquad (a \le t \le b)$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. In general, an *arc* is shown as a curve in the z-plane. When an arc does not cross itself, i.e., $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, it is called a simple arc or Jordan arc. For example, $z(t)=x(t)+i\, y(t)$, where $x(t)=t$ and $y(t)=1+t^2$ for $0 \le t \le 1$, is a simple arc.



An *arc* is called a simple closed curve, or Jordan curve, if it is a closed curve without any intersection.

Example

Consider the following arcs:

$$z = e^{i2\pi t} \text{ and } \quad z = e^{-i2\pi t} \quad \text{for} \quad 0 \le t \le 1.$$

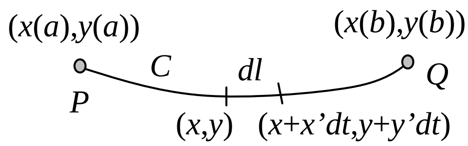
Although both the arcs represent the same unit circle in z plane, they are different simple closed curves, or different Jordan curves, since $z=e^{i2\pi t}$ and $z=e^{-i2\pi t}$ rotate in different directions, counterclockwise and clockwise.

Example

The arc $z = e^{i4\pi t}$, for $0 \le t \le 1$, rotates twice in the counterclockwise direction. It is not a Jordan curve.

Consider the *arc C*: $z(t)=x(t)+i\, y(t)$ for $a \le t \le b$ whose derivative is given as

$$z'(t)=x'(t)+i\, y'(t)$$



The length L of the *arc C* can be derived as below:
Place the *arc C* correspondingly on the x-y plane as shown.
The *arc C* is represented by the curve from $P(x(a),y(a))$ to $Q(x(b),y(b))$.
Now the differential from (x,y) to $(x+x'dt, y+y'dt)$ is

$$dl = \sqrt{x'^2 + y'^2}\, dt$$

Hence, the length is $L = \int_P^Q dl = \int_a^b \sqrt{x'^2 + y'^2}\, dt$, or $L = \int_a^b |z'(t)|\, dt$.

Suppose that $t=\phi(\tau)$ where ϕ is a real-valued function mapping the interval $\alpha \le \tau \le \beta$ on to $a \le t \le b$ and $\phi'(\tau)>0$. Hence, $z=Z(\tau)$ or $Z=z[\phi(\tau)]$. The derivative is

$$Z'(\tau)=z'[\phi(\tau)]\phi'(\tau)$$

The length L of *arc* C can be also represented as

$$\begin{aligned} L &= \int_a^b |z'(t)|\, dt = \int_\alpha^\beta |z'(\phi(\tau))| \frac{d\,\phi(\tau)}{d\,\tau}\, d\,\tau \\ &= \int_\alpha^\beta |z'(\phi(\tau))|\, \phi'(\tau) d\,\tau = \int_\alpha^\beta |z'(\phi(\tau))\phi'(\tau)|\, d\,\tau \\ &= \int_\alpha^\beta |Z'(\tau)|\, d\,\tau \end{aligned}$$

Clearly, the length is invariant under certain changes in variable.

An *arc* $z=z(t)$ for $a \le t \le b$ is said to be ***smooth*** if its derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and nonzero on the open interval $a<t<b$. A smooth arc is possessed of angle $\arg z'(t)$.

Now, wee define a ***contour***, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. When only the initial and final values of $z(t)$ are the same, a contour C is called a ***simple closed contour***.

***<u>Jordan Curve Theorem</u>*:**

The points on any simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded.

Based on the definition of contour, the integral of $f(z)$ along a given contour C: $z=z(t)$, for $a \le t \le b$, is defined as

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = \int_a^b f(z(t))z'(t)dt$$

where $z_1=z(a)$ and $z_2=z(b)$. Some properties concerning the contour integrals are listed as below:

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$$

$$\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

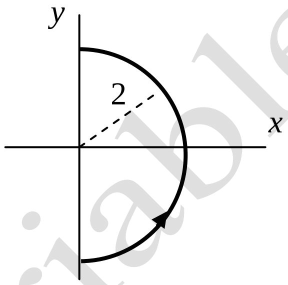
$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz, \quad \text{for } C = C_1 + C_2$$

Next, we will employ some examples to explain the contour integrals.

Example

Calculate $I_1 = \int_C \bar{z}\, dz$.



