Homework 9

Chapter14.7. :42

42. Maximize $f(x,y)=x^a y^b (100-x-y)^c$.
$f_x=ax^{a-1}y^b(100-x-y)^c-cx^a y^b(100-x-y)^{c-1}=x^{a-1}y^b(100-x-y)^{c-1}[a(100-x-y)-cx]$
and $f_y=x^a y^{b-1}(100-x-y)^{c-1}[b(100-x-y)-cy]$. Since x , y and z are all positive, the only critical point occurs when $x=a\frac{100-y}{a+c}$ and $y=\frac{100b}{a+b+c}$. Thus the point is $\left(\frac{100a}{a+b+c},\frac{100b}{a+b+c}\right)$ and the numbers are $x=\frac{100a}{a+b+c}$, $y=\frac{100b}{a+b+c}$, $z=\frac{100c}{a+b+c}$.

Chapter14.8. :1

1. At the extreme values of f , the level curves of f just touch the curve $g(x,y)=8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y)=c$ with the largest value of c which still intersects the curve $g(x,y)=8$ is approximately $c=59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y)=8$ appears to be $c=30$. Thus we estimate the maximum value of f subject to the constraint $g(x,y)=8$ to be about 59 and the minimum to be 30 .

Chapter14.8. :44

44. **(a)** Let $f\left(x_1,\dots,x_n,y_1,\dots,y_n\right)=\Sigma_{i=1}^n x_i y_i$, $g\left(x_1,\dots,x_n\right)=\Sigma_{i=1}^n x_i^2$, and $h\left(x_1,\dots,x_n\right)=\Sigma_{i=1}^n y_i^2$. Then $\nabla f=\nabla\Sigma_{i=1}^n x_i y_i=\left\langle y_1,y_2,\dots,y_n,x_1,x_2,\dots,x_n\right\rangle$, $\nabla g=\nabla\Sigma_{i=1}^n x_i^2=\left\langle 2x_1,2x_2,\dots,2x_n,0,0,\dots,0\right\rangle$ and

$\nabla h=\nabla\Sigma_{i=1}^n y_i^2=\left\langle 0,0,\dots,0,2y_1,2y_2,\dots,2y_n\right\rangle$. So $\nabla f=\lambda\nabla g+\mu\nabla h\Leftrightarrow y_i=2\lambda x_i$ and $x_i=2\mu y_i$, $1\le i\le n$.
Then $1=\Sigma_{i=1}^n y_i^2=\Sigma_{i=1}^n 4\lambda^2 x_i^2=4\lambda^2\Sigma_{i=1}^n x_i^2=4\lambda^2\Rightarrow\lambda=\pm\frac{1}{2}$.
If $\lambda=\frac{1}{2}$ then $y_i=2\left(\frac{1}{2}\right)x_i=x_i$, $1\le i\le n$. Thus $\Sigma_{i=1}^n x_i y_i=\Sigma_{i=1}^n x_i^2=1$. Similarly if $\lambda=-\frac{1}{2}$ we get $y_i=-x_i$ and $\Sigma_{i=1}^n x_i y_i=-1$. Similarly we get $\mu=\pm\frac{1}{2}$ giving $y_i=\pm x_i$, $1\le i\le n$, and $\Sigma_{i=1}^n x_i y_i=\pm 1$. Thus the maximum value of $\Sigma_{i=1}^n x_i y_i$ is 1 .

(b) Here we assume $\Sigma_{i=1}^n a_i^2\neq 0$ and $\Sigma_{i=1}^n b_i^2\neq 0$. (If $\Sigma_{i=1}^n a_i^2=0$, then each $a_i=0$ and so the inequality is trivially true.) $x_i=\frac{a_i}{\sqrt{\Sigma a_j^2}}\Rightarrow\Sigma x_i^2=\frac{\Sigma a_i^2}{\Sigma a_j^2}=1$, and $y_i=\frac{b_i}{\sqrt{\Sigma b_j^2}}\Rightarrow\Sigma y_i^2=\frac{\Sigma b_i^2}{\Sigma b_j^2}=1$. Therefore, from

(a), $\Sigma x_i y_i=\Sigma\frac{a_i b_i}{\sqrt{\Sigma a_j^2}\sqrt{\Sigma b_j^2}}\le 1\Leftrightarrow\Sigma a_i b_i\le\sqrt{\Sigma a_j^2}\sqrt{\Sigma b_j^2}$.