CV1 Complex Numbers: Basic Operations and Properties

"Complex Variables" is an important course for all the engineering students to learn since a lot of techniques are developed by the use of complex numbers, such as the famous Fourier transform and Laplace transform which have been widely applied in the field of signal processing, control theory, electrical circuit, and communication.

Here we will start from the discussion of algebraic and geometric properties related to the complex numbers. Unlike real numbers, a complex number z is often represented as

$$z=x+iy$$

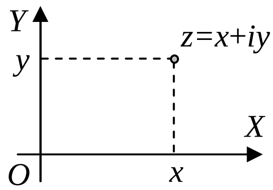
where x and y are real numbers and $i=\sqrt{-1}$ is the imaginary unit. Note that the symbol j is also used to represent the imaginary unit, i.e., $j=\sqrt{-1}$. In Matlab, both i and j can be used to represent $\sqrt{-1}$.

Matlab:

```
>> i
ans =
      0 + 1.0000i

>> j
ans =
      0 + 1.0000i
```

A complex number can be also interpreted as a point in the complex plane where the real part x is displayed on the real axis and the imaginary part y is on the imaginary axis.



Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Below list some basic algebraic operations commonly used in complex numbers:

(1) addition: $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$

(2) product: $z_1z_2=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)$

(3) commutative laws:

$$z_1+z_2=z_2+z_1 \quad \text{and} \quad z_1z_2=z_2z_1$$

(4) associative laws:

$$(z_1+z_2)+z_3=z_1+(z_2+z_3)=z_1+z_2+z_3$$

$$(z_1z_2)z_3=z_1(z_2z_3)=z_1z_2z_3.$$

(5) distributive law: $z(z_1+z_2)=zz_1+zz_2$

(6) additive identity 0: $z+0=z$

(7) multiplicative identity 1: $z\cdot 1=z$

(8) additive inverse: $z+(-z)=0$ where $-z=-x-iy$

(9) multiplicative inverse: $zz^{-1}=1$ where $z\neq 0$.

With the additive inverse, we can define the subtraction as

$$z_1-z_2=z_1+(-z_2)=(x_1-x_2)+i(y_1-y_2).$$

In addition, we can obtain

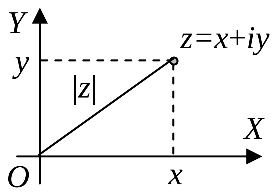
$$z^{-1}=\frac{1}{z}=\frac{1}{x+iy}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}.$$

which is derived the definition of multiplicative inverse

In the complex plane, the length of a complex number z to the origin O is defined as its modulus, denoted as

$$|z|=\sqrt{x_2^2+y_2^2}.$$

which is also called the absolute value or the magnitude of z.

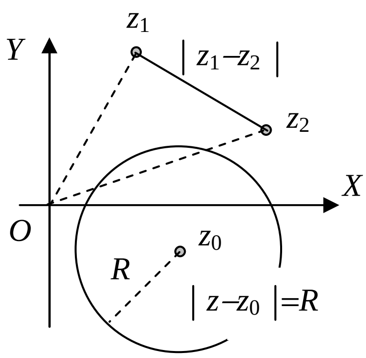


In complex numbers, the inequality $z_1< z_2$ is meaningless. Instead, the inequality $|z_1|<|z_2|$ is used to represent that z_1 is closer to the origin than z_2. Moreover, the length of the difference between z_1 and z_2 is

$$|z_1-z_2|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

which is the distance between z_1 and z_2. With the use of moduli, a circle centered at z_0 with radius R can be expressed as

$$|z-z_0|=R.$$



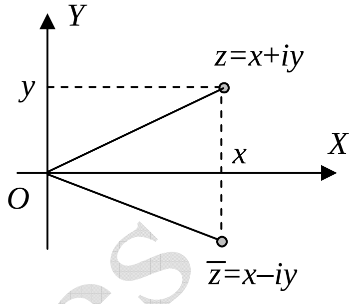
Similar to the real numbers, the complex numbers must satisfy the triangle inequality:

$$\big||z_1|-|z_2|\big|\le|z_1\pm z_2|\le|z_1|+|z_2|$$

For more than two complex numbers, the inequality

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

is also true for $n \ge 3$.



The other special operation is the conjugate of a complex number z which is defined as

$$\bar{z} = x - iy \quad \text{or} \quad z^* = x - iy$$

as depicted in the figure. Clearly, the conjugate $\bar{z}$ changes the numeric sign of the imaginary part of z. In Matlab, the conjugate of z is written as z'.

Matlab:

```
>> z=1+2i
z =
      1.0000 + 2.0000i

>> abs(z)
ans =
      2.2361

>> z'
ans =
      1.0000 - 2.0000i
```

According to the definition of conjugate, we have the following properties:

$$\bar{\bar{z}} = z,$$

$$|\bar{z}| = |z|,$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},$$

$$z \cdot \bar{z} = |z|^2,$$

$$|z_1 z_2| = |z_1| \cdot |z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

$$Re\, z = \frac{z + \bar{z}}{2}, \qquad Im\, z = \frac{z - \bar{z}}{2i}$$

which are important operations in the field of complex numbers.

Problems

P1-1 Reduce each of the these quantities to a real number:

(a) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$; (b) $\dfrac{5i}{(1-i)(2-i)(3-i)}$; (c) $(1-i)^4$.

Ans: (a) −2/5, (b) −1/2, (c) −4

P1-2 Use mathematical induction to verify the binomial formula:

$$(z_1+z_2)^n=\sum_{k=0}^{n}\binom{n}{k}z_1^{n-k}z_2^k.$$

Note: First let the formula be true when n=1. Then, assuming that it is valid when n=m where m denotes any positive integer, show that it must hold when n=m+1.

P1-3 Verify that $\sqrt{2}|z|\ge|Re\,z|+|Im\,z|$.

P1-4 Show that $\left|Re\left(2+\bar{z}+z^3\right)\right|\le 4$ when $|z|\le 1$.

P1-5 Show that if z lies on the circle $|z|=2$, then $\left|\dfrac{1}{z^4-4z^2+3}\right|\le\dfrac{1}{3}$.

Note that you can factorize z^4-4z^2+3 into two quadratic factors first and then use the inequality $\big||z_1|-|z_2|\big|\le|z_1\pm z_2|$.

P1-6 Using $Re\,z=\dfrac{z+\bar{z}}{2}$ and $Im\,z=\dfrac{z-\bar{z}}{2i}$, show that the hyperbola $x^2-y^2=1$ can be written as $z^2+\bar{z}^2=2$.

P1-7 Let $z_1=3+i$, $z_2=1-2i$ and $z_3=\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$.

Calculate the following terms by Matlab:

(a) $\overline{z_1-z_2^2}$, (b) $\left|z_2^3-z_1\right|$, (c) $\left(\dfrac{z_2-\bar{z}_3}{z_1+z_2}\right)^2$.

Ans: (a) 6-5i, (b) 14.0357, (c) −0.0613 − 0.0832i