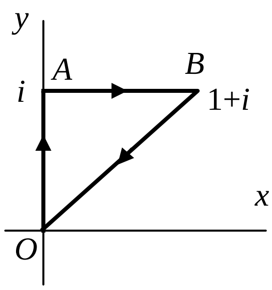
Since $z = 2e^{i\theta}$, we have $\bar{z} = 2e^{-i\theta}$ and $dz = 2ie^{i\theta}d\theta$. Hence,

$$I_1 = \int_{-\pi/2}^{\pi/2} \left(2e^{-i\theta}\right)\left(2ie^{i\theta}\right)d\theta = 4\pi i\,.$$

Example

Calculate $I_2 = \int_C f(z)dz$, where $f(z) = y - x - i3x^2$ and C: *OABO*.

Hence,



$$\begin{aligned} I_2 &= \int_O^A f(z)dz + \int_A^B f(z)dz + \int_B^O f(z)dz \\ &= \int_0^1 (y)(idy) + \int_0^1 \left(1 - x - i3x^2\right)(dx) \\ &\quad + \int_1^0 \left(x - x - i3x^2\right)(dx + idx) \\ &= \left(\frac{i}{2}\right) + \left(1 - \frac{1}{2} - i\right) + (-1 + i) = \frac{-1+i}{2} \end{aligned}$$

Example

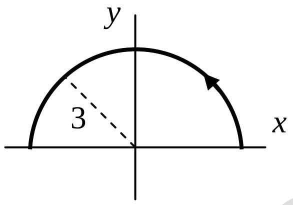
Calculate $I_3 = \int_C z\, dz$, where $z = z(t)$, for $a \le t \le b$ and C is an arbitrary smooth arc. Hence,

$$\begin{aligned} I_3 &= \int_C z\, dz = \int_a^b z(t)z'(t)dt = \int_C \frac{d}{dt}\left(\frac{z^2(t)}{2}\right) \\ &= \left.\frac{z^2(t)}{2}\right|_a^b = \frac{1}{2}\left(z^2(b) - z^2(a)\right) \end{aligned}$$

which implies $\int_{z_1}^{z_2} z\, dz = \frac{1}{2}\left(z_2^2 - z_1^2\right)$.

If a contour consists of a finite number of smooth arcs C_k, k=1,2,…,n, joined end to end, then

$$\int_C z\,dz = \sum_{k=1}^{n}\int_{C_k} z\,dz = \sum_{k=1}^{n}\frac{1}{2}\left(z_{k+1}^2 - z_k^2\right)$$
$$= \frac{1}{2}\left(z_{n+1}^2 - z_1^2\right)$$



Example

Calculate $I_4 = \int_C z^{1/2}\,dz$.

Since $z = 3e^{i\theta}$, we have $z^{1/2} = \sqrt{3}\,e^{i\theta/2}$ and $dz = 3ie^{i\theta}d\,\theta$.

Hence,

$$I_4 = \int_0^{\pi}\left(\sqrt{3}\,e^{i\theta/2}\right)\left(3i\,e^{i\theta}\right)d\,\theta = 3\sqrt{3}\,i\int_0^{\pi}\left(e^{i3\theta/2}\right)d\,\theta$$
$$= 3\sqrt{3}\,i\int_0^{\pi}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)d\,\theta$$
$$= 3\sqrt{3}\,i\left(\frac{2}{3}\sin\frac{3\theta}{2} - \frac{2i}{3}\cos\frac{3\theta}{2}\right)\Bigg|_0^{\pi} = -2\sqrt{3}(i+1)$$

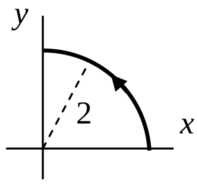
When consider the contour C: $z=z(t)$, for $a \le t \le b$, we have the following inequality:

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \le \int_a^b \left|f(z(t))\right|\left|z'(t)\right|dt$$

Since a nonnegative constant M always exists such that $\left|f(z)\right| \le M$ when $f(z)$ is evaluated along a contour, it can be obtained that

$$\left|\int_C f(z)\,dz\right| \le M\int_a^b \left|z'(t)\right|dt = ML$$

where L is the length of the contour C.



Example

Show that $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le ML = M\,\pi$

Since $|z| = 2$, we have $|z+4| \le |z| + 4 = 6$ and $\left|z^3 - 1\right| \ge \left||z|^3 - 1\right| = 7$.

Hence, $\left|\frac{z+4}{z^3-1}\right| \le \frac{6}{7} = M$. We have $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le ML = \frac{6}{7}\pi$.

