Calculus: Homework 4

March 20th, 2008

1. The following function

$$g(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} \mathrm{d}t$$

is called *elliptic function of the second kind*. Show that for $|k| < 1$

$$G(k) = \frac{\pi}{2} - \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \right)^2 \frac{k^{2n}}{2n-1}.$$

2. Assume that $a < b$ and let L denote the arc length of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Show that

$$L = 4bG(k),$$

where G is defined as in Example 1 and $k = (1 - a^2/b^2)^{1/2}$. Use the first three terms from the expansion from Example 1 to estimate the arc length when $a = 4$ and $b = 5$.

3. Let $\mathbf{e}_\theta = \langle \cos\theta, \sin\theta \rangle$. Show that $\mathbf{e}_\theta$ is a unit vector making an angle θ with the x-axis. Moreover, show that $\mathbf{e}_\theta \cdot \mathbf{e}_\psi = \cos(\theta - \psi)$ for all angles θ and ψ.

4. Show the following identity

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

5. Find the nearest point to $Q = (-1, 3, -1)$ on the plane $x - 4z = 2$.