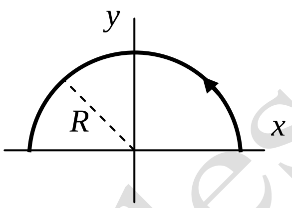
Example

Show that $\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}dz\right| \le M_R L = M_R(\pi R)$ where $R >> 1$.

Since $|z| = R$, we have $\left|z^{1/2}\right| = \sqrt{R}$ and $\left|z^2+1\right| \ge \left||z|^2 - 1\right| = R^2 - 1$.

Hence, $\left|\frac{z^{1/2}}{z^2+1}\right| \le \frac{\sqrt{R}}{R^2-1} = M_R$. We have

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}dz\right| \le \frac{\pi R\sqrt{R}}{R^2-1} \Rightarrow \lim_{R\to\infty}\int_{C_R} \frac{z^{1/2}}{z^2+1}dz = 0.$$



P10-1

Use the corresponding rules in calculus to establish the following rules when $w(t)= u(t) + i\, v(t)$ is a complex- valued function of a real variable t and $w'(t)$ exists:

(a) $\frac{dw(-t)}{dt} = -w'(-t)$ and (b) $\frac{d}{dt}w^2(t) = 2\,w(t)w'(t)$

P10-2

Evaluate the following integrals:

(a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$; (b) $\int_0^{\pi/6} e^{i2t}dt$; (c) $\int_0^\infty e^{-zt}dt$ ($Re\, z$>0).

P10-3

Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = \begin{cases} 0 & \text{when } m \ne n \\ 2\pi & \text{when } m = n \end{cases}.$$

P10-4

Show that if $w(t)=u(t)+iv(t)$ is continuous on an interval $a\le t\le b$, then

(a) $\int_{-b}^{-a} w(-t)dt = \int_a^b w(\tau)d\tau$

(b) $\int_a^b w(t)dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)d\tau$

where $t=\phi(\tau)$ for $\alpha\le\tau\le\beta$.

P10-5

Suppose a function $f(z)$ is analytic at $z_0=z(t_0)$ lying on a smooth arc $z=z(t)$ ($a\le t\le b$). Show that if $w(t)=f[z(t)]$, then $w'(t)=f'[z(t)]z'(t)$ when $t=t_0$.

P10-6

Evaluate $\int_C f(z)\,dz$, when $f(z)=(z+2)/z$ and C is

(a) the semicircle $z=2e^{i\theta}$ ($0\le\theta\le\pi$)

(b) the semicircle $z=2e^{i\theta}$ ($\pi\le\theta\le 2\pi$)

(c) the circle $z=2e^{i\theta}$ ($0\le\theta\le 2\pi$)

P10-7

Evaluate $\int_C f(z)\,dz$, when $f(z)=\pi\, exp(\pi\,\bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1+i$, and i, the orientation of C being in the counterclockwise direction.

P10-8

Evaluate $\int_C f(z)\,dz$, when $f(z)$ is the branch $z^{-1+i}=exp[(-1+i)log\, z]$ ($|z|>0$, $0<arg\ z<2\pi$) of the indicated power function, and C is the positively oriented unit circle $|z|=1$.

P10-9

Evaluate $\int_C z^m\bar{z}^n\,dz$, where m and n are integers and C is the unit circle $|z|=1$, taken counterclockwise.

P10-10

Let C_0 denote the circle $|z-z_0|=R$, taken counterclockwise. Use the parametric representation $z=z_0+Re^{i\theta}$ ($-\pi\le\theta\le\pi$) for C_0 to derive the following integration formulas:

(a) $\int_{C_0}\frac{dz}{z-z_0}=2\pi i$; (b) $\int_{C_0}(z-z_0)^{n-1}dz=0$, ($n=\pm1, \pm2, \ldots$).

P10-11

Let C denote the line segment from $z=i$ to $z=1$. By observing that, of all the points on that line segment, the midpoint is the closest to the origin, show that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$ without evaluating the integral.

P10-12

Show that if C is the boundary of the tringle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction, then

$$\left|\int_C \left(e^z - \bar{z}\right)dz\right| \leq 60 .$$

P10-13

Let C_R be the circle $|z|=R$ ($R>1$), described in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{Log\ z}{z^2}\,dz\right| \leq 2\pi\left(\frac{\pi + ln\,R}{R}\right),$$

and use ***l'Hospital***'s rule to show that the value of this integral tends to zero as R tends to infinitey.